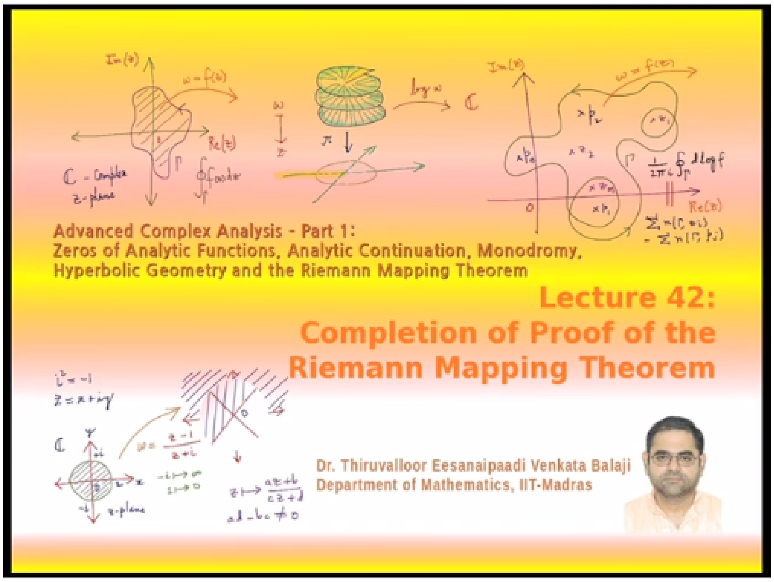
Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology-Madras

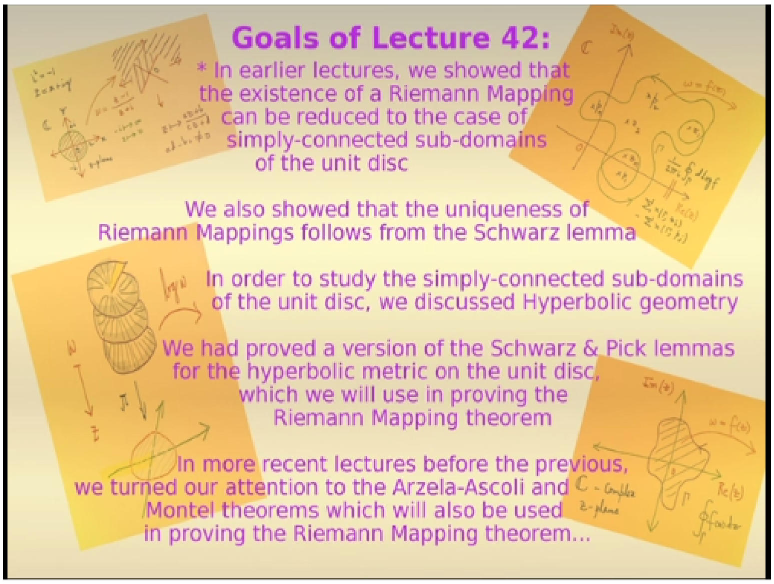
Lecture-43

Completion of Proof of The Riemann Mapping Theorem

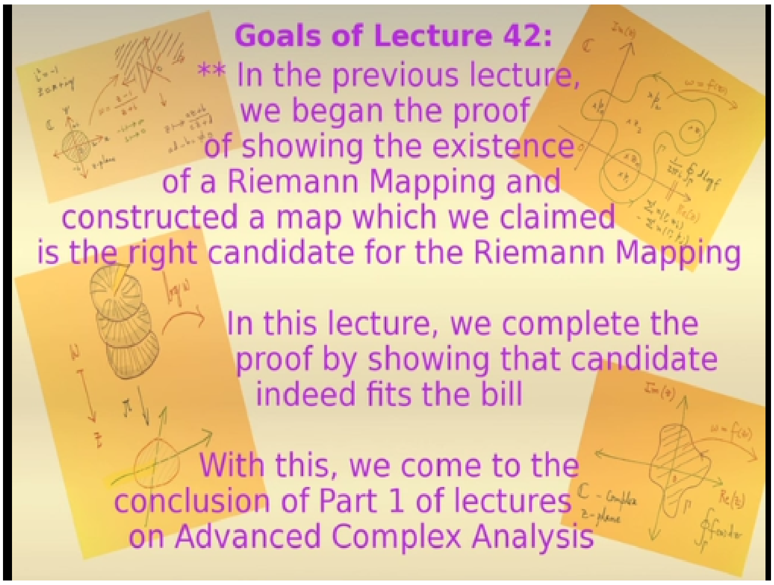
(Refer Slide Time: 00:04)



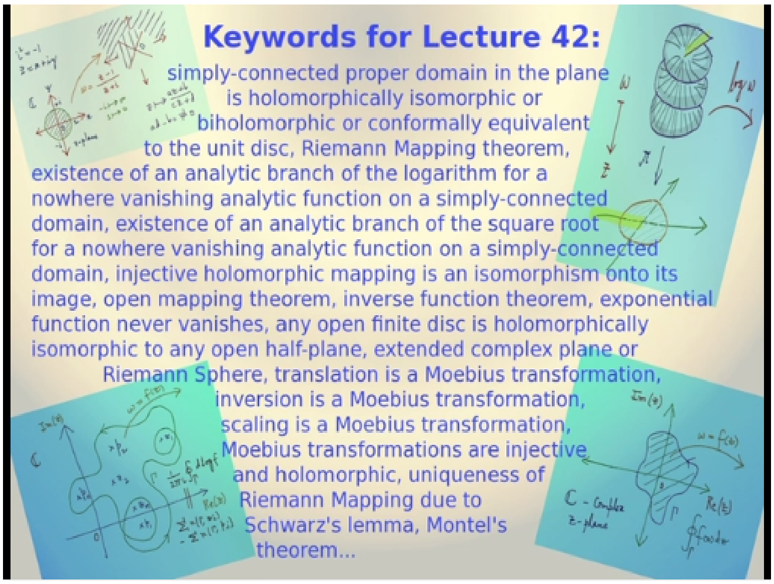
(Refer Slide Time: 00:09)



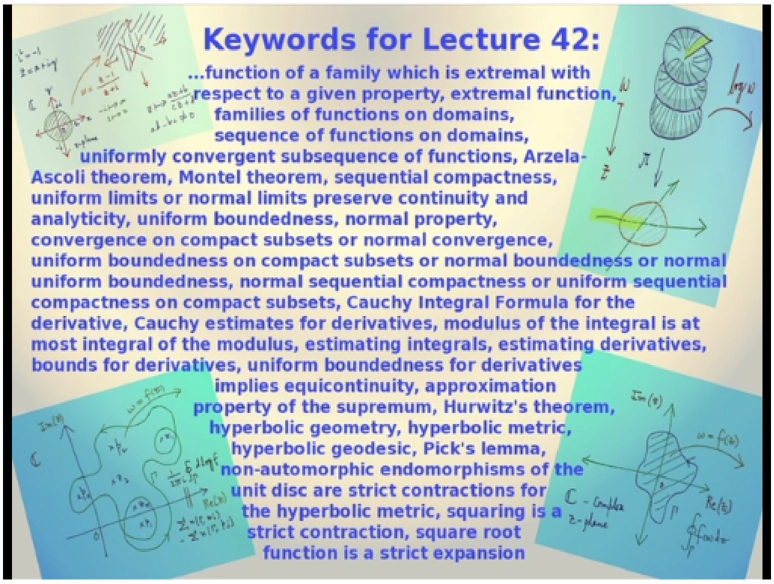
(Refer Slide Time: 00:16)



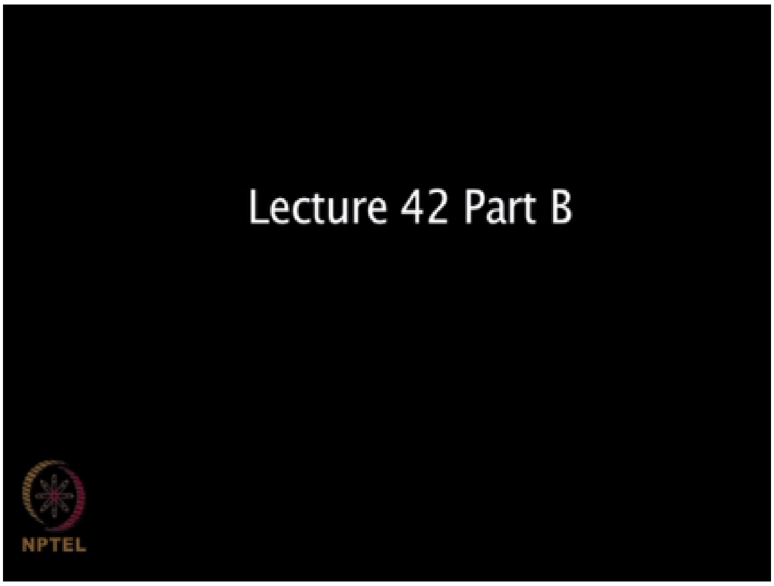
(Refer Slide Time: 00:23)



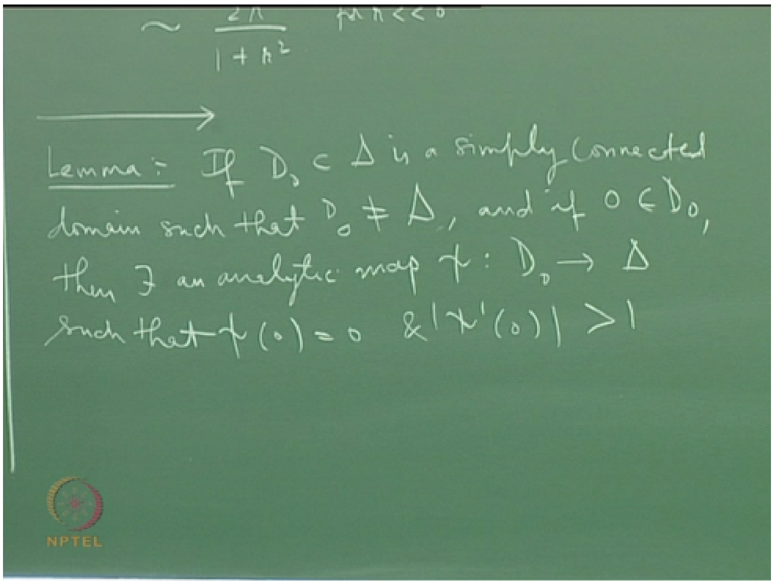
(Refer Slide Time: 00:31)



(Refer Slide Time: 00:38)



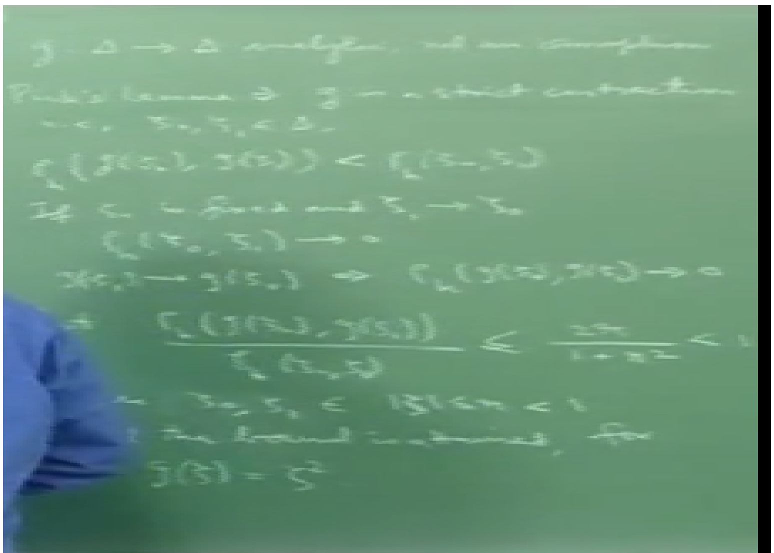
(Refer Slide Time: 00:41)



And how do you prove this, so it is in this proof that I need this hyperbolic geometry ok. So what you do is you know you first so here is you choose a point ok let me use b, you choose I will tell what we are going to do, it is very very simple, choose b in delta-D0 ok, choose a point b which is outside D not but inside delta.

This is possible because D not is not the whole unit disc ok and what you do is you take a mobius transformation that will map which is an atom of the unit disc and that will map the point b to the origin ok. So what you do is, so you know I am going to take a map, so I am going to take h a small h, so let h(zeta) zeta-b/1-b/zeta ok. You know that this is an automated of the unit disc that will map b-0 ok.

(Refer Slide Time: 02:04)



And you say it is an atom, so what this will tell you, it will tell you that you know h(d0) ok h(b0) and that will imply that h(D0) will not contain 0 because b is not in D0 h(b) will not be in h(D0) because h is the mind you it is a bijective map ok, since b goes to 0 and b is not is D0 ok, h(b) will not be in h(D0) that means 0 will not be in h(D0). So you know the effect of h on this thing will be something like this, now you can think of I am just drawing something figuratively.

You know this h will map D0 into something this is h(D0), it will push b to 0 and you know it on map h(D0), D0 on to h(D0) and this h(D0) will not be it will not contains 0 alright and since h(D0) does not contain zero I can find see whenever a domain is simply connected domain does not contain the 0 I can always find an analytic branch of the square root ok.

This you can the problem with finding branch of square root is that the square root of function you are trying to find that should not vanish and the domain why you are trying to define the square root. So h(D0) is a simply connected domain because it is a image under h of this simply connected domain D0 and it is an automorphism h is the holomorphism automorphism so it is a homeomorphism ok.

So h is an isomorphism topological isomorphism also from D0 to h(D0) and since D0 is simply connected h(D0) is also simply connected origin and h(D0) does not contain the origin and there is a branch of the ah hi mean there is the branch of the logarithm and you can use that you can find a branch of the branch of log eta and you can therefore find a analytic branch of root eta.

Where you know if you if you want to call neta as the variable here neta is h(zeta) ok, there exist analytic branch of log neta on h(D0) why because h(D0) does not contain zero and it is simply connected ok, so that is the analytic branch of log of neta ok and so and hence an analytic branch of the square root of zeta so let macro environment use capital G neta of root theta.

And you know route theta is e to the half log neta I mean the moment you get an analytic branch of log e to the half log will give you a square root right and and then so you know now you know therefore I have this G this G is a G is defined on this and G is a branch of the square root ok and if I take its image of course this does not contain zer0. So if take its image under G and I will gain get something that is not zero.

So you know if I apply probably get you know so I do not have enough space to draw this but any way I will draw it. So this is what will happen if I apply g ok and my diagrams are not accurate but they just help you to visualize what is going on. So this is what happens if I apply G ok. So G will move this hD0*GhD0 and that will be a of course the image of image under G(hD0) will be again an open set.

Because again open mapping theorem will always tell you the image of no constant analytic map will always be open. So you know h, so what I will get is I will get this domain with is G(hD0) h(D0) and I will get this domain and well all I can say about it is but there is even more see the fact is see the square root if you take the square root function ok.

The square root function is injective ok, the square root function is injective because you chosen analytic branch, choosing an analytic branch means of using a continuous branch that means you are choosing one square root only one of the square roots and that to you are choosing them choosing it in analytic way. So the square root is all is injective, the square root function when you take the analytic branch the square root that always be injective ok.

Therefore the moral of story is g is in fact holomorphic isomorphism because it is injecting analytic ok so G you see G from h(D0) 2D0 –G(D0) is a holomorphic isomorphism holomorphic or isomorphism. Because you know it is one to one it is injective analytic, it is an injective analytic function and inverse function theorem will tell you an injective analytic function is a holomorphic isomorphism.

Image will be open because of the open mapping theorem because it is a non constant function. In fact it will be isomorphic to the image, so g is the holomorphic isomorphism alright ok g is the holomorphic isomorphism. Now you see so you know so you know you have this h that moves this to this. So you know that is zero is was in D0 ok. Zero was in D0 that will now I will get the image of 0 under h, it will be h(0).

That will be the point in h(D0) ok and then I will get each further image under g, I will get a point here and this point will be h(g) what was used captain G G(h)(0) ok. So this is what I am getting, so let me get rid of this. This was confusing so let me know it ok. So I have this D0 which is moved by small h on to h(D0) which is and that is an isomorphism because h is an atom of some unit disc.

Then from h(D0) to G(D0) is also an isomorphism because is an isomorphism on h(D0) ok, but there is something funny about G. See this G is not defined on the whole unit disc. This G is not define on the whole unit this is only defined on this simply connected domain ok. And therefore you know what this G as far though it is see the pick's Lemma and hyperbolic geometry only tell you that if you are not an option of the unit disc.

Then it will be an isometric for hyperbolic machine. But G is of course an G is of course an isomorphism from HD0 to Gh D0, but G will not be an isometric which is hyperbolic, in fact G will be an expansion, G become a strict expansion, why because we see we already seen that G is the inverse of this function a squaring function which is the strict contraction. Therefore G will be a strict expansion ok alright.

So the fact that the squaring the square function is a is a strict contraction will tell you that it is inverse which will be a branch of the square root that will be a strict expansion ok. So you neta going to neta square is a strict contraction implies that G is a strict expansion. So so what does so what does this mean this means that is you see G(neta) so you know strict expansion with respect to what.

Of course strict contraction, strict expansion etc. are with simple hyperbolic metric. So you if you take the image under G of 32 points neta0, neta1, and take you take 2 points neta0 and neta1 in the in h D0 ok and you take the images under G. There will be 2 points in G(h) D0 and you will take the hyperbolic distance that will be you know that will be greater than or equal to c times a constant times the hyperbolic distance between the two original points for suitable c greater than 1.

This will happen see this is see expansion means that the distance between the image points hyperbolic distance between the image points is greater than the hyperbolic distance between the source points and there is a constant which will appear and the constant will be greater than one, you know what is a constant, that constant is actually the reciprocal of this constant which are replaced by root restaurant

Where route r is a root r is such that modneta less than or equal to root r that contains h(D0) ok. In fact C= 1+r/2roto r where modneta less than or equal to r modneta less than or equal to root r contains h(D0), that is in fact that will come from here ok that will come from here, note that after all small g capital G is an inverse for small g, small g is a square function alright.

Capital G is a square root, capital G is an inverse for small g ok. So you know if you in this in thins expression you call this as neta0, call this as neta1 ok. Then this will be gneta0 and this will be Gneta1 ok and you will get this expression where you will have to put Gneta to be inside this ok and if you want the image under G to the in the disc bounded by r then the source disc should be bonded by root r ok. So from this you automatically get this ok.

So the moral of the story is that you know if we need to use this, so alright so now we have the G is the strict expansion alright and then what you do is see now I still you know my original zero was there in my D0 ok and then I translated I mean I used I map this small b to 0 therefore then I took h(D0) 0 was not there ok and then I am using the square root function.

So 0 will not continue to be there alright, but then I would still go like to go back to the origin. So what I will do is I will apply another mobius transformation that will map GH to the origin ok. So I will bring this I will bring this fellow back to this to the origin ok of applying a suitable map and let me cal that map as have too many arrows here, so you know I have so I will I will apply map like this.

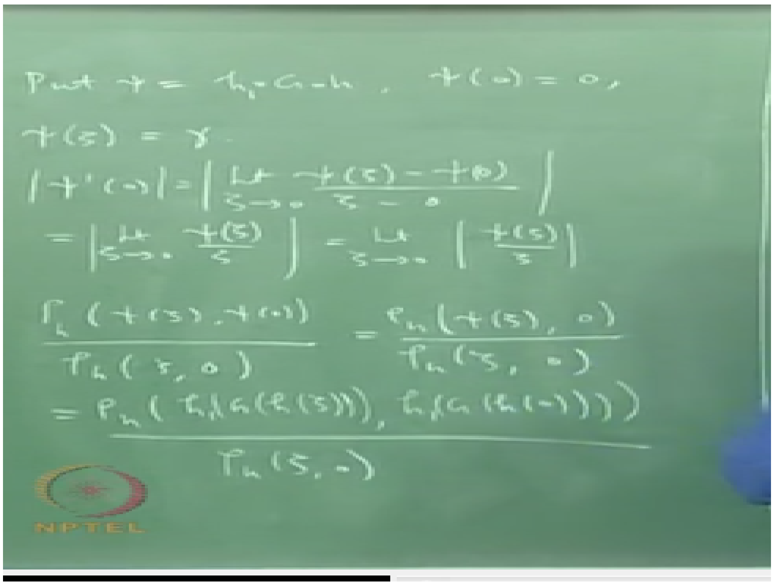
So consider so put for used small h and let me use h1, so h1 of omega to be you know I put omega- Gh0/1-Gh0/omega I will do this ok what this will do is an atom of unit disc that will map GH g(hD0) to the origin ok. So now I will do is now I will consider this composite from the so I have D0 I first apply h ok and I will land inside hD0 that is an isomorphism.

Then from hD0 I apply G and I will land GhD0, this also mind you automorphism I mean this also isomorphism because G is an analytic branch of square root is injective and injective holomorphic map is an isomorphism on to its image ok. And then and then I have applied this h1 that goes to hgh (D0) and mind h1 is also an isomorphism because h1 is atom of the unit disc is also injective.

So it gives an isomorphism from this ok. Now what you do and notice that 0 goes back to 0, so you know 0 goes to h0 h goes to G(h(0) and that goes back to 0. S0 0 goes here right and now the big deal is that you look at the derivative of this function so you know so let me continue from let me continue from here. So now look at put side to composition that is first apply h and then apply G then capital G and then apply h1.

Put to this ok, then shy of 0 full site will be equal to this then shy of 0=0 alright of course and now calculate the capital the derivative shy-of 0 ok you know I am trying to look at I am Lemma says that if you are smaller simply connected sub domain than the unit disc then I can find an analytic map which maps it isomorphic on to a sub domain of the unit disc which takes 0-0.

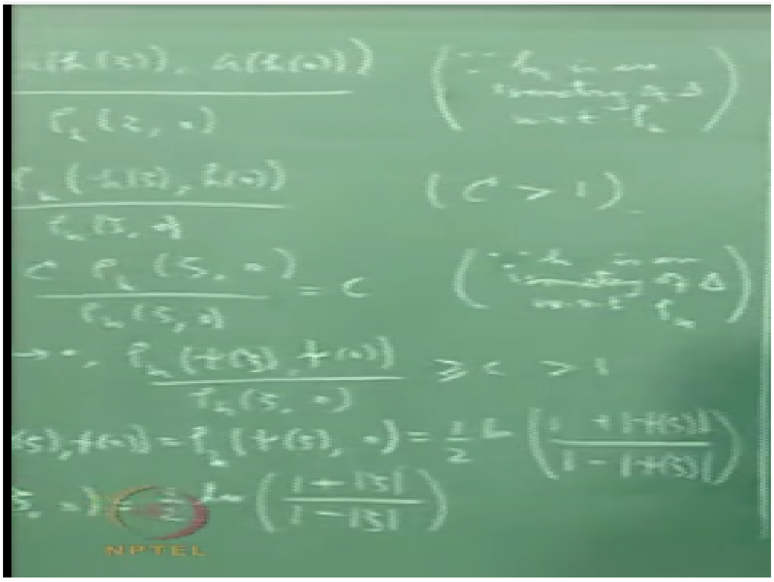
(Refer Slide Time: 20:06)



And whose derivate the origin is greater than ok, that is lemma right I trying to do that lemma I claim that this shy is the shy that I need, so it certainly a map it is of course shy is an first isomorphism because it is injective, it is composition of injective map, so it is an injective map, so it is an isomorphism it take 0-0 ok, calculate the derivative of shy at the origin let me use the correct variables my D0 what those mine variable on D0.

On D0 it is zeta and h(zeta) I call it as neta and neta goes to gneta which I have called as gneta is called as W and then I have h1 of w which I have to give another name so in short of so let macro environment use gamma ok. Normally gamma is used to denote the path ok, but in short symbol let me use gamma. So you know the variables or D0 it is zeta on h D0 this hzeta which is neta ok.

(Refer Slide Time: 22:03)



On Gh D0 is it is gneta which is W and again on h(g(h1(hD0) is the target variable is gamma, this is h1(w) ok. So write one more I have written it shy is the function of this starting variable is zeta and ending variable is gamma ok, please hold right and what is what is shy- what is mod of shy- of at the origin, this is limit theta times to 0 shyzeta/zeta, shyzeta-shy0/zeta-0 right.

And shy(0) is 0 and so I will get limit zeta times to 0 shy zeta/zeta ok I will get this ok certainly I can write this as limit zeta tends to 0 mod shyzeta/zeta I can write this because after all the map shy at the origin is analytic, so the derivative exist. So this limit does exist and if the limit exist I can take this the mod as the continues function, so we can write. But my aim is I want to show this si greater than 1.

Because that is the purpose of lemma, the lemma purpose of the lemma is that show that I can map a smaller simply connected sub domain of the unit disc I can I can map that isomorphic on to another similar smaller simply connected sub domain of the unit disc that counties the origin, but with the extra condition that the derivative of the origin can exit 1. The fact that I can the fact that I can make the derivative at the origin in mod exceed 1 is exactly due to the fact that I am working on simply smaller domain that the unit disc.

Because it had it been in the unit discuss show lemma will tell you that the differential version of lemma will proceed with this happening it will made the derivative at the origin to be less than or equal to 1. But the fact is because I am mapping my domain is not the whole unit disc, but as better simply connected domain smaller than the unit disc I can make the derivative at the origin greater than mark.

I can exceed the bound of the short film that is the whole point, so someone have to make this greater than the mark ok, now the fact is you know I have to somehow use this estimate ok, this estimate connected with the mapping in between which is capital g ok the mapping in between capital g which is the square root function, so analytic branch is square root is expanding ok.

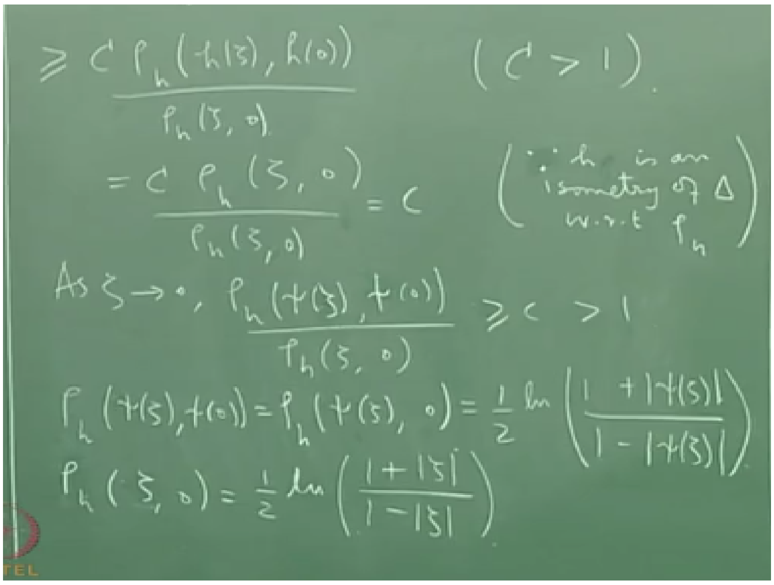
And I have to use that estimate ok to show that you know this happens alright. So you know let me do something, let let us keep as right let us try to apply that and let me try to apply, so you let us calculate rho h(shy) ah zeta, shy(0)/rhoh of zeta, h, ok. Well you see this is same as rho h(shy of zeta, shy of 0 is just 0/rho h(zeta 0) ok. But you see now but what is shy(zeta).

Shy(zeta) is h1(z(zeta), so this is rhoh(h1(g(hfzeta), let me keep let me write this here h1(g(h(0)/rhoH(zeta,0) ok. So I get this alright and now see I can knock of the h1 because the h1 is the atom of the some unit disc ok and therefore is an isometry with respect to the hyperbolic metric. So you know I can throughout the h1, so you know that will be equal to rho h.

Of course you know please do not confuse this subscript h with the h with this h because this h subscript of rho suppose to signify that I am taking hyperbolic discs ok. So do not confuse the subscript h of rho with the map ok. So I can knock this h1 out because h1 is an automorphism of the unit disc and is isometric with respect to the hyperbolic metric, that is the part of Pick's Lemma alright.

So I can simply write this rho h of g(zeta), g(h(0) divided by rho h zeta,0 ok I can I can write this since h1 is an isometric of delta with respect to the hyperbolic matrix rho h ok, so I can do this and then you know but g is I already have an expression for rho h of g of something divided by rho of that thing alright. So you know in this in this expression what you do is you put neta0=h(zeta) and you put neta1=h(0).

(Refer Slide Time: 29:00)



So I will end up with so you see this will be greater than or equal to c times but by using this estimate I will get c times rho h of what should I put in fact neta0 it is hzeta and it is of neta1 I will put h0 ok, here is where I am using this estimate, where c is greater than alright, I am I am use so this is very important I have to use this estimate which is reflection of the fact that this square root is expanding.

And that is because its inverse function which is the square function is contracting and why it is contracting is because it is define on the whole unit disc and it is not an isomorphism and any analytic function of the whole unit disc taking values in the unit disc which is analytic and which is not an isomorphism is necessarily strictly contractive ok, that is where see this is where I am using hyperbolic geometry throughout the portion ok.

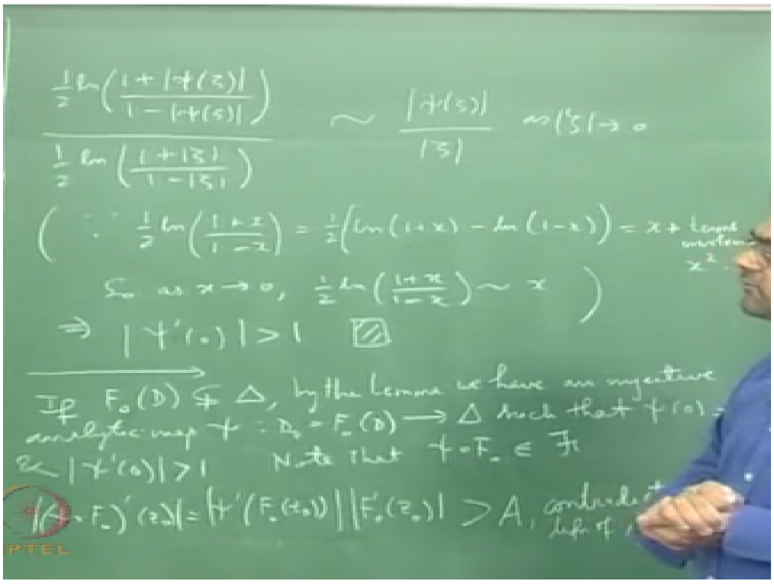
So I will get this, but you know if you look at small h the small h is an also an automorphism of the unit disc, so for small h is alos an isometry with respect to hyperbolic metric. So in teh numerator you can knock of that h. So the finally this series ration turns out to be this ration turns out to be greater than I am it is greater than or equal to c which is greater than 1 ok.

So now you know then I will have to compare this you know so you know as as zeta times to 0 ok what will happen is this ratio rho h of shy zeta shy(0)/rhoh(zeta0) this is this is greater than equal to c which is greater than 1 ok, you have this ok, if you combine all thisx together, this works if you let zeta10-0 alright, on the other hand my claim is a zeta tends to 0 this is exactly shyzeta/zeta.

This is this behaves like shy zeta/zeta therefore in other words as zeta tends to 0 you should take modshyzeta/zeta, this is the quantity which is greater than 1 and therefore the derivative of 0 is greater than one that is a thing. So what we need to understand is that this zeta times to 0 behaves like shy zeta/zeta ok and I think that that is that should be alright because you see rho if you calculate rho h of shyzeta, shy0 is rho of shyzeta, 0 ok.

And we know what this is, this is half we have seen of this is half lawn ah 1+modshyzeta/1-modshyzeta. This is what rho h ok right and rho h of rho h of zeta,0 will similarly be half lawn 1+modzeta/1-mod ok. We know this format ok and now again you know if you if you actually and if you actually divide and let zeta go close to 0 then the ration will be shy zeta/zeta.

(Refer Slide Time: 33:06)



Will behave like shyzeta/zeta just because of again because of if you want the locate also ok. So you know you know you should calculate half lawn of 1+modshyzeta/1-modshyzeta/half lawn 1+modzeta/1-modzeta. If you calculate this see what I want you to understand is half lawn 1+x/1-x is approximated by x, for x sufficiently small. See see if I expand half lawn 1+x/1-x where x is a small quantity ok.

Then you know what I am going to get is I am going to get half*lawn(1+x)-lawn(1-x) ok and this will see this will be x+ terms involving x square and so on because lawn1+x is x- you know x squared by 2+x3/3 and so on right. So you know if I if I expand both in power series for x sufficiently small alright, if I expand and subtract what I will get is I will get x+something and then here I will get –(-x) and so on.

So I will get 2x and then there is half outside, so I will get x=terms involving x squared and high of course ok. So as x becomes small this quantity is so as x tends to 0 half lawn 1+x/1-x is behaves like x o, I need this fact, see as x becomes very small ok half lawn 1+z/1-x can be approximated to x. So this can be approximated to numerator to be modzeta shy(zeta)/modzeta as zeta modzeta times to 0 ok.

This is an approximation, this comes very closetful to the behaviour of this alright, and therefore and the moral of the story is that if I now calculate if I let take the limit as zeta tends to 0 modshyzeta/zeta, the behaviour is like taking the limit as zeta tends to zero of this mod of this quantity and that is always greater than 1, so you know therefore what you will get is modshy-of 0 is greater than 1.

And that finishes the Lemma, that finishes this lemma I will see highly technically it is not very difficult but the point is the idea is involved they use lot of hyperbolic junction ok. Now you know now I can go back and say why the image of F(0) is delta ok, why the image of F(0) is less than delta and equal to delta because it is less than delta you can apply the lemma ok, if it is now it is now the proof is just 1.

If f(0) of t is proper properly contain in delta by the lemma we have an injective analytic map shy from D0 which is F0 of D to delta such that shy-z shy of 0 is 0 and modshy-of 0 is 1 I am sorry it is greater than 1 ok. Now I can apply this lemma, but the beautiful thing is if I combine if I apply this F0 and then follow it by this shy the resulting thing is continues to be in this family.

That is the big deal ok, note that first apply F0 then apply shy which is also in the family because after all it is also a map you first apply F0 and then you follow it by shy that continuously me a map from D to the unit disc ok because shy is takes from disc to the unit disc ok. So this composition is also map from D to unit disc first point. It is also analytic is combustion of analytic functions second point, third point is injective because both are injective.

Shy is injective and F0 is also injective therefore composition is injective, but now comes a big deal what is the derivative of shy circle F0 at z0 by the chain rule this is derivative of shy at F0 of z0 smallest times derivative of F0 I mean derivative at z0 if F0 ok and this is and you know derivative F0 is z0 is 0 and shy-of 0 mod is greater than 1, this is a, this will be greater than a ok.

Because this is greater than 1, this quantity is greater than 1 by the Lemma and this quantity is equal to a ok. So the product is greater than a now that is the contradiction because a was supposed to b a was suppose to be supremum of all the derivatives of all the functions in the family F, the modulus of supremum of the modulus of the derivative at the origin at z0 at the point z0.

You cannot find a for any number of the family if you take the derivative at z0 I will take the modulus it cannot exceed a, a is the supreme, but I have found something that exceeds a that is the contradiction, it is the contradiction to the fact that the size of the F0 is in the family script F. So this contradiction proves that F0(D) is delta, in other words the extremely function F0 fulls out the whole unit disc ok contradiction to definition of a.

So this proves the important fact that F0 does map this domain onto the unit disc isomorphic ok and of course you have already adjusted it to make zero z0 goes to 0 you can make the derivative at z0 to be positive by using a suitable rotation ok. So you can also make the derivative cost right and that finishes the proof of Riemann mapping theorem ok of course one has to also think about whether you can make the derivative at z0=1 ok .

But z1 that one has to wonder about but what we have is that given any simply connected domain D which is not the whole complex plane, you can find and you can find a holomorphic isomorphism of that on to the unit disc which maps any given point to endpoint is not of D onto the origin and you can make the derivative at the origin I am in we can make derivative at that point also to be a positive real number, so you can do this much ok, so that feels let me remember ok.