Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-41
The Candidate for a Riemann Mapping

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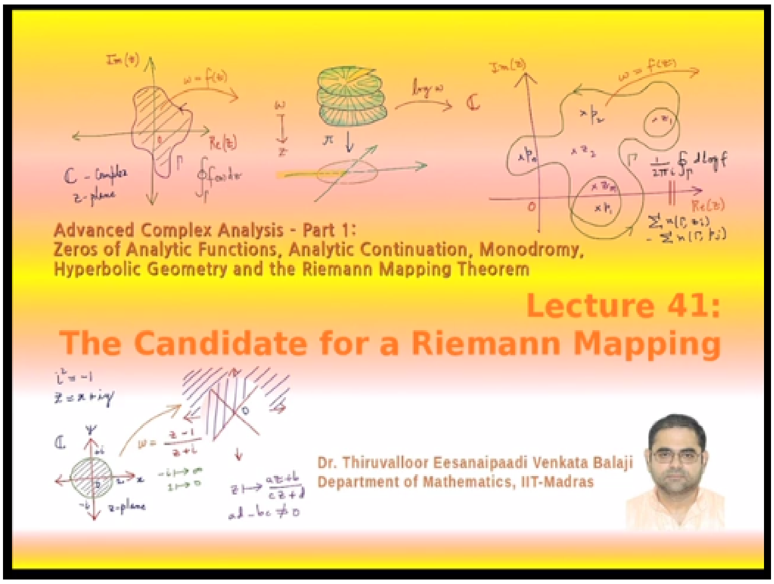

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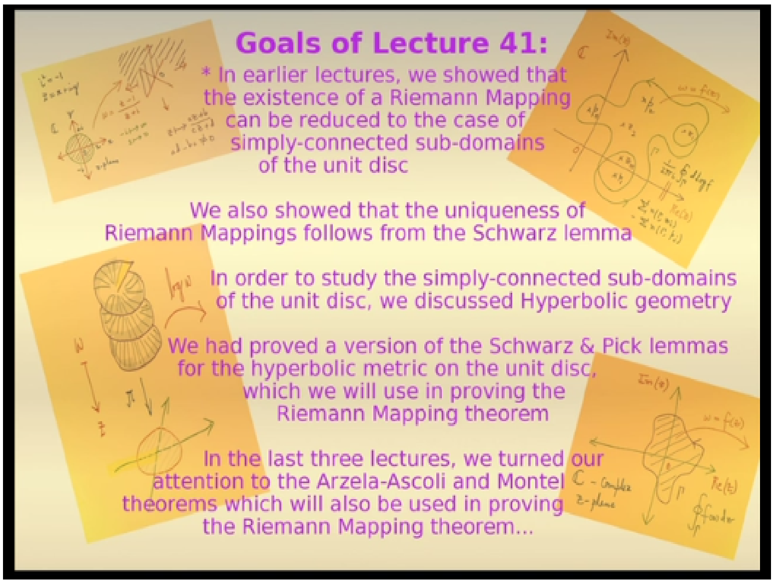

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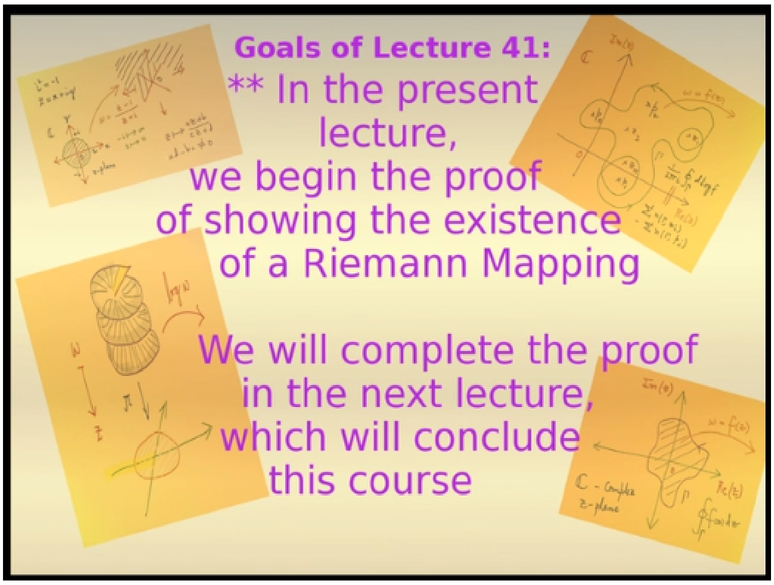

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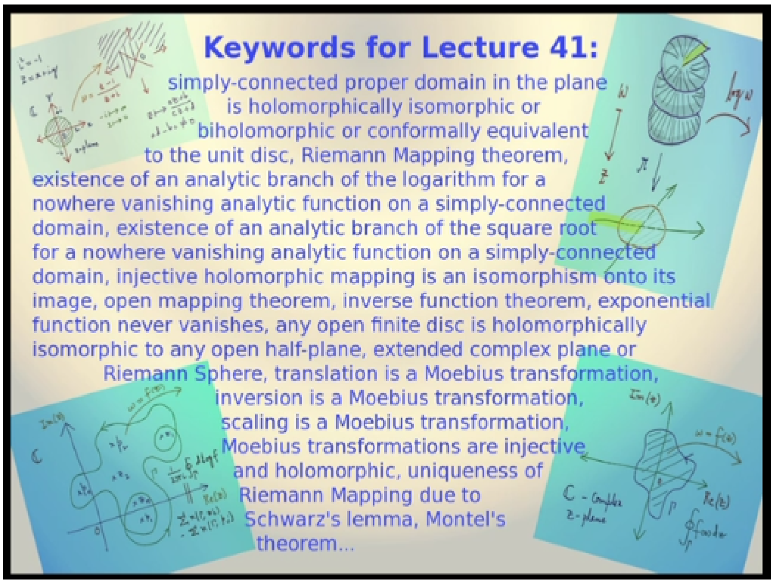

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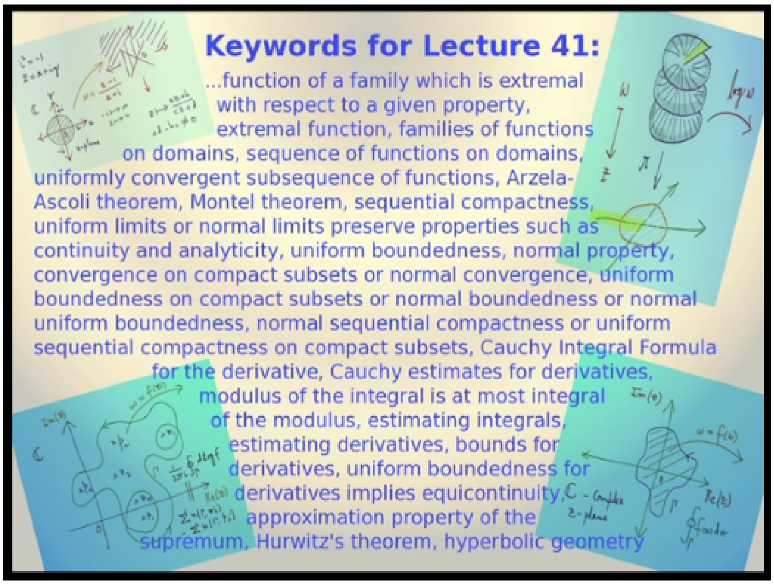

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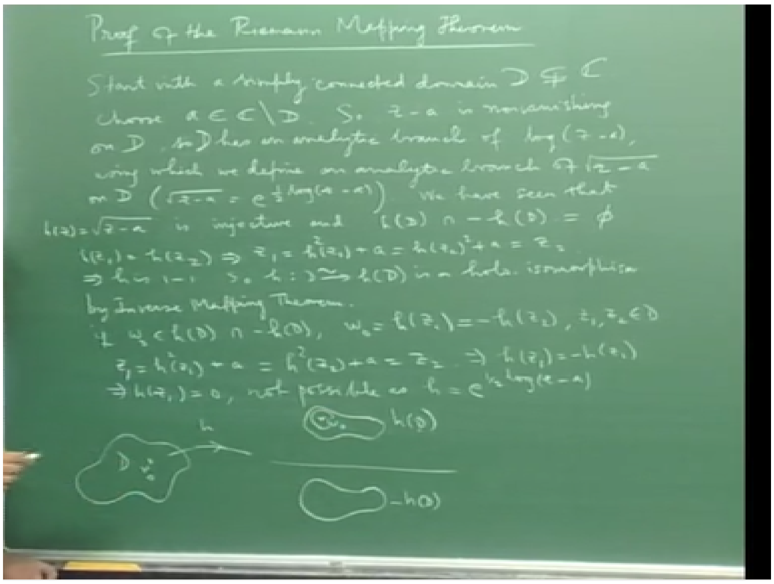

So start with a simply connected domain D which is not the whole complex plane and so choose an a which is in compliment of D ok, so D is D is not whole complex plane and it is simply connected and the Riemann mapping theorem is a statement that D is holomorphic isomorphic to use disc ok and in fact you can find such a holomorphic isomorphism which carries any fixed point of D to the origin.

And that such isomorphism is unique provided you fix the derivative of that map at that point which is D map ok. So now let me recall something that we all did you choose an point which is outside D ok let so F(z) so let me F(z), so z-a is non vanishing on D because a is not in D and it is analytic ok and if you have a lower vanishing analytic function on a simply connected domain you can find the branch of the analytic of the logarithm.

So has an analytic branch of log so D has an analytic branch of log of z-a using which we define an analytic branch of square root of z-a on D ok after all root of z-a is just e to the half log z-, say it is analytic branch of log of z-a then e to the half log z-a is also analytic and there is a analytic branch of root segments ok and we have seen that this function is root x analytic branch of root of z-a which we are simply writing as root of z-Armstrong

These actually one to one is injective and so you know if you call this as h(z) ok and h(d)) does not intersect –h(d) ok. So these are all things that we saw few lectures ago ok the image of D and H and –h this image should be distinct ok. So there is no intersection between HD and -HD and in fact and you know not very difficult recall the proof, see if z1 h(z1)=h(z2) to imply that you know z1 which is H(z1) squared+a will be h(z2) squared+a which will be z2.

So this will tell you that so this implies which is 1, 1 ok and h is suppose injective and h is of course holomorphic you know injective holomorphic map is always isomorphism on because at the inverse mapping theorem right therefore the injective holomorphic map, so it is a isomorphic holomorphic isomorphic on to it each made and so this is the fact that h is 1, 1 ok that is injective.

Then so h some from D to h(D) is a holomorphic isomorphism by inverse mapping theorem yeah inverse function theorem ok and the other fact is that there is no there is no point in hD and -hD and the reason is again the same kind of calculation if w0 belongs to hD and -hD then w0 is actually h(z1) and is also equal to –h(z2) where z1 and z2 part in D.

Remember that –h is actually the other branch of the square root ok h is one branch of the square root of z-a and –h is after all the other branch of the square root and all I am trying to say is 2 branches of square root at different ok because every number different from zero has two distinct square. Yes that is all that I will say. So you know immediately that is essentially the reason for all this.

So you know if you use this again you will get z1=h(z1)+a and that is equal to h(z1) I am sorry h squared of z1 is same as h squared of z2, so it will be could h squared of z2+a which is z2 ok and this will tell you that it give you this funny thing that you know h(z1)=-h(z1) and that will give the h(z1) is 0 and that is the contradiction because g, a h is after all an exponential.

H is h(z) is a square root of z-e is e to the some alright and you know exponential function can never take the value z. So this is this is not possible as h is e to the half log z-a ok. Therefore the contradiction tells you that there is no point in common the h(D) and h(-D) aright. So you know so you know that so the diagram is that you have this this is I will not draw this D simply connected domain and I have this map h then you see if you draw.

So I will draw D bounded, but it will not be bounded ok just because the domain is holomorphic isomorphic the unit disc does not mean that it is bounded because you should remember that unit disc is holomorphic isomorphic to any half plane and no half plane is bounded ok. So you should not expect a simply connected domain which is and the domain which is holomorphic isomorphic the unit disc do not expect it to be bounded ok.

That is the only mistake one makes, but it will be bound if you can consider as a domain in the excited Riemann play ok, and the excided complex may be namely on the Riemann the upper half plane will also look like a disc ok, so it will be bounded in the external in the external complex plane ok , but not on the complex plane right. So when I am drawing D to be bounded just. So that you know it easier to draw.

And then so you know the picture is like this you have so you have h(D) and then you know you have h(-b) which is just its reflection I think it is a pretty bad picture. So this is –h(D)and they do not they do not intersect alright, this si the situation and now what you do is you know you take a point z0 here and then you take your point w0 there ok and choose a small enough this surrounding w0 ok.

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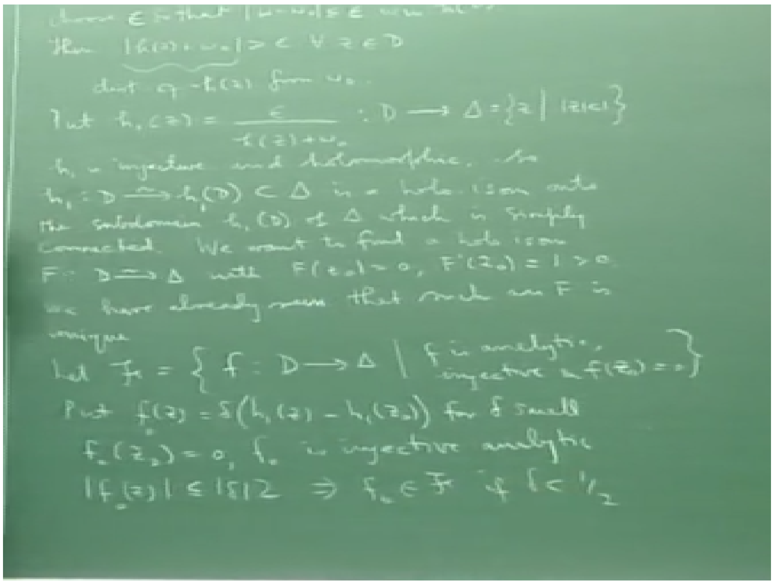

So choose rho, so that you know modw0 so the modw-w0 less than rho is in h(D) mind you D from and h from D-h(D) is a holomorphic isomorphism because h is injective homomorphism and injective holomorphic map is a holomorphic isomorphism ok. So h is the holomorphic isomorphism of D-hD ok and I am taking a point w0 in hD is an image of a point z0D and h and I am taking this small disc close this centre at w0 which is inside hD ok.

Then you see then what you will get is rho if I take this I mean if I take small enough disc small enough disc which I take centre at w0 ok with radius of epsilon, then what will happen is that modh is at +w0 be always be greater than Epsilon for zAD, this happens because you see the difference of hz from w0 I mean this is actually distance of hz, this is the distance of -hz from w0. This quantity is distance of –h(z) from w because the distance is modulus of -hz-w0 which is same as modulus of Hz+w0 ok.

And but you know –h is disjoint from this, so if you take – if you take any z here ok then hz will lie here. So you know –hz will lie will be reflection, it will be in -hD and certainly therefore the distance of –hz from w0 will be certainly greater than this Epsilon ok, that is all I have done. But now this is what I am one can use to produce holomorphic mapping of the domain into a subdomain of unit disc. So what you do is you put h1(z)=Epsilon/Hz+w0.

Look at this now, this is a map from D to the unit disc why because modh1z is mod of this and mod of this is less than 1 because and mind you h1 is of course injective why because you see h is injective and h(z)+w0 is injective inversion is mobius transformation that is injective ok and then multiplying the epsilon also mobius transformation. So I am getting h1 from h by applying a series of mobius transformation.

So I first take h I translate by w0 that the mobius transformation to get a hz+w0 then I apply a inversion zeta going to 1/zeta that is also mobius transformation ok and then I multiply the epsilon multiplying by constant is also mobius transformation. Therefore I get h1 from h by applying a series of mobius transformations. Therefore h is injective, therefore h1 is also injective.

So h1 is injective and holomorphic therefore again by inverse mapping theorem h1 will give an isomorphism of D onto h(D) which will now be a subdomain of delta ok and it will be a simply connected because all holomorphic isomorphism are also homo options and holomorphic image of simply connected a and z connected ok. So h1 is injective and holomorphic, so so h1 from D to h(D) h1(D) it is a delta is a holomorphic isomorphism onto a subdomain h1(D) of delta which is of course simply connected.

And you know now so we have having using all this somehow we are able to map this D into the into subdomain of the unit ok. sorry this should be F yeah as I first I wrote modw-w0 less than rho. But then I change the rho to epsilon, so this should be Epsilon ok., So Epsilon sufficiently smart. So you know finally I have list a map D into they are simply into isomorphic onto holomorphic on to the subdomain of the units.

But what you want is you want a map that whose image is the whole units ok plus you also want the map with the property that it takes a given point that is z0 to the origin ok and you also want the derivative to be some fixed at the derivative at Z0, you want also have the map such that the derivative at z0, see something at possibly on this ok these are everything. So we want to find holomorphic map holomorphic isomorphism f from D to Delta with FFz0=0.

So positive wavelength ok, you want to find holomorphic isomorphism ok and we already that such a holomorphic isomorphism we have already seen that such an F easily ok. The holomorphic map of the simply connected domain is not the whole complex plane on to unit disc can be made unique if you specified its value at one point and the derivative at that point. So what we do if we fix a point if you fix a point is not in the domain.

And specify that the value of the function at z0 is which we usually take F0 and you specify the derivative at that point z0 and usually you take the if you want you can take it one you can make the derivative one if you want but for that matter with make it any real number right, so if you want you can make it 1 then such an F is unique because you know if we have 2 such Fs F1 and F2.

Then if you compose if you have 2 F1s and if I if you have 2 set maps F with satisfy these conditions, then if you compass F1 inverse and then followed by up to you will get an automorphism of delta ok which will take 0-0 ok and whose derivatives whose derivative you will see that delta you get an automorphism of delta and this derivative condition will tell you that it has to be just identity automorphism.

And that will tell you (()) (19:09) ok this is something what that all we see. So such an F is unique with that something at you all we see because you know we already know how that automorphism of them is a delta look like. So I have to find this F. So you know what we do is we completely changed our view point ok, see you do not try to get this F directly ok.

What you do is you look at all possible holomorphic maps into the unit disc ok and which they which are injective ok. You look at all possible maps like this which map D intro subdomain of the unit is ok and which are injective plus you also had that condition that z goes to 0 ok you look at that family and then up essentially apply Montel's theorem to that family cleverly ok.

And then you will get this F as an extreme function ok. So now we will change the view point, so what will do it let script F so here is a family here is where I tried to bring in something on which I can apply on Montel's theorem, so let us script F b the set of all h let me use small f anywhere yeah so centre for small f from D to delta such that f is analytic injective and f(z0)=0.

You look at this family, look at all mappings of the simply connected domain D into unit disc which are injective and you say z0 is zero, see this family is non empty the reason is because see h I already have already constructed injective holomorphic map from D into the unit disc, but I can adjust it so that I can make z0 goes to 0 ok. So F is non-empty see if you put F let me put F0 to be you know you put F0(z) to be some .

You know let me use deltaxh1(z-h1) for delta small ok see h1 so I have this h1 ok which maps d*unit disc alright I will just inject it. Now what you do is you define F0 to be delta times h1z-h1z0 you know why I am putting this h1z-h1z0 that is because if I plug in z=z0 become zero. So it will map znot 0 because it is a condition for functions in F but the other thing is I want this function also to go into the unit./

And for that if you have to take I think if you have to take just enough to take delta less than half probably, see mod modF0 so then see I do not have taken delta very small probably you see because F0 of z0 is zero ok and F0 is of course F0 is injective analytic because you know h1 is injective, so h1-constant is injective and Epsilon having some constant multiplied by injective functions is also an injective function.

So long as a constant is not zero alright therefore it is an injective analytic function you know problem of that and the only thing you have to worry about is whether it takes values in the unit disc, so you know if you have modF0 of z0 you know triangle inequality this will be less than or equal to mod delta*modules of –this which is less than mod of this+mod of this which is 2 ok.

Because h1 takes values in unit disc, so any value of h1 has modulus less than 1, so triangle inequality will give this. So if you choose delta less than half I mean less than half then you are done. So this implies F0 is seen your family F if delta is less than half delta sithere as an half then modF0 strictly less than 1 which means that F0 lands in inside the unit disc and that means F0 is in this family script F.

So therefore this family is non in I have non-empty family and now comes the now you know the whole purpose of defining the family is because you know Montel's applying Montel's theorem your family should be family of analytic functions which are uniformly bounded that is the only condition you what ok. The point is automatically uniform rebound because is there all functions into the unit disc.

Therefore the modulus of functions always bounded by 1 ok whenever you consider a family of analytic functions taking values in a bounded domain ok it automatically uniform bounded.

So the script f is automatically uniformly bounded that is the whole point ok and you apply Montel's theorem ok.

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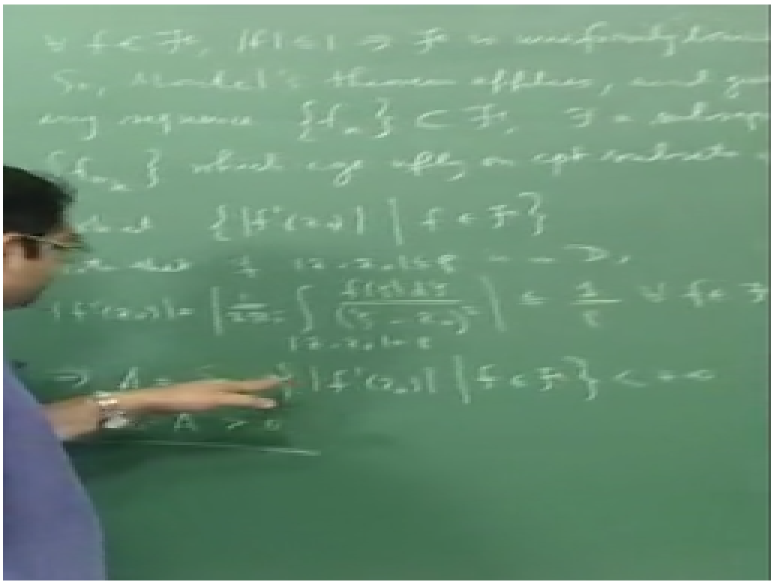

For all functions small F script f mod F is less than or equal to 1 implies this script f is uniformly bounded ok in fact I in the in Montel's theorem I need only uniformly bonded and compact success which is here for but it uniformly bounded throughout and as because the target domain is bounded ok and so know if I so the moral of the story is that Montel's theorem apply.

And you give me any sequence of functions in script F ok have always be a subsequence with converges uniformly and compact subsets of t ok. So Montel's theorem apply and given any sequence fn in f and there is a subsequence fn k which converges uniformly on compact of D I have to give same compact subsets of D because I do not know D is bounded D need not be bounded ok domain that is holomorphic isomorphic unit disc need not be bounded.

For example it could be there a half plane is not unbounded, so whenever you are working with an unbounded domain always uniform convergence extra you should only expect on compact substance. So now so the question is which sequence I am going to apply this I can sequence I am going to apply. So you know what I am going to do is I am going to take a look at at a set of all derivatives at z0 and their module i for f small f.

And look at this I am taking all these function these functions are all holomorphic maps that map is not to the origin and the map D holomorphic on to simply connected subdomain of the unit disc and looking at all the derivatives at all at z at the fixed point z ok what you must understand is that you know what is seen during the course of the proof of Montel's theorem that you know if you look at a family .

If you look at a uniformly bounded family of analytic functions ok and then you take a point the domain, then you know in a sufficiently small neighbourhood of that point all or are you and at that point at the given point uniformly bounded because of cautious estimate ok. So you note that if modz-z0=rho less than or equal to rho is in D ok then modf-z0 is going to be and just repeating what I wrote.

I am just applying to this formula and this is less than or equal to 1 by rho ok where that is because modf is also less than or equal to 1 ok. So normally when you apply course is estimate what will get is will get is m/the you get the bound for the derivative at the centre of the circle should be equal to m/the radius of the circle ok where m is the maximum is the bound for the mod of the function on the circle.

And of course the bound on the circle also be a bound inside the circle because of the maximum things ok so it will be a uniform bound, but for the functions f in f for the function smaller f script the boundaries one. So I will get this, but what does this tell you this tells you that this will tell you that this will tell you that this set is bounded above. This is for all this is for all of encrypted f script this over for yeah there is the square of the denominator.

But you get 1 yeah thank you that is right because if you look at n to the derivative you should get vzeta-z0 to the n+1 ok. So if you look at first derivative zeta-z0, so you get this, but now so what this tells you this implies that if you take supermom of this of all this derivates at z0 where small f is script f, this is if you call this as a this is finite, because you know this si set of nonnegative real numbers.

It is bounded above by this 1/rho ok and therefore it is therefore it is supremum exists and is finite alright and of course you know the point is it the same rho works for all f ok. So all the derivative at z0 are uniformly bound, the family of derivatives of functions in f and all the derivatives are uniformly bounded at z0 ok and therefore this is a finite quantity.

I said that I will so this is standard property of the substance of the real line know the details of the real line which is bounded above then it has a supreme ok and that is equivalent to actually the completeness of the real life ok. So this a is finite alright and now you also know that **so** so one of the thing that I want to tell you that this see you see this a is so the thing I want to say that this a is positive ok.

I want to say that a is positive ok that is one this a is a positive number that something that I want to say and that is because you see so the claim I want to say is that a is positive I think that more or less clear because you know if a is 0 then the supreme of these are all non negative numbers, the supremum is 0 then each of 0 ok, then all these derivatives at z0 have to vanish right.

Because each of these numbers is less than or equal to the supreme supposed to be released, so each modm-of z0 is greater than or equal to zero and it is less than or equal to a. So if a is zero then all the derivative at z0Z vanish and that does not that is not possible because you know all the see all the Fs I am considering there are injective ok for an injective you know an injective analytic function is an isomorphism on each fs.

So derivative cannot vanish ok, derivative cannot vanish because there is a inverse function. So the injectivity of all these Fs will force that you know a cannot be seriously a as be positive ok since F each since any F is script F is 1-1 ok, so a is the positive take have to 0 alright. So a is some positive number and now you know I am going let me go back to your good old real analysis.

where you know the supremum has a approximating property, so the supremum of set of numbers is can be obtained as the accumulation point of the set of numbers ok in other words what is the property of implement it is there supremum is called the least of a upper ball, so it is an upper bound and any number less than that is not an upper bound ok, so which means that you take any number less than that you can find a member of the set which exceeds that.

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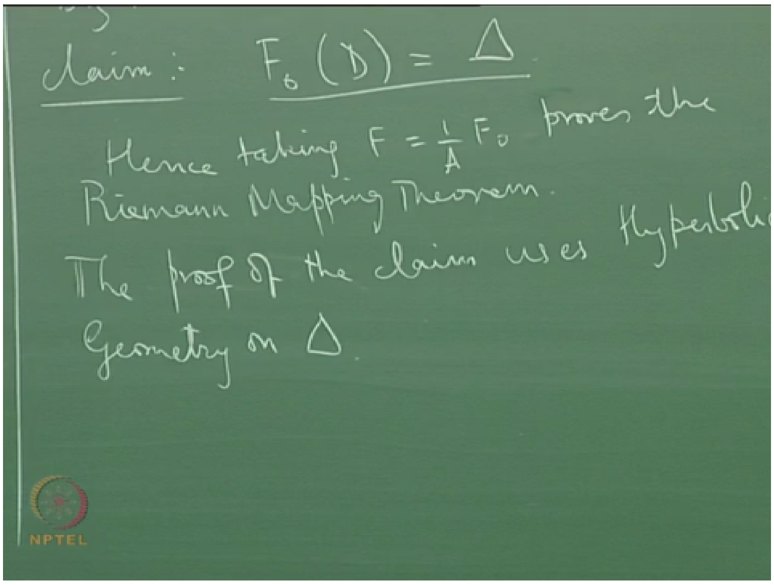

And that number less that you take that could be Epsilon less than that and in this way you can cook up a sequence of elements from this set which tends to be ok and it is that sequence of F that to which I am going to apply Montel's theorem ok. So we can we can find fn such that fn inside script f such that fn-z0 in modulus times you can do this because of the approximation property of the supremem.

That is because the supremum is a accumulation point and I can therefore get the sequence of elements here with tends to supremum and that sequence corresponds will give rise to sequence that correspond sequence of functions apply to the point z whose derivatives are apply to the points and that the sequence I want, that is the sequence which I am going to apply Montel's theorem ok.

Now apply Montel's theorem by Montel's theorem fn converges to a function f ok my claim is that this is my function ok probably maybe I may have to modified. So let me call it as f0 ok, so by Montel's theorem, so what is Montel's theorem it tells you that whenever you have family analytic functions which is uniformly bounded a compact subsets.

Then given any sequence in this collection, there is always a subsequence which will converge uniformly a compact subsets. So it means there is I will find a subsequent fnk of fn which converts is uniformly on compact subsets. So basically converges to a function that functional will be analytic and the analysis is because the convergence is uniform and compact se ok. So you know that if you have a uniform limit of analytic function is analytic ok.

And more generally if you do not have uniform do it on the whole domain you have uniform limit only on compact surfaces si enough because that allows you to check within a close this question compact ok. So uniform limit of a normal limit of analytic function is analytic ok. So is f0 is analytic clearly you see clearly you can see that f0 modf0 is less than or equal to one because each of the functions here are into the unit disc.

The unit function will also be in the unit disc ok and the derivative of f0 at z0 will be a ok because you know under normal convergence not only does the sequence of functions converts to a given function is analytic the derivatives will convert the derivative of the remote function and this will happen for any nth derivative. Because normally it is a very powerful alright.

Therefore what will happen is this converges to this, so it derivative will converge to its derivative but its derivative at z0 converges to a, so it is derivative at z0=a ok. So you know what we have so more or less what we have done is we have found a function which is mapping into the unit disc because modf0 is less than equal to 1, but mind you less than or equal to 1.

But mind you if it is less than or equal to 1 it is not exactly in the unit disk it is fixed with close units ok, it is our function which maps into close unit and so if I, so I have this function which maps the domain D into delta bar ok because see each for each function you know in fact each function maps into the unit disc ok in fact this called here alright.

And therefore here also we have strict inequality alright, but here I cannot claim the limit I cannot probably claims strict inequality alright but more or less I will get strict inequality simply because of the open mapping theorem ok , anyway but let it be as it as, let me not worry about it, the first thing I want to fix is that you know I want to fix is that I want to say this F0 is I want to say this first of all I want to say this F0 is a non constant function.

It is non constant analytic function and I want to say F0 is also injective, now why is F0 not non-constant because its derivative is at z0 is non z0 a is a is possible, so this implies that F0 is not constant ok and now comes under the beautiful thing, see the fact definitely is injective come from Huruvitz's theorem ok, see Huruvitz's theorem there is one version of Huruvitz's theorem will says that you know if you have a sequence of analytic functions which are universe which are injective.

If that sequence converges normally ok then the limit function is either constant or reducing again injective ok see the original Huruvitz's theorem the standard version of the Huruvitz's theorem will say that you know if you have a sequence to analytic function converges normally F to a given to close function and then that you should take a 0 of that limit function of multiples to m.

Then you know the 0s for such beyond a certain stage all the sequence of in the in sequcne analytic function that you are considering beyond a certain stage all the functions in your sequence will have zeros of the total number of zeros in a small neighbourhood of the zero of the limit function of multiple m will be again multiple pm, that is what the Huruvitz's theorem says.

It will say that if the limit function has 0 of it has a 0 of order m at z0 then beyond a certain stage all the functions in the sequence will also have M0s in neighbourhood of z0, sufficiently small neighbourhood of that z0 and all these zero will accumulate at the 0 of the limit function and that is the original version of Huruvitz's theorem, but another version of Huruvitz's theorem is if you apply this original version carefully you get the other version of Huruvitz's theorem which says if you have a sequence of univalent that is injective analytic functions ok.

If it converges normally to a unique function then their either the limit function is constant and if it is not constant the limit function is convince to be injective alright, the injectivity is just the fact it is just the fact that it take very value 1. So the multiplicity is 1. So basically Huruvitz's theorem says that multiplicity will be preserved ok the multiplicity will be preserved.

So if all the functions in the sequence are injective they are their multiplicities are all one ok, that a limit function will also have multiversity one if it is not constant ok, that is huruvitz's theorem. So if you apply it here you see F this f0 is actually uniformly limit it is normal limit of these function and these functions are all injective ok. There are and the limit function is not in you know therefore the Hurivitz's theorem F0 is injective ok.

So like Huruvitz's theorem F0 is injective ok, now you know we are more or less than we are more or less than here more or less than ok and you see the there is only one more thing left and the claim is the following, the image of F0 is the whole unit disc ok. So this is the finite, see you know F0 is an non constant analytic function and it is injective, so it is a holomorphic isomorphism.

So it is a holomorphic isomorphism of D into close units alright, but the image has to be open because image of a holomorphic map is open, so in open sets inside the close units ok and the claim is that open set is the open unit disc, in other words F0 achieve the job of mapping the given simply connected domain is not the whole complex plane onto the whole unit and prove Riemann theorem ok.

And there is of course the derivative at z0 if I wanted to be one I have it as a so if you take insert if you take instead of F0 if you take 1/a times F0 that will have derivative 1. So you can always adjust the derivative to be 1 at z0, that is not a big deal, a big deal is which shows the disc F0 free out the image fills out the whole units open units ok and here is where the big deal comes.

The big deal come from hyperbolic geometry on the disc, that is what helps us to ensure that this F0 has to fill out the whole unit disc ok and that is because this F0 is extremal function, it is extreme because this limit achieve this limit ok, so whenever you are familiar functions and you know if you look you look at some numbers define based on that family ok.

And you take a limit ok then if you are able to find a function in that family which achieves this number it is called as extremal function ok. So I am able to find member of F script F ok namely F0, F0 is a member of script F such that the derivative at z0 is exactly this extreme values, this limiting values which is supremum, mind your supremum is also a limit ok. So and the point is because of this extremal property of a F0 it has to fill the whole unit.

That is that is where hyperbolic geometry comes, so hyperbolic geometry comes in touch me tell you that the image F0 is exactly unit, so let me write that F0 taking F please remember Riemann mapping theorem. The claim uses rather the proof of the claim instead hyperbolism ok. So let me stop here.