**Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem**

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**Lecture-06**
**The Open Mapping Theorem**

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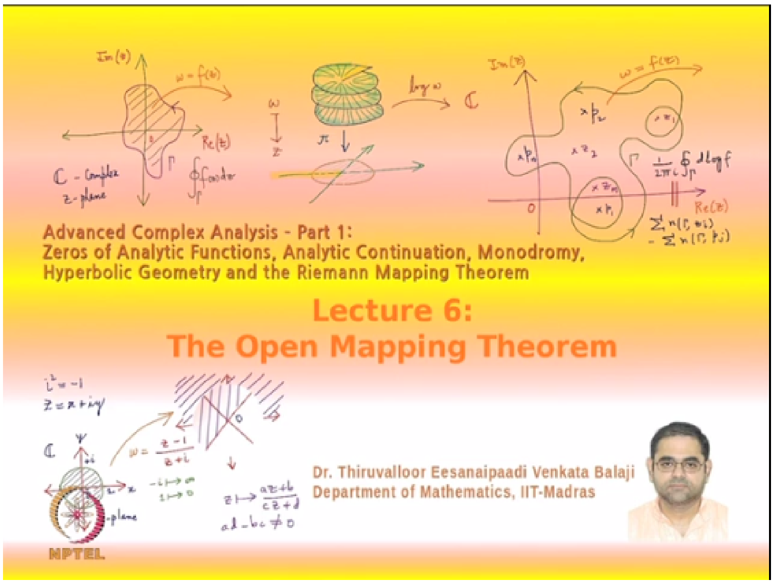


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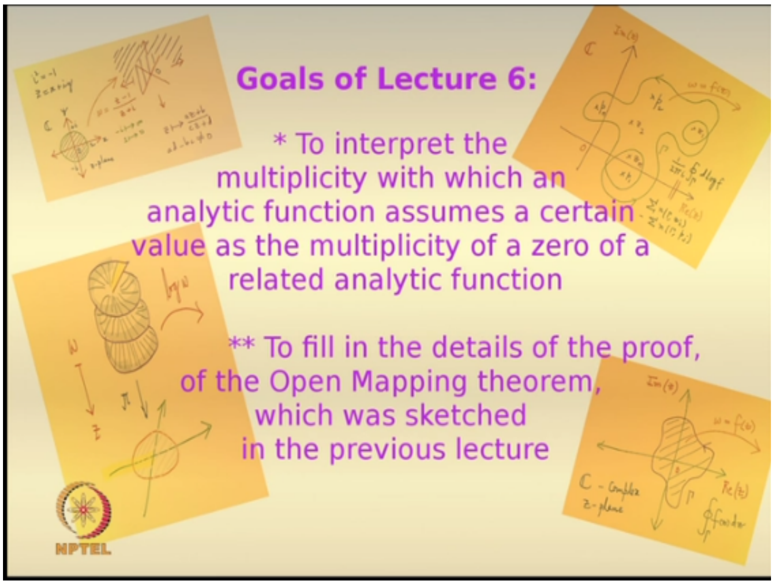


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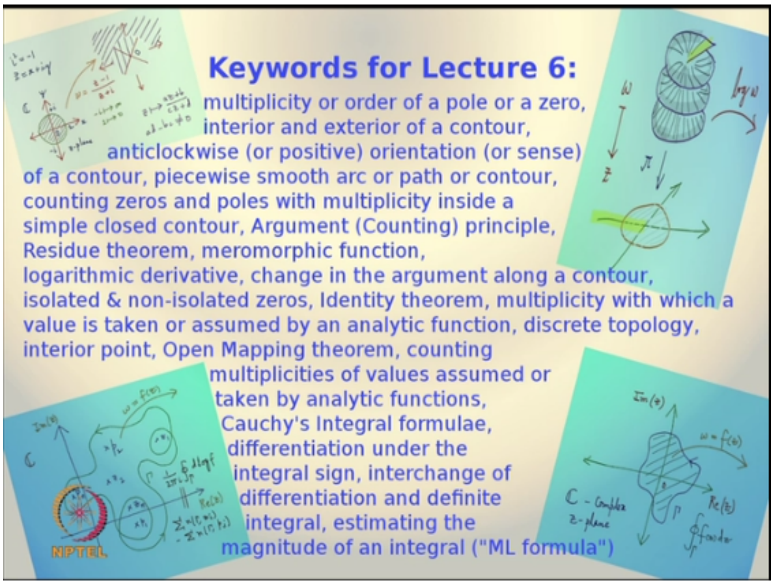


Continue with our discussion of the various important theorems concerning zeros of analytic functions. So, what I described at briefly at the end of the previous lecture was the so called open mapping theorem okay. So, let me go ahead and try to give a proof of that and I must say that once you look at the at the proof that you will see that from that proof you can also get the proof of the so called inverse function theorem okay.

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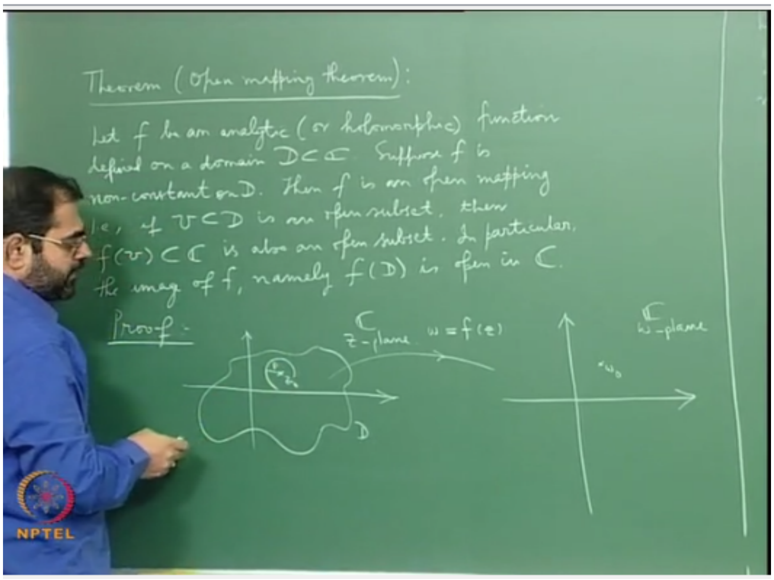


So, so we start like this you so the theorem let me state the theorem open mapping theorem, so what is this say let f be an analytic function analytic or holomorphic function defined of course defined and analytic on a domain D in the complex plane then . Suppose f is non constant on D okay, then f is an open mapping, that is if u inside D is an open subset.

Then f of u inside complex plane is is also an open subset, so in particular the image of f namely f of D is open in C, so this is open mapping theorem. It says that non constant analytic function non constant holomorphic function carries open sets to open sets it is a very deep theorem and the proof of the theorem of course uses the argument principle which is actually the residue theorem applied to the logarithmic derivative of a suitable function okay .

So, so you know so let me draw a diagram, so let me tell you what the proof, so let me go ahead with the proof. So, here is the complex plane this is the source complex plane C, this is the z plane and here if you want my domain D of course the way I have drawn it my domain D looks like a bounded domain mind you let me recall that a domain is an open connected subset okay, it is a subset which is both open and which is connected.

Of course the way I have drawn it looks like a bounded domain bounded by this boundary curve I will just drawn like that for simplicity but it not look like this okay, it need not be bounded at all okay. And then I have this mapping w=fz which takes any if I start with a point z0 then it will take it will map it to a complex value, the values are taken in another copy of the complex plane which is called the w plane or the omega plane.

And well z0 goes to some w0 okay and what we want to show is that the image of any open set is open okay. So, what I am going to do is so the as I told you the point is so let me write that down.

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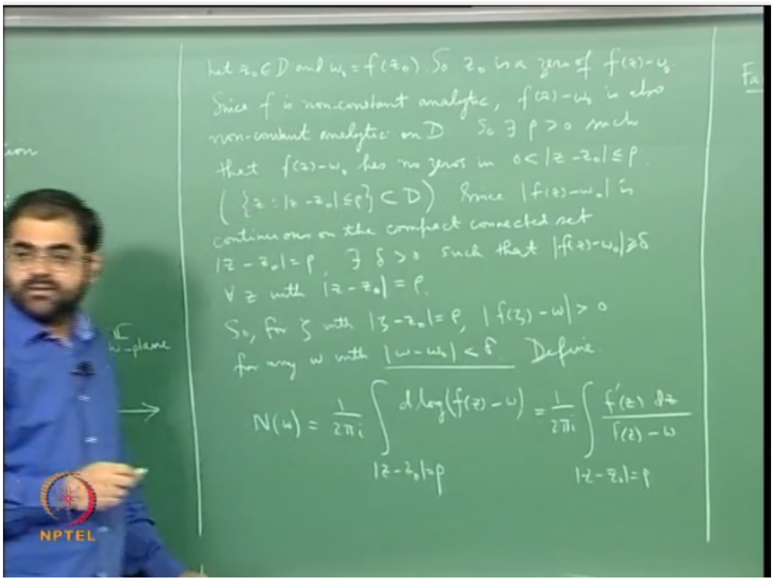

Let z0 be a point of D and of w0b=f of z0, so we are always going to see the whole point is you are going to think of values of a function as zeros of a suitable okay. So, that is always the idea, so values of so w0 is a value of f of z okay that should be translated as 0 w0 is a value of f of z at z=z0 that should be translated as z0 is a 0 of f of z-w0 that is how you should translate everything in terms of zeros okay.

And why do you translate everything in terms zeros because you can then apply the counting principle the argument principle which allows you to count the number of zeros you, so that is a whole point okay. So, thus so z0 is a 0 of f-f 0f z-w0 right but notice that f of z is analytic it is holomorphic and non constant okay . So, f of z-w0 is also holomorphic it is also analytic, it is also non constant.

So, since f is non constant analytic fz-w0 is also non constant analytic on D okay and this is the whole point you have a non constant analytic function and you have a 0 of that you know then that the zeros of a non constant analytic function are isolated okay, that is the that is why I made this observation to begin with that it is non constant analytic okay and that since zeros are isolated it means that there is a disc surrounding the 0 z0 of this function.

Such that in the disc, inside the disc and on the boundary of the disc there are no other zeros of the function f-w0 okay. So, so there exist rho positive such that f of z-w0 has no zeros in zeros strictly less than mod z-z0 less than or equal to rho, so the only 0 is that z=z0 okay and there are no zeros in this deleted closed disc right. So, if I draw it in the diagram here so you have a disc D I mean you have this small disc centred at z0 radius rho okay and perhaps this diagram to small let me try to draw slightly bigger diagram.

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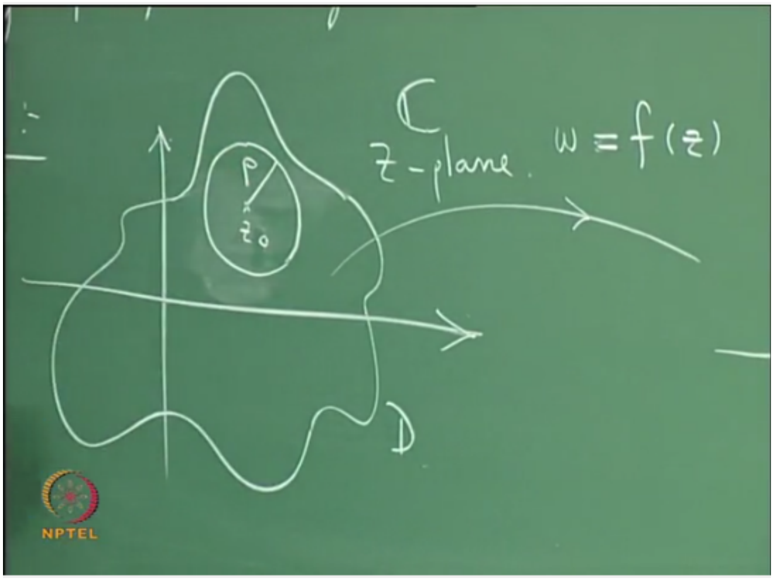


So, so here is z0 z0 and here is the disc centred at z0 radius rho and in this disc of course this disc 0 set of all points in the disc the set of all z such that mod z-z0 is less than or equal to rho is of course contained in D. of course you are looking at a disc inside D okay I mean what is the theorem on isolation of zeros of an analytic function it says you take an analytic function which is not constant on an open set, given a 0 you can find a disc surrounding that 0 where there are no other zeros.

And this disc is of course in that open set on which the analytic function is defined. Therefore this disc is chosen inside D that you must remember. So, this is because of the fact that the zeros of an analytic function are isolated okay and which if you go back and recall it is actually another way of saying the so called identity theorem which says that you know.

If you have an non isolated 0 then the only possibility is in the function is throughout 0, identically equal to 0 which means it becomes constant and the but that is not true it is we have assumed it is true the function to be a non constant function okay. So, also you know we have use this idea many times before if you take the function and restricted to this boundary circle okay.

If you in particular look at the modulus of the function the boundary circle then that modulus will have a lower bound and an upper bound that is because mod of f-f of z-w0 will be a continuous real valued function on this boundary circle okay which is compact and a continuous real valued function which is compact a continuous real valued function defined on a compact connected set the image will be a closed interval okay.

Therefore it will have a minimum value, it will have a maximum value, so since mod of f of z-w0 is continuous on the compact connected set mod z-z0=rho namely the boundary circle there exist at delta positive such that mod of f of z-w0 is greater than or equal to delta for all z with mod z-z0 =rho okay. This is just the fact that f of z-w0 is analytic, so mod f is continuous okay, f of z-w0 is analytic is an analytic function of z it is a holomorphic function of z.

So, it is certainly continuous because analytic functions are continuous and then modulus of a continuous function is again continuous because you are composing the function with mod function and mod function is continuous it is a composition of 2 functions, so it is continuous. And this is a continuous real valued function in fact it has non negative real values because it is a mod and if you restricted to this compact connected set the image will be again a compact connected subset of the real line.

So, that will be a closed interval on the real line it will be a finite closed interval on the real line and it is and the minimum value of the interval it is exactly what I am calling as lamda and mind you the I am sorry as delta and this delta is positive because the function does not vanish on the boundary circle okay, the only place where it vanishes at the centre it is the only 0 of it has okay, so you have this.

Now what you are going to do you do the following thing, so you see take any w in the target complex plane with the property that the distance from w to w0 is less than delta okay. So, from w0 you draw a disc okay centred at w0 radius delta and look at any w inside this disc okay what you must understand is that if you take a point.

If you start with a point zeta here on the boundary circle then the modulus of f the modulus of f of z –w0 at zeta which is mod f zeta-w0 that is greater than or equal to delta that will tell you that f zeta is going to lie outside because you see what is the assumption, the assumption is you take if this zeta is on the boundary circle and for points on the boundary circle the function value-w0 mod is greater than or equal to delta.

So, mod f zeta-w0 is greater than or equal to delta, the distance of f zeta the value of f of z at zeta –w0 mod of that is greater than or equal to delta that means that f zeta will goes outside this disc. In particular what this tells you is that you know f of z f zeta cannot be equal to any w in this disc okay, so let me write that down. So, for zeta with mod zeta-z0=rho the modulus of f of zeta-w is greater than 0 for any w with mod w-w0 lesser than delta okay, this is a very simple observation.

But why this is so important is because it can define the following thing, define N of w to be 1 by 2 pie i integral over mod z-z0=rho of d log f of z- w0 –w, I want you to understand this I mean in other words what is this I mean this just 1 by 2 pie i integral over mod z-z0=rho f of f dash of z by f of z-w dz this is what it is, that is the logarithmic derivative okay.

First of all what I wanted to know is that the what does the argument principle tell you, the argument principle tells you that whenever you take d log of an analytic function and integrate it over a curve okay and divide by 2 pie i what will get is the number of zeros-number of poles inside that curve okay. So, if you go by that I will get number of zeros-number of poles of f of z-w but there are no poles.

Because f of z-w is of course it is analytic okay and f of z-w is not 0 mind you, f of z-w for z on the boundary circle, that is for a value of z=zeta that mod zeta-z0=rho f of z the value of f of z-w will become f of zeta-w and that is positive. So, this quantity is never going to vanish this quantity it is modulus is never going to vanish on the boundary curve, in particular it means this quantity itself is not going to vanish on the boundary curve.

Therefore this integral is well defined okay and the function f of z-w has no poles what it has is only zeros. So, what this will give you is if you this yeah if I take f of z-w0 then what I will get is the number of zeros of f of z-w0 which is N w0 okay which the number of times the value w0 is taken okay it is a multiplicity of the 0 of f of z-w0 okay.

And if I install that if I put w what I will get is the number of the times the value w is taken see the number of times the value w is taken is as same as the number of zeros of f of z-w okay, the number of times a value w is taken by f of z is the same as the number of zeros of f of z-w okay. So, this is the number of so you should think of this as number of times the value w is taken okay, so you this is just a application of the argument principle, the counting principle.

And it is purely which purely is just the residue theorem nothing more than that okay, now but the beauty of this thing is that you can count you have a formula you have a nice formula for number of times an analytic function takes a value okay. So, now the this where the fact is that following.

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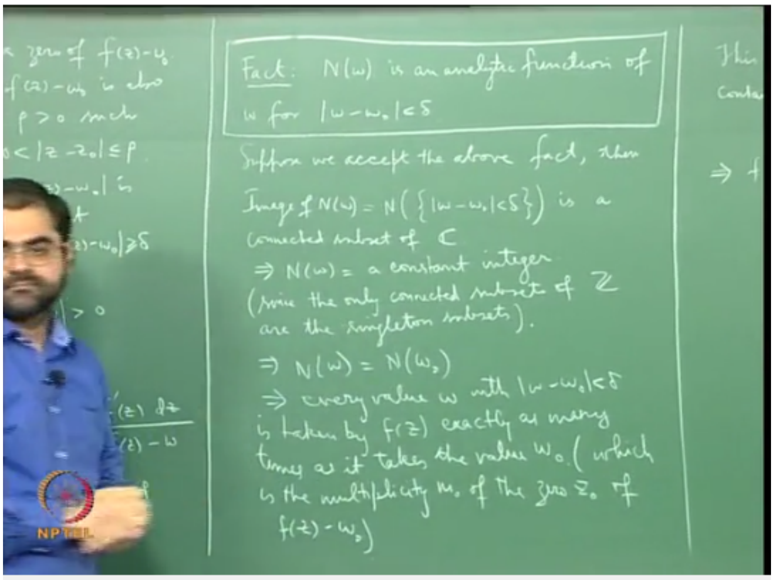


So, I will tell you what is the fact is, the fact is that N of w, so I should tell you let me again remind you where is this N of w is defined, it is defined for all w with this property okay, so this N of w is going to be defined on this disc okay, N of w is going to be defined on this disc and what are the values it takes, it takes integer values because N of w counts the number of times the value w is taken okay. So, what you are going to have is let me continue let me extend this diagram.

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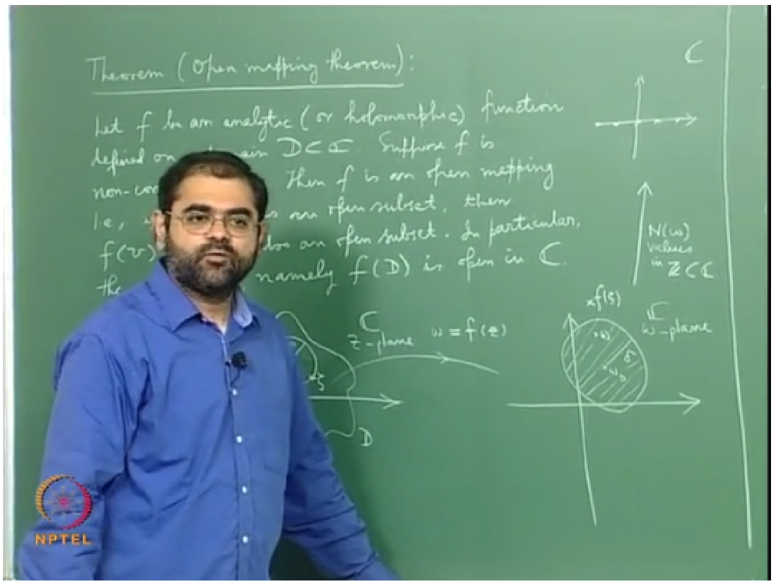

So, you have a map like this, this is N of w and this map goes again into the complex plane if you want but actually it goes into integers, so I will put the complex plane here. But actually values in set of integers which is contained in complex numbers, so it takes values only on the real axis okay and the values are only the integer values takes only integer values, so it takes only discrete set of values okay.

Now the fact is a following, you have a function from defined on this open disc okay which takes values in the complex plane of course the values are integer values forget the fact they are integer values for the moment think of this function as simply taking values in the complex plane. The important fact is this function depends on this w as w varies over this disc, the important fact is this N of w is an analytic function of w, that is the crucial fact, N of w is an analytic function of w for mod w-w0 strictly less than delta, this is the fact okay.

Now this so I am telling you the heart of the theorem is in this statement okay, the heart of the proof is in this statement. So, let us accept this for the moment and see how you can get the theorem and then go on to prove this separately alright. So, you see suppose we accept the above fact then if I take the image of N that will be N of the whole disc.

If I take the image of N under this disc you see N of w is an analytic function of w you know analytic is of course continuous, so you are taking the image under continuous map of this disc okay. But what is this disc, this is disc both connected okay it of course the point that I have wanted z is connected and if you take the image of a connected set under a function it is again connected.

So, this is going to be is a connected subset of the complex plane okay, it is a connected subset of the complex plane. But mind you what are the values it is taking, the values it is taking are integer values okay. So you are going to get a connected subset of integers but you know the integers are discrete points. The only connected sets are the individual points okay, the only connected subsets of the set of integers are the single time sets.

If I take a subset having more than 1 point that will be disconnected because it can be broken down into union of single points and each single point is closed and it is also opened because it is the discrete upon a function okay. So, there is some topology here, so this implies that N of w=a constant integer because the only connected subset of integers are singleton subsets since the only connected subsets of zee are the singleton subsets.

Of course when I say only connected subsets I am not worrying about then null set okay it can be the null set is always a if you want topologically maybe one can take null set as a connected set okay . But the point is I am not worried about the null set here, the fact is that the N is taking N is going to take some value okay, N is going to give me some value, the value is an integer.

That is assured because of the argument principle, so it going to take a value, the so the image is this is not going to be empty it has to take a value and fact is that can take only that value because N is analytic. So, the only thing I am using here is that N is continuous I am not using anything more than that, I am not using the full power of the fact that N is analytic, the only weaker thing that I am using from here is that N is actually only contiguous okay as a function of w, that will give me N of w is a constant okay.

But what is this imply, this implies N of w is the same as N of w0 because N of w does not depend on what w is, why I can put w=w0 okay, I can put w=w0 because w can vary anywhere inside this disc w is a variable point in this disc. So, I put w equal to the central point w0, but what is N of w0? N of w0 is a number of times the function takes the value w0 I mean the number of times the function f of z takes the value w0 which is the same as the number of zeros of f of z-w0 okay.

So, what does this tell you, this tells you that for every value w in this disc N the function f of z does take that value and it takes it as many times as it takes the value w0 that is what it is says. So, this implies every value w with mod w-w0 strictly less than delta is taken by f of z exactly as many times as it takes the value w0, that is what it says which and of course the number of times it takes the value w0 is going to be the multiplicity of the 0 of f at w0 of **f** f-w0 at z0 okay.

So, let me write that which is the multiplicity m0 of the 0 z0 of f of z-w0 okay, in particular what does this tell you this tells you that this whole disc is in the image all these values are taken by the function, this whole disc in the image. So, go back to our argument I started with an analytic function depending on a domain okay I assumed it is non constant, I took a value z0 and took the value of the function z0 that is omega0.

And what I went up proving is a there is a whole discs surrounding w0 full of values of the function that is means is this whole disc is contained in the image of f okay. So, for w0 is of course in the image of f because it is a value of f what is the image of f it is the values of f, the set of subset of values of f okay. So, w0 is certainly in the image of f and what this argument says is give me w0 which is in the image of f I have a whole disc surrounding w0 which is also in the image of f.

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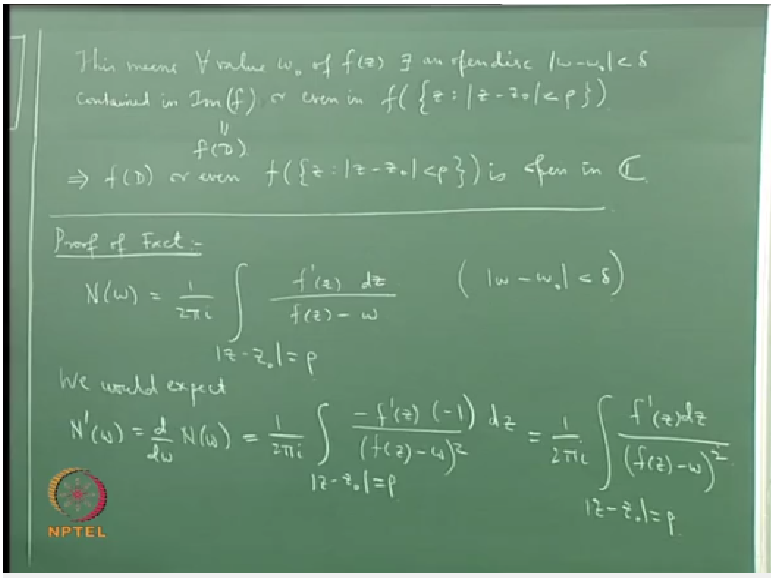


But that is a condition which says that every point of the image of f is an interior point but that tells you the image of f is open okay. So, what this tells you, so this implies thus this means for every value w0 of f of z there exist an open disc mod w-w0 lesser than delta contained in image of f okay. And in fact you see and these values what are points at we are looking at these values you are the points at which I looking these values of the points in this disc okay.

So, in fact what you are saying is the values are counting are the values inside this disc okay, so for every point z in I mean if you look at the function restricted to this open disc there itself the function takes those many values okay. So, in fact it is in the image of that disc or even in f of the set of all z such that mod z-z0 is less than rho.

In fact the image is inside this which is of course the subset of image of f which is f of D, so what this tells you is that f of D or even f of is open in the complex. And that finishes the proof of the open mapping theorem okay that finishes the proof. So, the only thing that I will have to check is this fact, so this fact is the fundamental thing that I will have to check this fact okay.

Once you check this fact you have done okay, so how does one do, so let us check that fact proof of the fact. So, you know, let us let me write this down Now is defined to be 1 by 2 pie i integral over mod z-z0=rho and mind you whenever I write integral over a curve you are always taking the anti-clock wise positive orientation. So, whenever I when I am integrating over this curve.

Of course if you want I will put a circular, I mean I put an arrow on this circle saying that you are going the positive direction which is the anti-clock wise direction, the direction is inside in such a way that the area in which you are counting the zeros is lies to the left as you walk on the curve in the direction specified. So, if you walk on this in this direction then to your left lies this interior of the curve and to the right is the exterior of the curve okay.

And it is in the interior of the curve that you are counting the number zeros okay more generally the residue theorem you have counting the number of zeros-number of poles okay and that is in the interior of the curve. So, the interior is very very important the interior specified by the orientation, so whenever there is an integral please remember there is always an orientation okay.

And in this you usually assume only the anti-clock wise orientation, so let me write that down let us f dash of z dz by f of z-w alright. So, you see what now let us try to you know scientifically I mean let us try to think of a proof of this you want show N of w is there analytic function of w that means you have to show that N of w is a is differentiable with respect to the variable w.

So long as w lies inside this disc, the set of all points centred at I mean points whose distance from w0 is strictly less than delta okay, see suppose it is differentiable okay, so think you suppose it is differentiable think of w and z as variables that do not depend on each other okay, then how do I get the derivative here, I just get the derivative here by applying d by dw okay.

And if I apply d by dw I will get N dash of w which is the first derivative of w and so that is the same as applying d by dw on this side. But this is the integration is only with respect to the variable z, the integration has got nothing to do with the variable w. So, if you can push the differentiation inside if you can switch the integration and differentiation then you can guess what you should expect as the derivative of the of N of w okay.

We would expect N dash of w=d by dw of N of w to be equal to 1 by 2 pie i integral over mod z-z0=rho differentiate with respect to w, if you differentiate this with respect to w what you will get is, you will get f dash of z is of course a constant it does not involve w the only term that you know w is this okay and the differentiation of that will be well you will get f of z-w the whole square because differentiation of 1 by t is -1 by t square okay.

So, I will get a – here and then I have to differentiate this f –w with respect to w and I will get a -1 and I will get these, this is what, so in other words you will get it expect 1 by 2 pie i integral over mod z-z0=rho f dash of z dz by f of z-w the whole square this is what you could expect okay. But what have I but what is it that you used use the fact that you can interchange the integration and differentiation by w okay.

And this needs justification because it is a technical thing you cannot blindly interchange integration and differentiation okay and there are many ways to prove that this is correct okay many ways to prove this that this is correct, that if you prove this then you are proving that w is differentiable okay all this is happening with mod w-w0 strictly less than delta okay, that is where you are that is disc on which your w is varying okay.

So, if this is if you can prove this then you have N of w is differentiable everywhere in that disc but that means it is analytic because analytic is differentiable not just at a point but differentiable everywhere in a disc surrounding the point okay, this is what you would have expect and this is correct okay. So, you should remember that even the the usual Cauchy’s integral is actually a statement that involves this kind of idea, see the usual Cauchy integral formula is also actually a statement of this type.

Because you know you should think about it so let me I mean I am telling you so that it will help you to you know it will help you to understand that this is the standard thing that happens and you should recognise it when it happens, so you know if you want .

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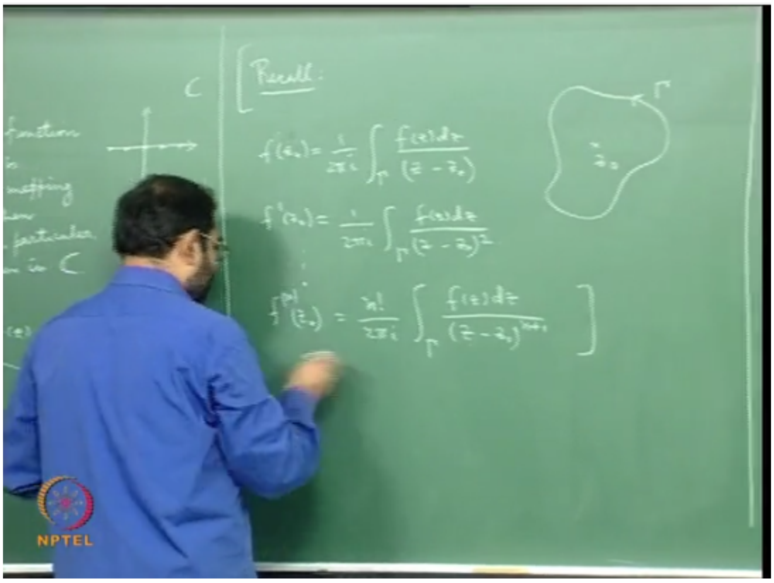


So, I will say recall suppose you know suppose we have a point z0 and suppose we have a simple closed curve gamma surrounding z0 then you know that f N of so you know that f of z0 is given by 1 by 2 pie i integral over gamma f of z dz by z-z0 this is your usual Cauchy theorem this is for N=1. Then if you calculate f dash of z0 you will get 1 by 2 pie i integral over gamma f of z dz by z-z0 the whole square.

And more generally you will get fn of z0=n factorial by 2 pie i integral over gamma f of z dz by z-z0 to the power of probably I will get n+1, this is more generally what this is your Cauchy integration. But what is how do you get this formula actually see if you start with this formula then you can get all the other formulas simply by noting that f dash of z0 is differentiate this with respect to z0, think of z0 as a variable.

If you differentiate this with respect to z0 so you have differentiate this term with respect to z0 okay but then assume that you can interchange integration and differentiation then you will be differentiating this but if you differentiate this you will get this. And inductively if you do it you will get all these formulas, so the whole philosophy of a Cauchy integral formula itself is the fact that you actually differentiate under the integral sign okay.

And is a same kind of idea that I want to use here and that is the same kind of idea that given me this okay. So, in fact what I want to say is that if you know the proof of the Cauchy integral formula, then you know the proof of that there is nothing different, that is really nothing different. But nevertheless let me do it okay just to tell you how this techniques are proved I mean how this techniques work**.**

So, you know it also gives you proof of Cauchy integral formula if you want, so you know I want to show the derivative of this is this okay. So, what does it mean how do I prove that okay let us go back to that, you see.

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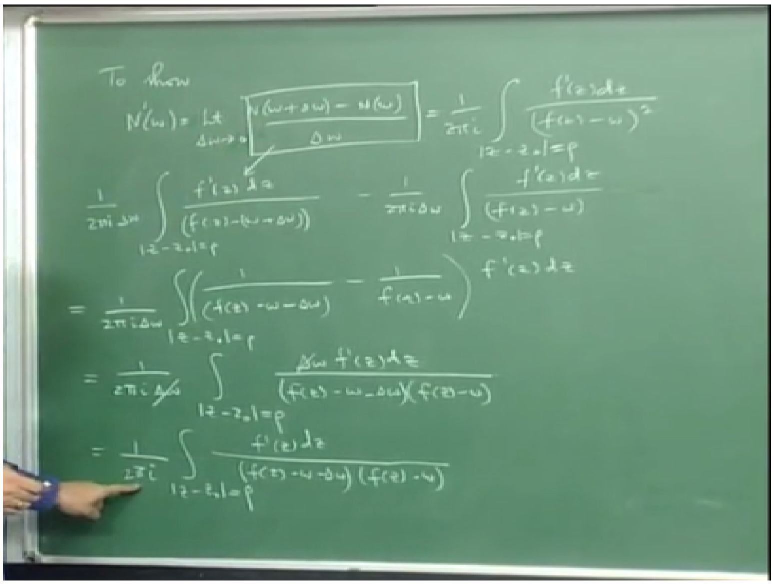


Now what is N dash of w by definition you want to show this is limit delta w tends to 0, N of w+delta w-N of w by delta w this is the derivative, this is the usual definition of derivative okay. And of course all this is happening w+delta w and is also in this disc okay, this is how we do the derivative this is for the definition of derivative and you want to show that this is equal to this 1 by 2 pie i integral over mod z-z0=rho f dash z dz by f of z-w the whole square, this is what you want to show.

So, what is this quantity okay, if you write it down it is 1 by 2 pie i into delta omega integral over mod z-z0=rho and N of w+ delta w is f dash of z dz divided by f of z-w+delta w- will get why again 1 by 2 pie i delta w integral over mod z-z0=rho f dash of z dz divided by f of z minus w okay this is what it is which if I combine 1 by 2 pie i delta w outside I will get integral over mod z-z0=rho 1 by fz-w-delta w-1 by fz-w into fz of z dz, this what you get.

And then if you cross multiply take LCM then what you will get is essentially 1 by 2 pie i delta w integral over mod z-z0=rho the numerator I am going to get a delta w*fz of z dz divided by fz-w-delta w*fz-w okay. And since delta w has got nothing to do with the variable of integration z you can bring it outside and cancel this with that. so, effectively what you will get is simply 1 by 2 pie integral over mod z-z0=rho f dash of f dz by f of z-w-delta w into fz-w.

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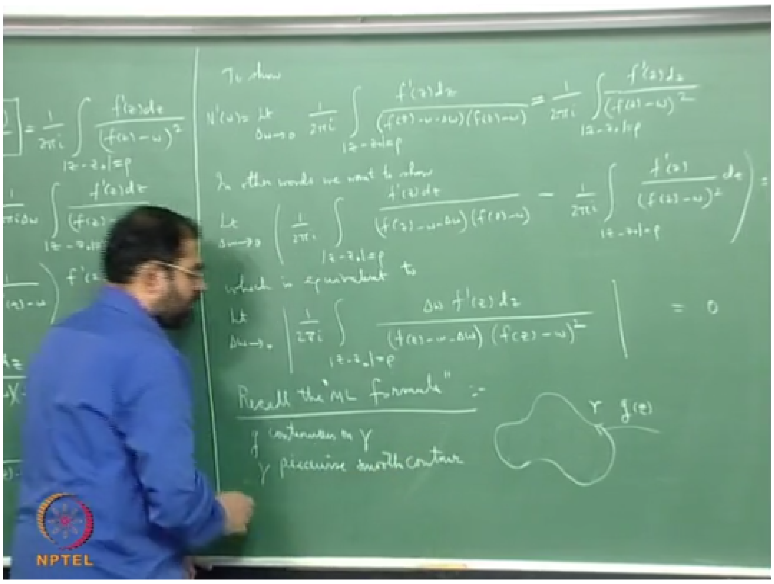

So, this is what this the quantity in the box reduces to this okay, so you have to show let me rewrite it. So, to show N dash of w=limit delta w tends to 0 of this quantity 1 by 2 pie i integral over mod z-z0=rho f dash of z dz divided by fz-w-delta w into fz-w that is equal to 1 by 2 pie i integral over mod z-z0=rho f of z dz by fz-w the whole square this is what you have to show okay. Now it is a statement that looks quite obvious because you know if for a moment if you can if you assume that you can push this limit inside the integral okay.

Then and if you take delta w if you let delta w tend to 0 you will obviously get this okay, so essentially this what you have to show is believably correct because if you can agree to switch the integral and the limit okay but technically you cannot do that all the time right, that also has to be proved okay. And how does 1 prove that you want to show the limit of the certain quantity is some other quantity.

Then it is enough to show that the limit of this difference is 0 okay, so we go by that. so, in other words we want to show limit as w0 goes to 0 of this quantity 1 by 2 pie i you combine both the integrals, the integral over mod z-z0=rho if I combine both I will get 1 by fz. So, let me write it like this f dash of z dz this into fz-w- 1 by 2 pie i integral over mod z-z0=rho f dash of z by f of z-w the whole square dz =0, this is what you want to show.

And mind you limit of this whole quantity is equal to 0, so I have pushed this quantity in should inside the limit I have brought the this quantity was in the right side I brought to the left side and I have push into the limit and I push it into the limit because this has no delta w term okay. So, this term is independent of delta w, so applying limit delta w to this does not give anything else except itself okay you get back this term.

And you know so I have to show as delta w tends to 0 I have to show that this quantity is 0 goes to 0 and to show that complex number goes to 0 it is enough to show that is modulus goes to 0. So, which is so this is equivalent to **to** limit delta w tends to 0 of modulus of this quantity push to 0 but even but let me do one more simplification let me take this 1 by 2 pie i integral common alright.

And if I go if I combine both the integrants okay what I will end up with is I will get 1 by 2 pie i integral over mod z-z0=rho I will get if I take the LCM it will be f of z-w-delta w into fz-w the whole square this is what you will get and here on the numerator I will get an Fz-w and for the atom I will I am fz-w-delta w, if I subtract I will implicate it delta w.

And what was there on top is already a common fz of z dz okay and you want to show that this is equal to 0 again I mean the limit of this quantity has delta w tends to 0 is 0 is what you want to show okay. And but to show that this quantity is 0 it is enough to show that the modulus of this quantity the limit of the modulus of that quantity is 0 okay.

So, now is where so you know you have to estimate this integral in magnitude okay that is you have to estimate the modulus of the integral as delta becomes very small and show that the estimate also goes to 0 okay or it is bounded by something that goes to 0 as delta w tends to 0. So, you see finally it boils down to estimating an integral and this is the trick or technique that you would have seen even in a first quotient complex analysis we use the so called ML formula.

So, what is the you recall the ML formula the so called ML formula which is in other words in it is also called as I mean you also remember it as modulus of the integral is lesser than the integral of the modulus okay what is that formula, see if you have a simple closed curve g I mean gamma. And you have a function f of z or let me even write as a g of z which is defined an analytic on the region enclosed by the simple closed curve and the boundary okay.

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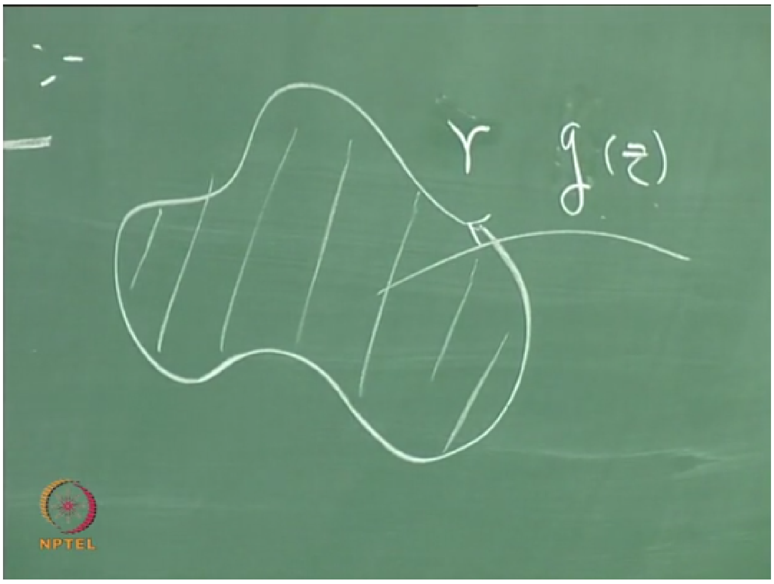


Perhaps I do not even need g to be you know analytic and all that it is enough if g is just continuous on the boundary it need not even be defined inside okay in I do not have g to be analytic I do not need to be worry about the interior of the curve okay. So, g continuous on the curve gamma okay gamma is of course piece wise smooth contour which means that a gamma is a continuous curve.

And when you parameterise gamma in as a function of the parameter it is first derivative is piece wise continuous okay. Then what the ML formula actually tells you is that if you integrate, if you calculate integral over gamma with this orientation it is some orientation okay.

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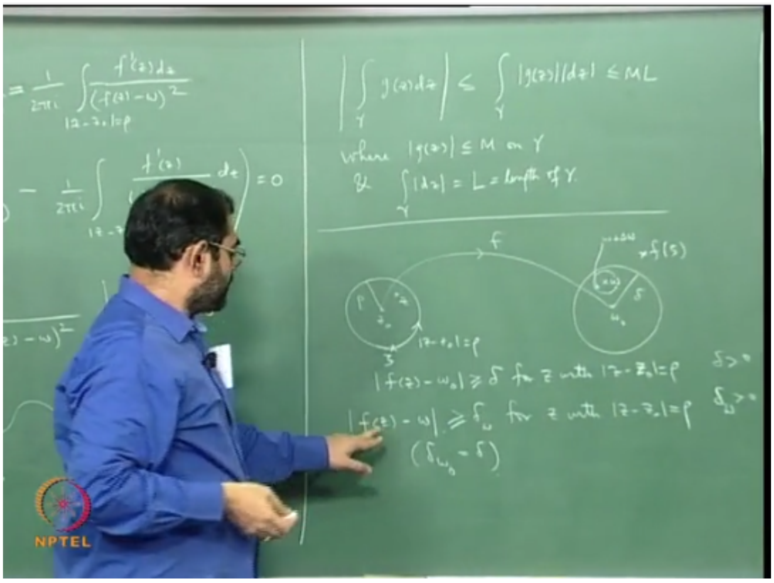

Since I am not worried about whether about the interior or the exterior of the curve okay, I am really not worried about the orientation it can be either this way or the other way alright.

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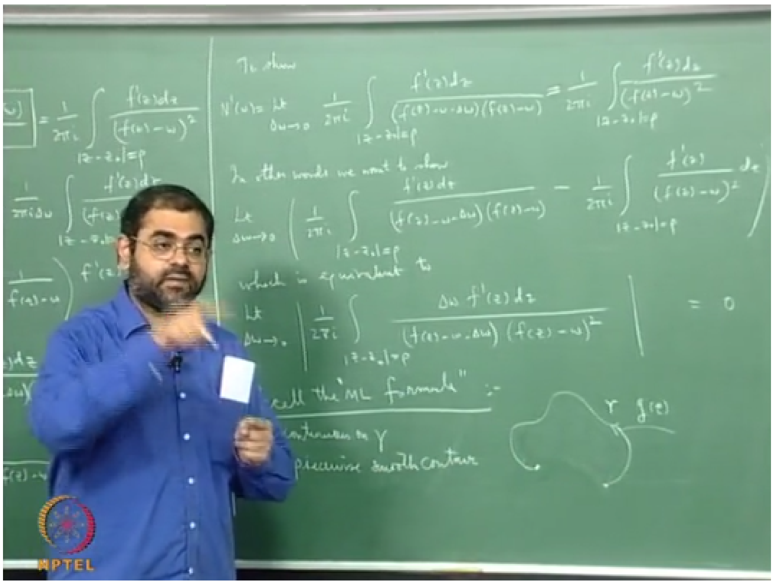

And for that matter I should also tell you that the curve gamma also need not be a closed curve, it can just be any path starting from some point ending at some point, it need not even be a could be just any path okay, it need not even be a closed curve. So, then the if you calculate the integral over gamma of g z dz okay this make sense okay. This integral over gamma if you calculate the modulus is always lesser than or equal to modulus of the integral is less than or equal to integral of the over the modulus.

And this is less than or equal to ML where mod gz is less than or equal to M on gamma okay and integral over gamma mod dz is actually L which is length of gamma okay. So, this is something that you should have seen on the first quotient complex analysis see the fact is g is continuous okay, therefore mod g is continuous. And mod g is continuous it is a continuous function on this path okay but the path is both compact and connected.

Therefore mod g is bounded a continuous function on a real valued function on a compact connected set is you know uniformly continuous and it is image is an interval if it is real valued okay. So, that interval will have a maximum value and that maximum value is M, so that mod so mod g will be just the continuity of g will tell you that mod g is bounded by M, so I can instead of this mod g is I can put M and that M can come out the integral.

Then what I will be left with will be integral over gamma mod dz an integral over gamma mod dz is actually the length of the arc gamma that is the meaning of integral of mod dz okay. Because mod dz is an element of length along the arc okay, so this is actually the formula for arc length if you recall from the first quotient complex analysis. So, this is the very standard formula and this comes from the corresponding formula that you have for real valued functions of 1 variable.

That the modulus of the integral is less than equal to integral of the modulus okay, you can reduce this from that okay. And this is often used and this is what we are going to use in this case also, so let us use that here. If you use that here, so you know I have to estimate this integral okay, now you see look at let us look the situation that I am in the point is so how this is so let me draw the diagrams again .

So, there is this point z0 there is this point originally I started with z0 and there was in the z in the in a in an open disc centred at z0 radius was if I remember it was rho okay, z was somewhere here. And then the function f took the value z0 to the value w0 and I was again considering a disc centred at w0 and radius delta okay and w was the value here inside this disc okay.

And if you remember the delta was chosen in such a way that if you take z on this boundary circle which is given by mod z-z0=rho then f of z will go outside. Because f mod fz the delta was chosen in such a way that mod fz-w0 is always greater than or equal delta for z with mod z-z0=rho this was this is how we started out if you remember okay. This was just from the fact that z0 is a 0 of fz-w0, fz-w0 is a non constant analytic function.

So, z0 is an isolated 0, so there is a disc surrounding z0 where there are no other zeros of fz-w0 and in particular on the boundary if you restrict fz-w0 it has minimum value it is again a continuous function real valued function on a compact connected set. So, it is image will be a closed interval in the real line and then you are looking at it is minimum value okay and it is this delta that is used here alright.

And it is only because of this delta that I am using here that N of w is properly defined alright. Now see my w is here and my delta w is also somewhere so when I calculate the derivative at w I am actually taking a smaller neighbourhood. So, the there is a w+delta w actually lies in this neighbourhood it is a smaller neighbourhood of w that lies inside this delta neighbourhood of w0 okay.

And what I wanted to know is that see if you look at mod of so you know I am trying to estimate these quantities okay. It is if you look at the function mod of f of z-w okay, now what I want it to understand is when I calculate the derivative N dash of w, w is fixed when I calculate N dash of w, my w is fixed and it is only delta w that is varying okay, delta w is a small change in w.

So, w now is fixed mind you okay and it is delta w that is changing if I fix w then I know mod fz-w this quantity on mod z-z0=rho is also greater than or equal to a certain delta w deltas of w which for w=w0 gives me deltas of w0 is my delta okay. Because you see fz-w does not vanish for z on the boundary okay, fz-w will not vanish for z on the boundary why is that so because for z on the boundary fz-w0 is greater or equal to delta okay.

So, the distance of fz from w0 is outside this disc ok. So, f, so if I fix a zeta here then f of zeta will be outside it will be a wildly outside this disc. Because the distance f of zeta from w0 has to be greater than delta and my w is inside, so that f of zeta can never be equal to w. In other words fz-w will be greater than or equal to a certain minimum value for z with mod z-z0=rho.

This is again by the same reasoning namely that fz-w mod fz-w is a continuous function, real valued function and it is not vanishing it is a non 0 function. And it is when you restrict it to this circle which is both compact and connected it has to be uniformly continuous it has to the image has to be a closed interval and this delta w is the least is the left hand point of the closed interval it is the least value.

And it is positive mind you delta is positive, delta w is positive and if you put w=w0 then delta w is a original delta you started with delta w0 is actually delta that is how we got delta okay. So, I am saying you are getting delta w is just in the same way as you got delta but only thing is delta was gotten for w0 same argument you apply to some w you will get a deltas of w and of course this deltas of w depends on w.

But mind you when I am calculating the derivative with respect to w, w is fixed what is varying it is only delta this capital delta w okay. So, yeah so I say delta but sometimes I mean this small delta sometimes I mean the capital delta which is the change in w. So, please make an effort to distinguish between the 2. So, you see you know the reason why I want this inequality.

Because you know so that I can write the reciprocal of this is less than or equal to 1 by delta w, so it helps to take care of this term okay.

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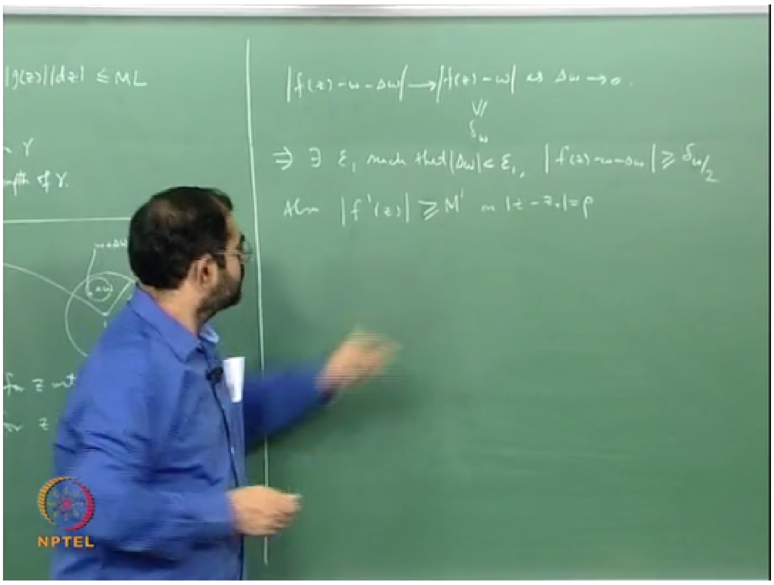

And what about this guy, you see f z-w-delta w this will tend to f of z-w as delta w tends to 0, this is just continuity alright. Therefore what this will tell you is that modulus of this will tend to modulus of this as delta w tends to 0 okay. And but you see this guy is greater than or equal to deltas of small of deltas of w. So, what this will tell you is that I can choose delta w sufficiently if I choose delta w sufficiently small I should able to make this greater than or equal to say delta by 2 okay.

So, there exist epsilon 1 such that mod if you make mod delta w less than epsilon 1 then you can make mod of fz-w-delta w greater than or equal to delta w by 2, see after all there is a which is going to a which is greater than or equal to delta w. Then at some point it has to be greater than half that value see in the limit this quantity is going to be greater than or equal to this oaky as delta w tends to 0.

That means at some point it has to be it has to exceed half delta w okay, so beyond a certain point it has see values are coming close to delta w. So, at some point it has to exceed half of delta I will chosen half of delta w but you could have chosen delta w by any chain I mean by you could have taken delta w into some constant okay into some fraction right I have chosen to just for fun okay.

So, this is true, so what this will tell you also mod f dash of z is greater than or equal to some M dash on mod z-z0=rho which is you know which is pretty straight forward because you know f is analytic then you know it is derivative is there also analytic in particular f dash is continuous okay and a continuous function if you take the modulus no I should not put greater than.

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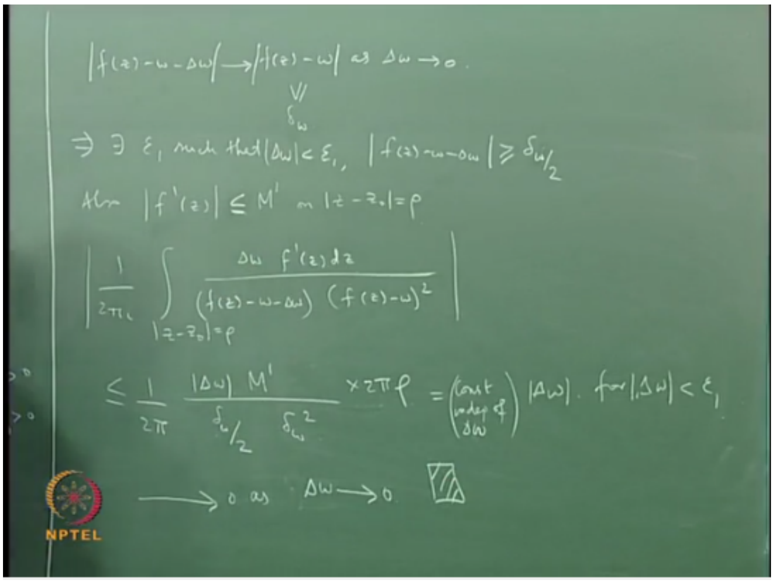


I should put less than or equal to okay, greater than or equal is also correct, so again same I have argument you take f dash of z that is continuous function and that continuous function if you restricted to this compact connected set is uniformly continuous the image of that it being a real valued function is a closed interval on the real line in the positive x axis.

And if you take the maximum the right end point of the interval you call that as M prime then you see that this is the one okay. So, you will put all this together you will get modulus of 1 by 2 pie i integral over mod z-z0=rho of that quantity delta w f dash of z dz divided by f of z-w-delta w into fz-w the whole square, this by the ML formula you will get 1 by 2 pie which is a modulus of this.

And I will get a mod delta w on top, mod f dash of z is less than or equal to M, M prime and then I will get a mod d is then for this quantity I am going to get a delta w by 2 and for this quantity I am going to get a delta w square and for what is left that is 1 by I mean integral over mod z-z0=rho of mod dz I will get the length of the arc which is in this case a circle, circle of radius delta sorry it is a circle of radius rho.

So, I am going to get 2 pie rho, so I will get this into 2 pie rho and if you calculate it I will this is just going to be constant independent of delta w which is that constant this is M 2 M prime by delta w q okay that as got nothing to this the change in w times mod delta w okay and this of course you know this is true for mod delta w lesser than this epsilon 1 okay. So, if you take mod delta w less than epsilon 1 and if you further let delta w tend to 0.

Then this will go to 0 which means this integral will go to 0 as delta w tends to 0 and that is what you wanted to prove and that ends the proof okay. So, this tends to 0 as delta w tends to 0 that is the end of the proof of the theorem okay. So, that proves the open mapping theorem okay. The next thing that we need to understand is how the proof of the open mapping theorem actually gives the inverse function theorem okay and probably we will look at it in the next lecture right.