Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-05
Local Constancy of Multiplicities of Assumed Values

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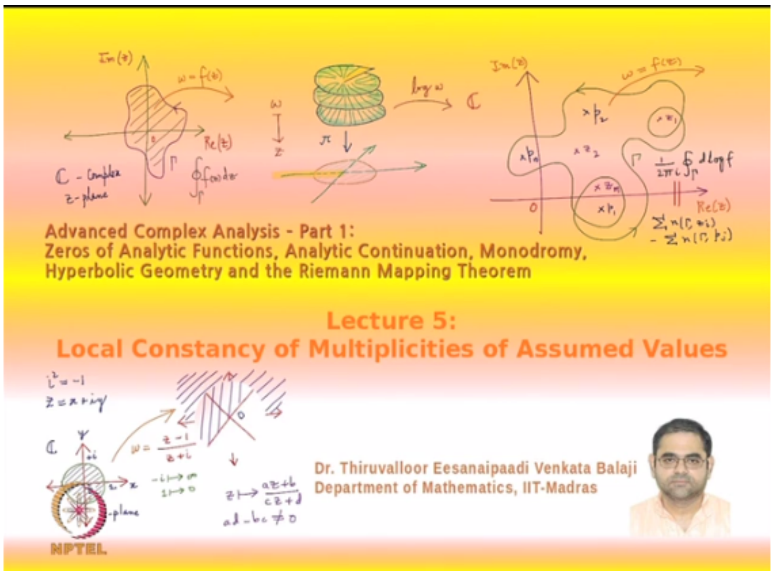

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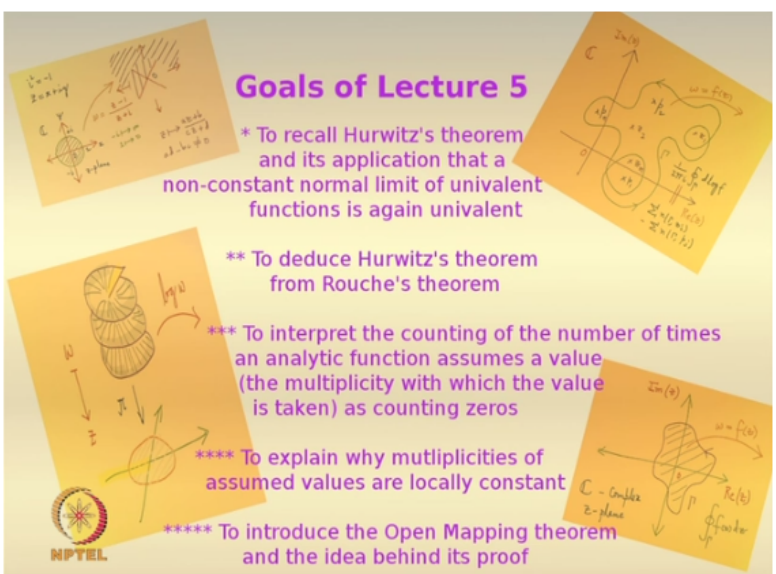

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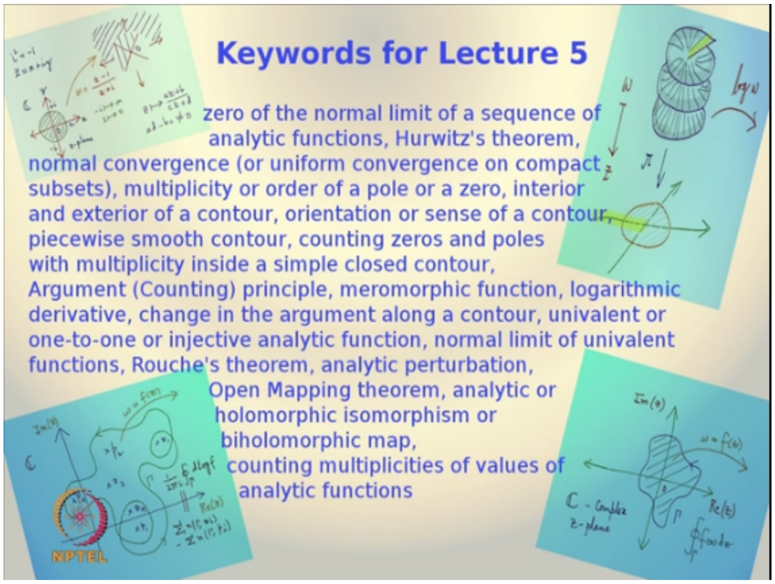

Okay, so let me recall what we did in the last lecture, so if you remember the theme that you have discussing is about zeros of analytic functions okay and as you know the residue theorem allows you to compute the number of zeros okay. And that is throughout the so called argument principle and then we saw Rouche's theorem which tells us that if you take an analytic function and change it by small amount that you made.

That is you add a smaller function to it, function that is smaller on the boundary curve in of course in magnitude or modulus then the there is no change in the number of zeros okay and then what we discuss in the last lecture was Hurwitz's theorem, Hurwitz's theorem which says that a 0 of a limit of analytic functions is coming from zeros of the functions in the limit okay.

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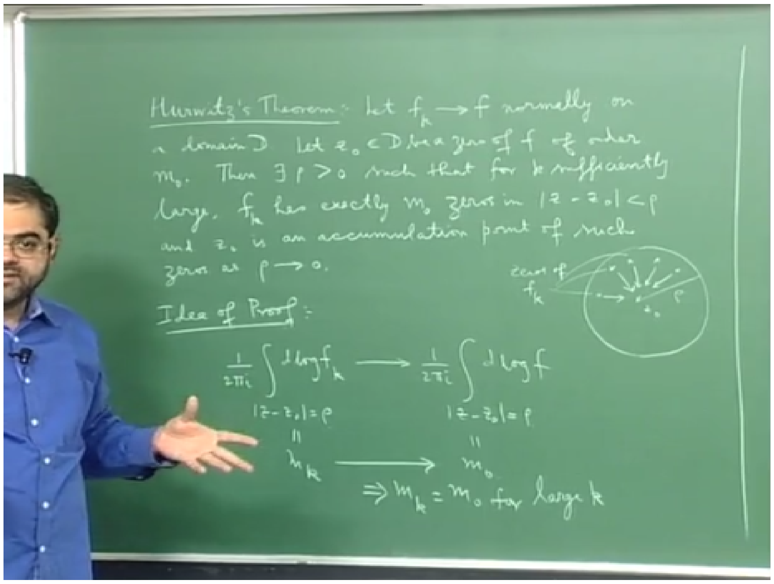

So, let me just recall that so here is Hurwitz's theorem, so you assume that fk is the sequence of analytic functions which converges to the function f normally in normally on a domain D. so, let me recall that domain it is an open connected set may not be bounded and of course subset of the complex plane and all these fk's are analytic function defined on D.

And this f that this the statement that fk converges to f normally means that the convergences uniform on compact subsets okay and then what Hurwitz's theorem says that if you take a 0 of f let z0 belonging to D be a 0 of f of order m0. So, as I explained the last lectures it follow that f is analytic okay because of normal convergence an normal limit of analytic function is again analytic okay.

That is essentially because of the uniform convergence on compact subsets and since analtyic function has zeros which are isolated okay you can always find a given any 0 you can find a disc surrounding that 0 where there are no other zeros okay. So, I am suppose I pick 0 of the limit function and suppose the 0 is of order m0 okay . Then there exist a rho greater than 0 such that for k sufficiently large fk as exactly m 0 in mod z-z0 strictly less than rho.

And z0 is an accumulation point of such zeros as if you want rho attains to 0 okay, so this is Hurwitz's theorem and I explained of the of this theorem and basically the proof of course use the argument principle okay in fact I mean the basic idea of the proof was, the idea of the proof I give last time was you just calculate 1 by 2 pie i integral over mod z-z0=rho of d log fk.

And show that this tends to 1 by 2 pie i integral over mod z-z0=rho of d log f where of course d log fk stands for fk prime, the derivative first derivative of fk divided by fk the z and similarly d log f stands for f prime by f the first derivative of f divided by f the logarithmic derivative dz okay. And of course the most serious point of the proof was that you will have to show that this is the fine, this is the fine and then this converges to that.

And of course argument the argument principle will tell you that this is actually m0 and this if you call this as mk then the argument will tell you that mk then of course this m0 is of course the value of this integral is m0 which is the number of zeros of f in this inside the region bounded by this circle and of course you know you do not I mean you choose this disc as I told you in such a way that no other zeros of f.

And similarly this quantity if you call this as m's of k that will be the number of zeros of fk inside this disc and this argument tells you that this mk converges to m0 and but then mk being a sequence of integers when you say sequence of integers converges to an integer it means that beyond a certain stage the sequence of integers is just that constant integer which is the limit okay, so that means that mk=mo for k sufficiently large and that is the conclusion of the theorem.

And of course the argument if it is works for certain rho it is start it will work for smaller rho's okay if you take smaller it will work. So, **so** this is m0 implies that mk=m0 for large k okay and of course diagramatically what this means is that you see if z0 is the point here where you have a 0 of f this is the disc centred at z0 radius rho then you can find all the zeros of these are zeros of fk .

And all these zeros they converge to z0 as you make rho smaller okay now what I wanted to discuss is the true thing that I want to discuss 1 is that there is another proof that you can give which actually uses Rouche's theorem okay and then I also wanted to discuss about the application of the application which says that if you take a normal limit of univalent functions and the limit is non constant. Then the limit is function is again univalent where of course univalent means 1 to 1, so let me first do that.

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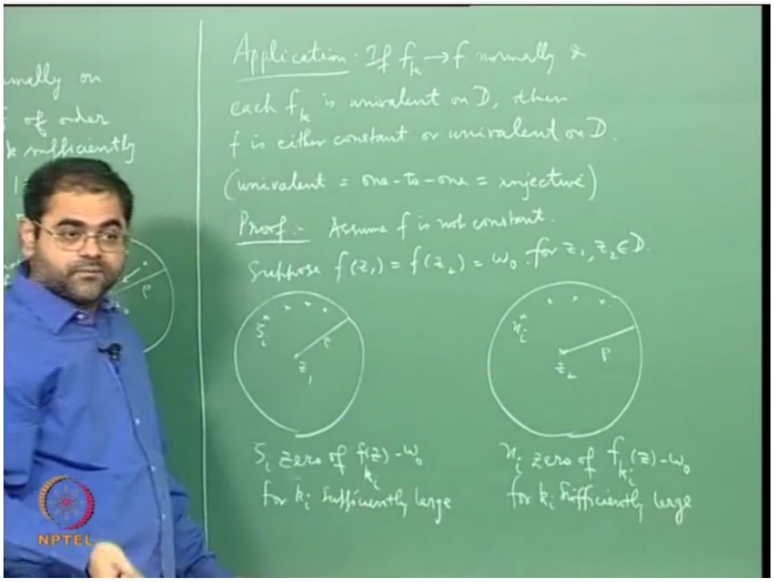

So, let me look at this application again if fk converges f normally and each fk is univalent on D then f is either constant or univalent on D where of course univalent means 1 to 1 which is also as injective okay. So, this is the this is an application of Hurwitz's theoerm and I just wanted to look at this proof see so I will I know that each fk is 1 1 I want to show that f is 11.

And of course I the the result says that f is 11 provide f is not constant, so you assume f is not constant okay, assume f is not constant. So, you know I want you to remember that the moment I say that f has a 0 of order m finite order I am assuming that f is not constant because if f is constant then f has to be identically 0 and if f is identically 0 then basically you do not you will not get a you will not able to find disc surrounding a 0 where there are no other zeros.

Because every point is a 0 okay, so you must understand that this Hurwitz's theorem applies only to a non constant it applies only to the case when f is a you know non constant analytic function okay. So, even the theorem on the set of zeros of an analytic function being isolated assumes that you are working with the an analytic function is not constant for a non constant analytic function the zeros are isolated okay.

So, non constant is always there in the at the back of a back of all this okay. So, **so** assume f is not constant but it is important there is a reason why I am insisting sometimes we might be careless not enough to write it or insist but it is very important it is there in the background. So, assume f is not constant suppose f of z1=f of z2=omega0 I will have to show that for z1, z2 in D, I will have to show that z1=z2.

And what do I do I apply Hurwitz's theorem, so here is my z1 and you know there is a rho and there is disc surrounding rho, so that I can find zeros zeta I can find a 0 zeta i **of** of zeta i is 0 of f of z-omega0 okay. So, what you must understand is fk converges to f, so fk of z so i should be fk, so fk of z converges to f of z, so fk of z- omega0 converges to f-omega0 and it is again normal convergence.

So, I am applying Hurwitz's theorem to not to f not to this but I am applying it to the sequence with – omega0 added on both sides okay **.** So, I can find a 0 zeta i of fk of z-w0 and of course you know I **I** let may write it as ki for ki sufficiently large abd you know the same way so there is also th epoint z to I can take a similar disc of radius rho.

Of course you know this rho here is chosen, so that z z0 z1 is the only of f of z- w0 this disc and here also I am try to choose rho of the same type that is 1 in which in this disc z2 is the only 0 of f of z-omega0 but what I want to tell you is that to begin it with this rho1 these rhos maybe different this maybe rho1 that maybe some rho2 but then I am just saying take the main if you want to take the minimum rho.

And do it for the minimum value of rho okay and so that rho is the minimum value okay and what you do is here again Hurwitz's theorem will tell you that I will get eta i, so I will get neta i 0 of f of ki of z-omega0 for ki sufficiently large and you see the rigid point note is that I am choosing the same ki okay this ki that I got for this maybe different from that ki okay.

So, in fact I should call this as if you want ki and k prime i but then it holds for all values beyond certain stage then I can take the maximum of those 2 and call that as ki okay. So, that is the adjustment I make, so what you must understand is that this rho it fixed as right as rho1 and that

I should write rho2 okay and this I should write as ki and that I should write as ki prime okay but I can choose the maximum of ki and ki prime.

And replace that ki call that as ki okay and I can take the minimum of rho1 and rho2 and call that as rho okay. And then you know what you can do is once you have done it for rho you next do it so **so** you know maybe I will first call it as let me first call it as let me do it for okay.

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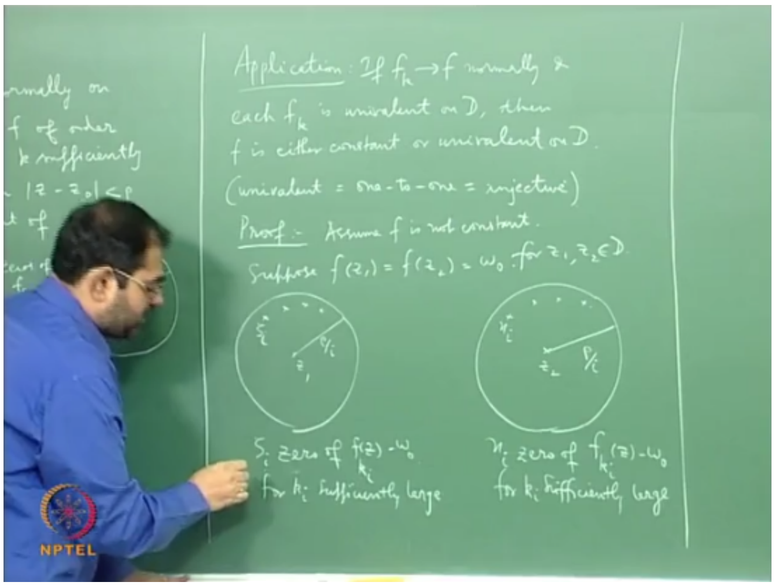

So, let me do the following thing here instead of rho let me put rho by i okay, instead of rho let me put rho by i right. So, the point is that the reason why I am putting rho by i is that you know the distance between zeta i and z1 is less than rho by i which as i takes to infinity goes to 0 which tells you that the zeta is will converge z1 and the eta i will converge to z2 okay.

So, I can do I can take this these rho rho i's okay and you must think that as I increase this i okay then the for example if I put i=1 it is just rho if I put i=2 it is rho by 2 okay then it is becomes by 3, you get smaller and smaller and smaller discs okay. So, with this kind of thing what you get is the following you get that okay, so I should write here and this is eta i.

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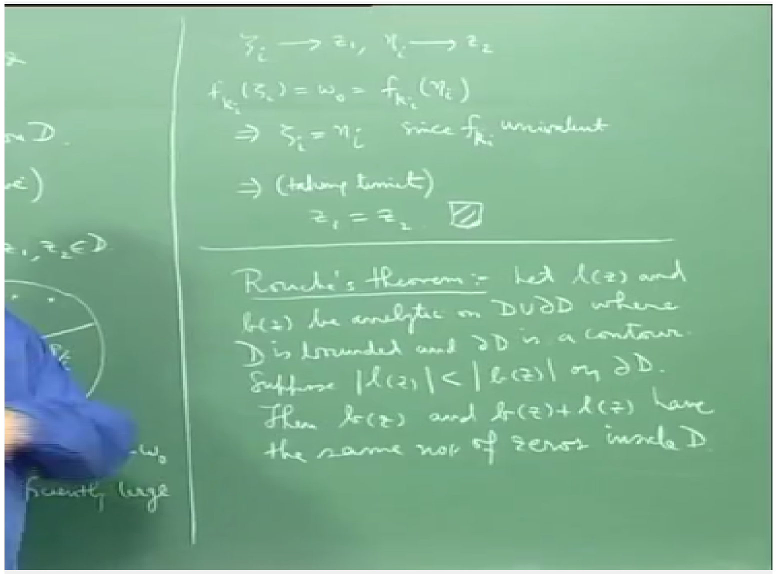

So, you see zeta i converges to z1 eta i also converges to it converges to z2 fk i of zeta i is actually w0 which is equal to ki of eta i because zeta i is a 0 of fk i of z-w0 and is also a 0 eta i is also 0 of fk i of z-w0 and but then fki is give to be 1 to 1. So, this will tell you that zeta i=eta i since fki univalent and this implies taking limits that limit zeta i is limit neta i.

But that means you will z1=z0 and that finishes the course okay. So, what I want to tell you is that you have to do be little carefull in choosing the zeta i's and neta i's okay. And you so you should so basically you should choose a set of sequence of zeta i's which converges to z1 and sequence of neta i which converges to z2 okay as i tends to infinity okay, so that is the proof.

Now what I want to discuss next is another proof of Hurwitz's theoerem which is which actually uses the Rouche's theorem okay. So, recall what was Rouche's theorem see Rouche's theorem basically says that if you have a the number of zeros of an analytic function in a simple closed inside a simple closed curve is not going to change if you add to the analytic function another analytic function which is a smaller function on the boundary okay.

So, let me write that so let L of z and d of z, d analytic on d union tou D where D is bounded and tou D is a contour okay, so it is a piece wise smooth contour and the both function show l is suppose to be taught of us the litle function b taught of to be taught of it is a bigger function okay suppose that the little function is lesser than the bigger function in modulus strictly lesser than on the boundary okay.

Then b of z and b of z+l of z have the same number of zeros inside D okay this is the this is Rouche's theorem where you think of so what it says is that if the number of zeros of bz is the same as number of zeros of bz+lz now that bz+lz iz thought of as a small perturbation of b because you have added the error term that you have added is l of z which is analytic force. But the point is that l of z is strictly smaller than b of z in magnitude on the boundary okay.

So this is as if you remember we proved this very easily using the argument principle but the point is this also yields a beautiful proof of Hurwitz's theorem okay. So, how so the reason why I am doing this is these discussion of you know zeros of analytic functions essentially uses residue theorem I mean argument principle. And all these ideas are inter-related okay, so if you understand how each idea you know is kind of connected to another okay.

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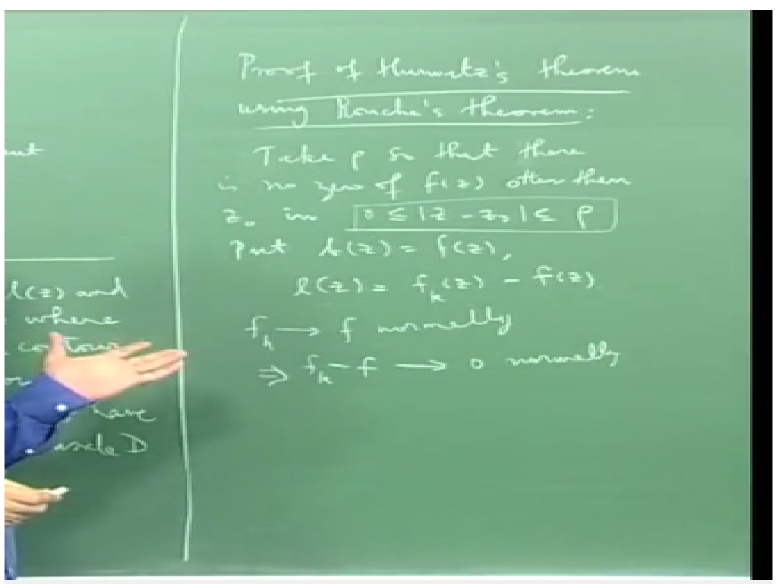

So, you see so proof of Hurwitz's theorem using Rouche's theorem, so you see it is a suppose I want to prove this using Rouche's theorem then it is very easy to guess what you have to do you see what what does Rouche's theorem actually want to say, it wants to say that you know in a disc like this f and fk have the same number of zeros that is what you want to say.

So, you see so it is very clear that you know you have to take one of the big functions the big function has to be f okay and the small function should be chosen, so that when you add it to the big function you get fk see the Rouche's theorem says that the big function and the big function+a smaller function they have the same number zeros okay.

Now if you want to get this from that then you but here I want f and fk to be the 2 functions for which the number of zeros are the same beyond a certain stage. So, the big function has to be f and the big function+a small perturbation must be fk okay, so the small perturbation has to be fk-f it is very simple to see that. So, what you do is put take rho so that mod so that there is no 0 of f of z other than z0 in well 0 less than or equal to mod z-z0 less than or equal to rho.

Choose such a rho of course as I told you this is possible because you are assuming that f is analytic I mean you have that f is analytic okay, that is because f is a normal limit of analytic functions right and then put big function to be f of z little function to be fk of z-f of z okay. Then you know of course if I add the 2 I will get fk okay and of course to apply Rouche's theorem I am the domain on which I am apply applying Rouche's theorem is this disc.

And the boundary is just the boundary circle okay, so I am applying Rouche's theorem here okay I am just applying Rouche's theorem here alright to the little function and the big function and I will get that the big function which is f and the sum of the big function the little function which is fk they will have the same number of zeros okay provided the little function is really little than the big on the boundary okay.

But you see it is that is something that you can easily see because you see fk of z fk converges to f normally this implies that you know fk-f goes to 0 okay fk-f goes to 0 that is what it means and that also normal okay. And of course in this case normally means that it will be uniformly in this region. So, I should say so in fact I can rub of this normally here and simply write uniformly.

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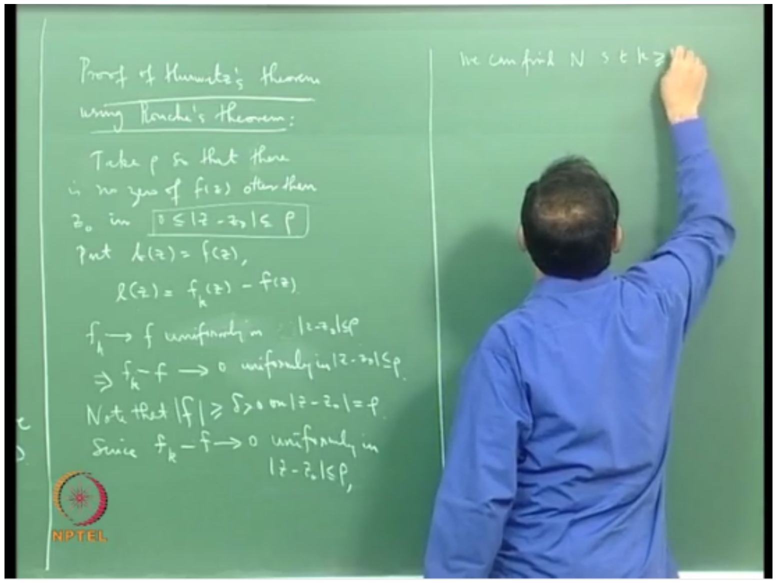

In this and this is also uniformly goes to 0 uniformly in mod z-z0 less than or equal to rho because the convergence is uniform on compact subsets okay and z-z0 less than or equal to rho is a compact subset it is closed and bounded. So, but what does this mean, this means that the **th** modulus of this can be made lesser than any small quantity that is what it means and you see f note that f mod f is greater than or equal to delta on mod z-z0=rho okay.

This is the fact that we also use in during the proof of Hurwitz's theorem because you see **see** mod f is a continuous function okay it is a continuous real valued function and when defined and when you restricted to mod z-z0= rho, mod z-z0=rho is a circle centered at z0 radius rho that is compact. Because it is closed and bounded okay, so we have this fact from analysis you take a real valued continuous real valued function if you restricted to a compact set.

Then it will be uniformly continuous and it will attain it is bounds, so in particular mod f will have lower bound it will have an upper bound and it will take the lower value and it will take the upper value also okay. And delta is a lower value okay on this compact, the circle boundary circle okay and of course is delta is positive that is because mod f is positive mod f vanishes only at the center and it does not vanish anywhere else.

So, it is on the boundary it is positive therefore the minimum value is also positive okay and that is because the minimum value is taken by a mod f okay. And mod f cannot be 0 if mod f is 0 then f is 0 and f is not suppose to be f is not suppose to vanish anywhere in that closed disc except at the center that is the choice of rho okay. So, mod f is greater than or equal to delta but then so you know fk-f converges to 0 uniformly means that I can choose a you know index large enough index N.

Such that for k greater than or equal to N, fk-f in modulus can be made less than delta I can do that okay. Since fk-f converges to 0 uniformly in mod z-z0 less than or equal to rho we can choose, so let me continue here.

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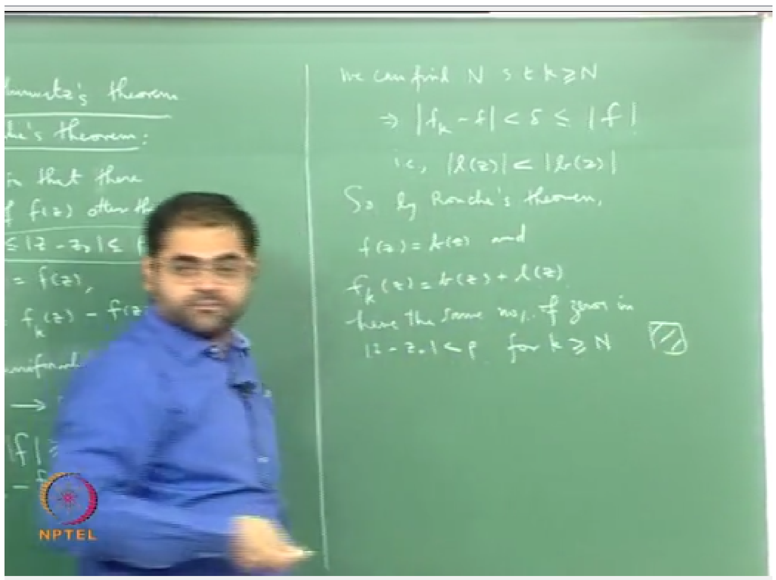

We can choose we can find N such that k greater than or equal to N implies that mod fkf can be made less than delta I mean this is just uniform convergence okay. And this is I am not writing fk of z-f of z because all this is done independent of z and this independence of on of z is exactly the uniformness of convergence okay, so this N does not depend on z, it does not depend on what value of z you plug in where z is in this closed disc okay.

That is the uniformness that I am using their alright and but you see this is but delta is less than or equal to mod f. so, what you get is you get the modulus of the little function is strictly less than modulus of the big function by our choices and that is precisely what you need to apply Rouche's theorem okay. So, by Rouche's theorem f of z which is bz and fk of z which is b of z+l of z have the same number of zeros in mod z-z0 strictly less than rho for k greater than or equal to rho.

And that is exactly Hurwitz's theorem okay, so you see you get Hurwitz's theorem has a consequence of Rouche's theorem right fine. So, I mean then this what I want to do next is I want to a topic which is called as I want to go to the topic of open mappings okay. So, I want to prove the very important open mapping theorem, the open mapping theorem says that any non constant analytic function maps open sets to open sets okay, it is a very deep theorem.

But the point is that somehow the proof of that theorem also involves ideas of this type, it just involves it is again about zeros of analytic functions okay and it falls again literally involves the if you want you know the residue theorem in the form of the argument principle okay.

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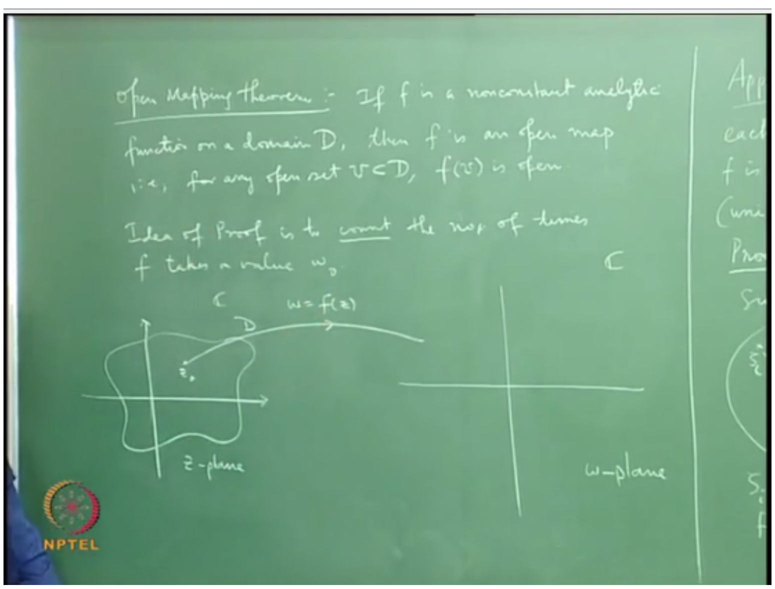

So, here is the open mapping theorem it is a very deep theorem very important theorem if f is a non constant analytic function on domain D, then f is an open map that is for any open set u in D f of u is open okay. So, this is the open mapping theorem it says that a **a** non constant analytic function if you take the under non constant analytic function, if you take the image of an open set you will again get an open set okay, this is the very deep theorem.

Because you see you cannot find any counterpart for this in for example functions of 1 real variable okay, it is rather I mean it is god an it is beautiful normally you cannot expect a map to take open sets open sets which is a very important condition, it is an important condition because along with this if you put the condition that f is 1 to 1 okay.

Then it means that since f is 1 to 1 f inverse make senses set theoretically and saying that f is open will tell you that f inverse is homeomorphism okay. And what it will tell you is that that is what we are going to see after this there is an inverse function theorem which will tell you that f inverse itself it is analytic okay, that is the next step okay. So, put all so for all these things.

So the final statement is that if you have an injective analytic map then the image of the source domain will be an open set and f inverse on that open set will again be analytic. So, that means that f is an analytic isomorphism okay what it tells you is that an injective analytic map is a isomorphism on it is make which is open analytic isomorphism an inverse has an inverse which is also analytic okay.

So, the starting point is this for even for the inverse to be even continuous the fact that f is open caps okay. So, yeah how does 1 prove this, so basically you know we do the idea of proof is to count the number of times f takes a value omega0 okay. So, it is again a it is again the counting principle the argument principle okay. So, in fact so not only count the number of times f takes a value omega0 in fact you also let this omega0 to vary okay.

So, let me explain that so you see so here is so let me draw a diagram so here is my source complex plane and well here is some domain and here is a point z0 and here is my function f, f is non constant and this is of course the this is the z plane. And the target is the is also complex numbers but it is a omega plane where omega=f of z okay for is it is 1 real variable you write y=f of x okay, since it is 1 complex variable you write now you now write omega=f of z.

And suppose you take a value omega0 which is f of z0 okay, now what one does is how will you count the number of times f takes the value omega0 okay. So, that means you know you have to look you have to think of z0 as a 0 of f of z-w0 okay, you think of z0 as a 0 of f of z-omega0 you see that is the idea that we have been using all the time.

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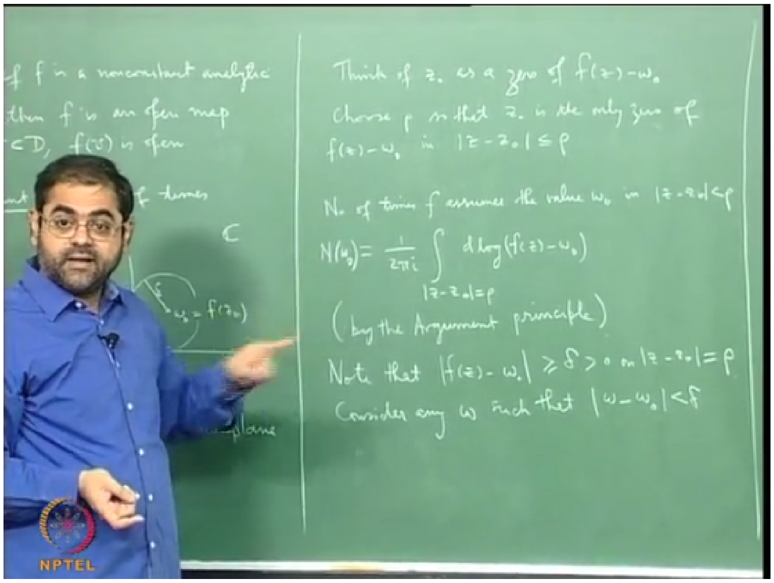

So, think of z0 as a 0 of f of z-omega0 okay and notice that f of z-omega0 is also a non constant analytic function because if f of z-**z** omega0 is constant that will tell you that f of z is constant but I have assumed f is non constant. And so after all f of z-omega0 is the analytic function f with –omega0 added to it, -omega0 is just a constant you have added, adding a constant to analytic function continues to keep it analytic okay.

So, if you want because the constant function is trivially analytic okay and the sum of analytic function is again analytic. So, **so** again f of z-omega0 is a non constant analytic function and z0 is a 0, so the number of times it assumes the value z0 is given by the argument principle in a disc surrounding z0 where there are no more zeros other than z0.

So, what you do is that you choose a disc of radius rho okay, choose rho , so in this case okay, so choose rho so that z0 is the only 0 of f of z-omega0 okay in mod z-z0 less than or equal to rho in this disc centered at z0 to radius rho. So, z0 is a only 0 of f of z-omega0, this you can do because f of z-omega0 is a non constant analytic function.

And the zeros of a non constant analytic function are isolated, so the 0z0 is isolated, so you can find a small discs surrounding z0 where there are no other zeros okay even on and you can disc small enough so that there are no zeros on the boundary as well on boundary circle as well okay. Now so you see so what is the number of times f assumes the value not assumes a value omega0 in mod z-z0 less than rho how is this given by the let me call this N's of w0.

This will be 1 by 2 pie i integral over mod z-z0=rho d log f of z-w0 this is just the argument principle, the argument principle tells you that d log of something if you take and then you integrate over a simple closed curve and divide by 2 pie i you will get the number of zeros of that inside the closed curve. So, I will get this will actually give me the number of zeros of f of z-w0.

And they will be exactly the number of points inside this in the region they enclosed by this circle namely the disc centered at z0 radius rho where f takes the value w0 okay. This is just again by the argument principle or so which is a counting principle okay, so now what you do is you see note that mod f of z-w –omega0 is say again greater than or equal to delta greater than 0 on mod z-z0 on z mod –z-z0=rho.

So, this is again the same kind of argument that we used earlier namely fz-w0 does not have zeros on the boundary circle because in this closed disc the only 0 of fz-w0 is at z0 at the center. So, there are no zeros on the boundary circle at the boundary circle is closed and bounded, so it is compact-and mod fz-w0 is a continuous function but it restricted to this compact set it has it is uniformly continuous.

And it will have a minimum and maximum value and delta is the minimum value and the minimum value is positive because it does not vanish okay. Now you see the trick is what you can do is consider any omega such that mod omega- omega0 is less than delta okay. So, you see it is a same delta I am using, so what you do is you now take a disc centered at omega0 and radius delta okay.

Then you know I can completely replace omega0 in this equation by omega and that will give me the number of times f takes the value omega in the disc mod z-z0 less than rho.

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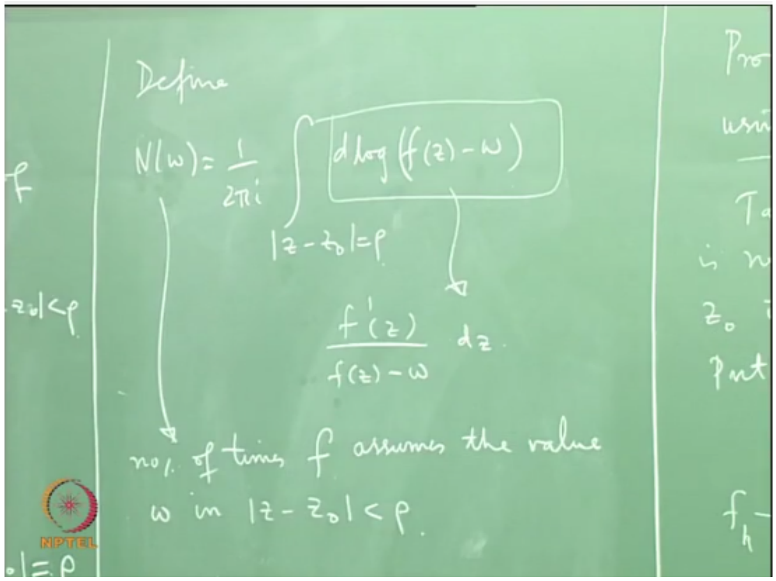

So, define N of omega to be 1 by 2 pie i integral over mod z-z0=rho d log f z-omega okay, mind you this makes sense because you see what is d log f of z what is this, this is actually f derivative of this which is f dash of z divided by fz-w d this is what is okay and mind you, you see the fz-w cannot vanish on the boundary okay, fz-w cannot vanish on the boundary, why is that so that is because of this squares of w okay.

The choice of w is that see the choice of w tells that the distance from w to w0 is less than delta whereas the distance of fz from w0 is greater than or equal to delta okay. So, this will tell you that fz the modulus of fz-w cannot be seen. Therefore this is well defined, this integral is well defined and what does it give, it gives you the number of times the function f assumes the value w are omega in the disc in the disc centered at z0 radius rho.

So, this is number of times f assumes the value w in mod z-z0 less than rho okay, this makes sense. Now after having written all this let me tell you that the whole point is that you see if you think of w as now a complex variable okay. Then this N of w is a function of w okay the amazing fact is but it is amazing but easy to prove, the amazing fact is that N of w is actually an analytic function of w okay, it will turn out that N of w is an analytic function of w okay.

And that will mean that N of w is constant because you see it is an analytic function but it is values are in integers okay and you know the image of if you have an analytic function if you take the values of an analytic function okay, if for example when I say N of w is analytic function of w in mod in this disc then this disc is of course connected.

So, if I take the image of this disc is I have should get a connected disc but on the other hand the values I are integers. So, I should get a connected set of integers okay but what is the connected set of integers has we only a single integer, so what it will tell you is that N of w is a single integer and that is irrespective of w. So, it will be the same integer as N of w0 okay but then what does that tell you it tells you that if f assumes a value w0, N w0 times.

Then f assumes every other value w, the same N w0 times in this disc mod z-z0 less than rho what this tells you therefore is that this whole disc is in the image and that is the proof that the image contains an open discs centered at z0. So, if you take a point centered at wo, so if you take a point w0 in the image then you get a whole disc centered at w0 in the image and that is exactly saying that every point is an image is an interior point of the image.

And that means that the image is open and that is the proof the open mapping theorem, so the technical point is to show that this is an analytic function okay and everything follows from that okay and mind you the idea is very simple we are just using the counting principle the argument principle okay. So, I expand upon this in my next lecture I will explain how to show N of w is an analytic function okay, so I stop here.