Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-40
The Proof of Montels Theorem

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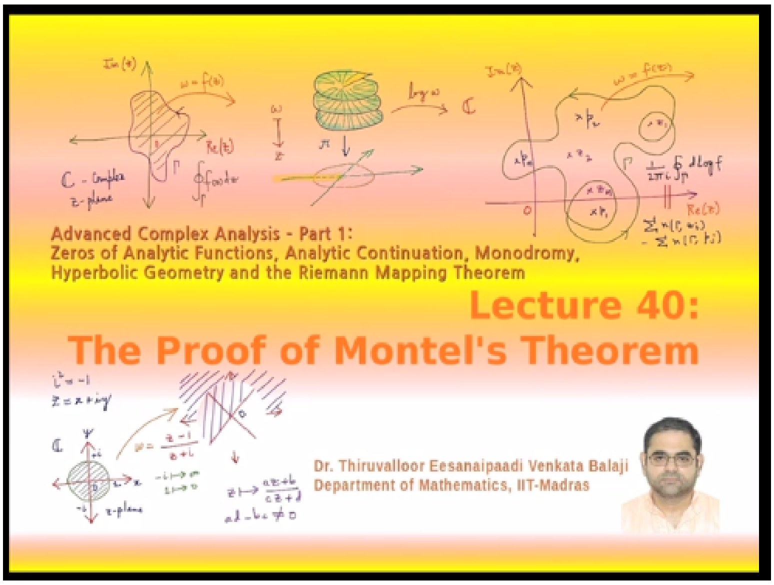

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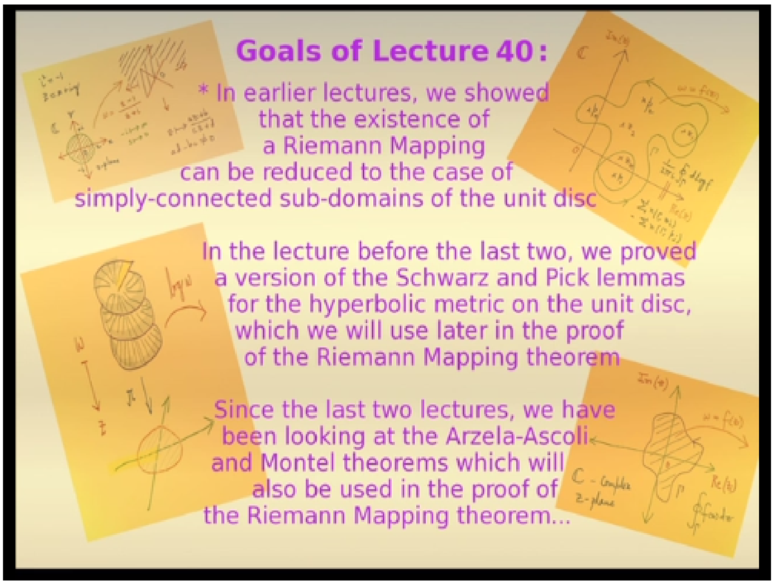

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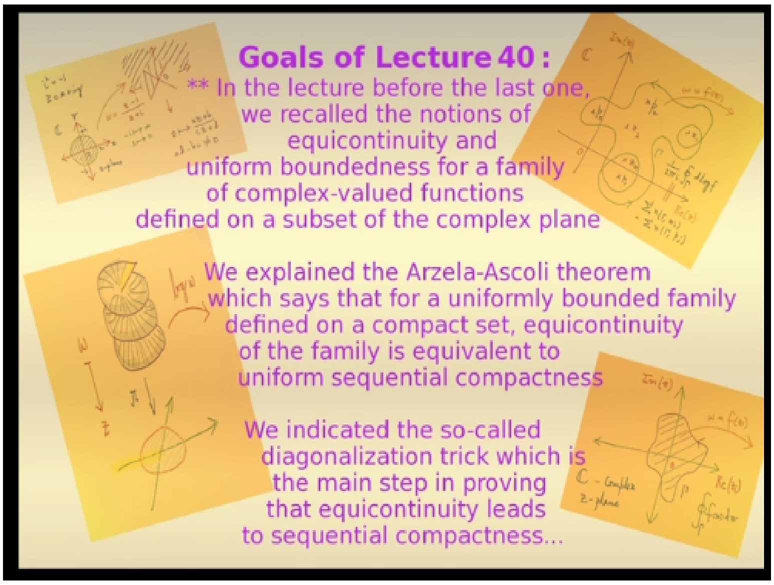

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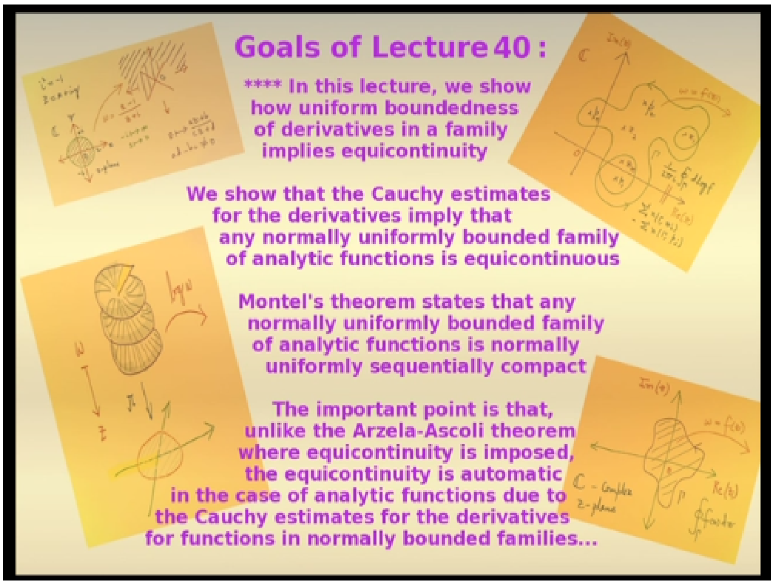

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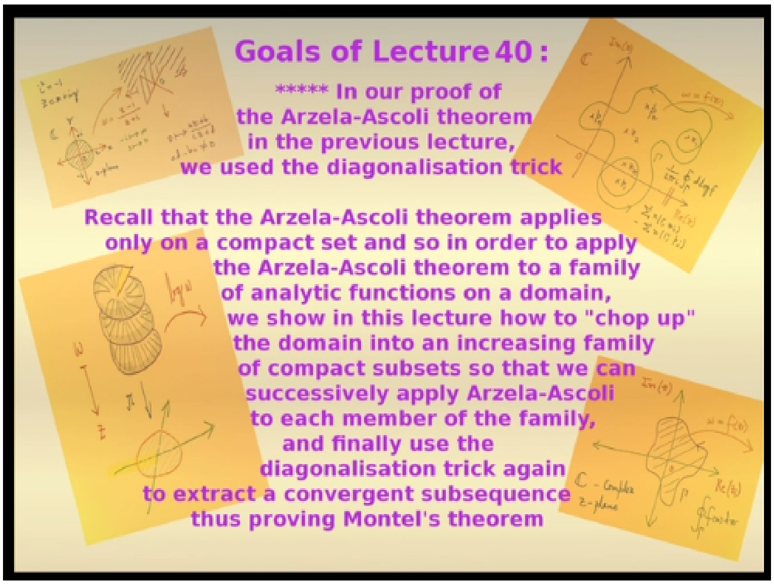

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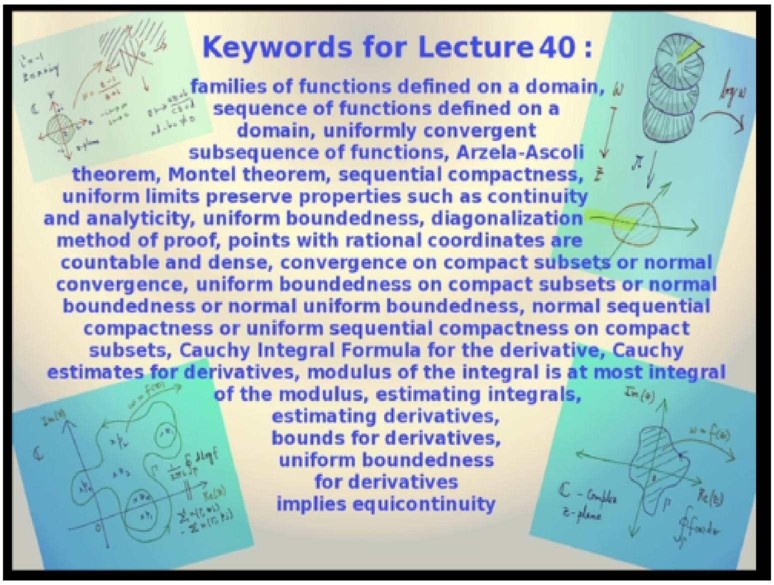

So so let us continue with our discussion of Montel's theorem. So you know so this is so this is essentially Montel's theorem. So what you do you take so it applies you know it is a version of Arzela Ascoli theorem adapted to the case of analytic functions ok and so as I told you that Arzela Ascoli theorem everything happens on a compact alright unique compactness on the set on which your functions are define alright.

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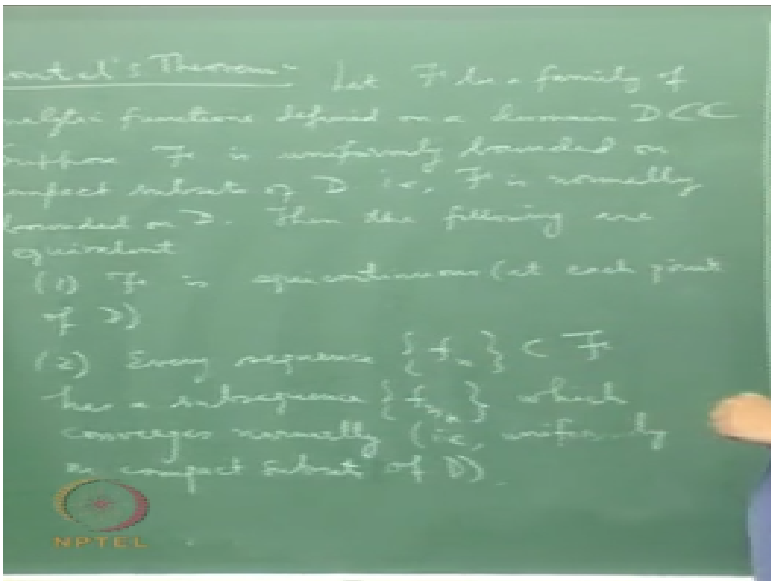

But here you know so analytic functions are defined in an open set on domains ok and we consider them on open connected sets in unit domain. So what you will have to do you have to put all the requirements only on compacts of sets ok. So let me write this F family of analytic functions define on a domain D ok inside the complex plane. So these an open connected sets and all the functions in the family script F they are define on the domain.

And your analytic function ok suppose script F is uniformly bounded on D, so here is the unit if I firstly wrote down something that is too much to expect. So you know when you want to when you do the Arzela Ascoli theorem we will say your family of functions defined on a compact set and is uniform bounded on the complex ok. Then the Arzela Ascoli theorem is an gives you an equivalence between 2 statements.

One statement is equicontinuity of the family at each point and the second statement is that every sequence of functions in the family has a uniformly convergent subsequence ok. Now so you know to expect a family of analytic functions to be uniformly bounded on a whole domain is too much ok. So you should modify a uniformly bounded on compact subsets of a domain ok.

So I can see this is too much on compact subsets, subsets ok. So see a properties that force the compact subsets is called a normal property, so if you have convergence and compact subsets call normal convergence. If you have uniform boundedness on compact subsets is called normal normally boundedness ok. So you know F is normally bounded on D ok. Then so you know what is so what is your so what your Arzela Ascoli theorem in usual will say that you know if you have this uniform boundedness.

Then equicontinuity is this is equivalent to the fact that every is equal to the statement that every sequences uniformly convergent subsequence, then the following requirement or requirement number 1 scripted is equicontinuity at each point of D ok and the second condition will be every sequence in this family script f has uniformly convergent subsequence.

But now again you should not expect uniform convergence on a domain always you should only expect normal convergence it is usually expect uniform convergence only on compact sets. So the second statement should be written carefully, you should say that every sequence fn in that has a subsequence fnk which converges normally that is uniformly on compact subsets of D ok.

So you know the statement if you compare this Arzela Ascoli theorem the statement is in the Arzela Ascoli theorem your family of functions is not analytic in a general Arzela Ascoli theorem the family functions is not analytic is only continuous family functions ok. But they are complex valued of course. So the analytic condition is not there but you have a weaker condition is just continue ok.

Then and in Arzela Ascoli theorem the functions are not defined on a domain, they are define on a compact subset of the complex plane and the condition on F is that it is a uniformly bounded on the compact set. Now that is replaced, now the compact is replaced by a domain ok, therefore the condition of uniform boundedness is restricted only to compact subsets of the domain alright.

And then you have then when you have uniform boundedness then the Arzela Ascoli theorem the philosophy is that equicontinuity is the same as the existence of subsequent that convergence uniformly. So the equicontinuity condition is going to anyway which is continuity condition, so it remain as it is but this the existence of a subsequence which converges uniformly.

That also you can you should expect only on compact subsets. So that is why we put this we say that the for every sequence get a subsequence at converges normal right, so this is Montel's theorem and actually again you know again you know the implication 1 implies to is what we are going to prove the implication 2 implies 1 says you can prove the same thing and if you know 2 implies one can be proved.

Just in the way that we have done that that one could do for Arzela Ascoli theorem ok, it is a proof by contradiction ok, so what I want to tell you is that condition one will always be true ok, condition one will always be true because of Cauchy's integral formula for the derivative of analytic function which will give bound. Therefore what will happen is that one will always be true if the family is uniformly bounded.

And therefore this is always to refer this always ok, so what I want a state is that this first condition is is superfluous the first condition always true ok simply because you are not working with just continues function they working with analytic and an analytic functions analytic functions you know for analytic function you have a good bound for the derivative and in fact you have bound for all orders of derivative.

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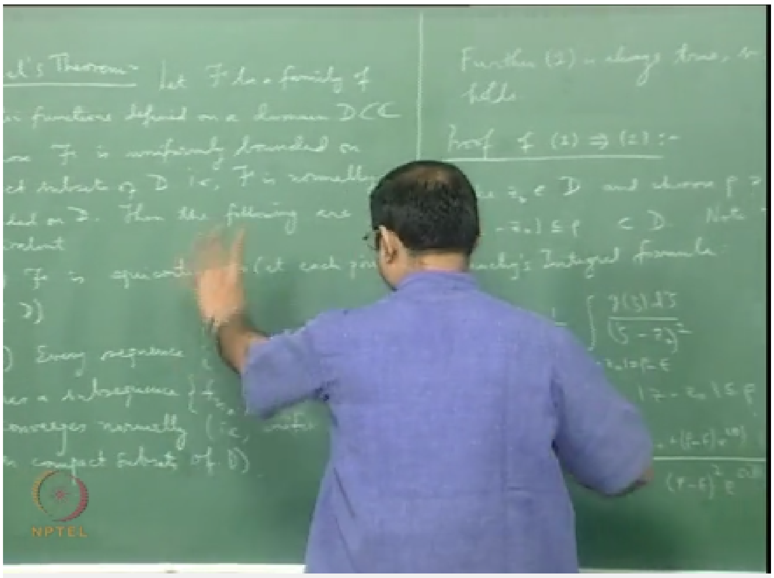

Derivatives of every order at a point because of Cauchy integral formula ok. So one is always true and then therefore 2 is always true, so is so 2 always false ok. So let us look at the proof of this so what I will do is I will just I give the proof of 1 implies 2 which is the which is the slightly technical team ok, take a point z0 in domain and choose rho greater than 0 such that that is modz-z0 less than or equal to rho it contain in domain ok.

So certainly you can do this, we being since the domain is an open set z0 is an interior point, so that is a disc surrounding z0 is containing the domain and you take a slightly smaller disc it is closer will also be contained in the domain and call that radius of rho ok. Then note that by Cauchy integral formula what you will have is you see you know my diagram is like this here is my domain D and here is here is my point z0.

There is this I am taking this centre at z0 and I am going to take a radius to be equal to rho-Epsilon where epsilon is a very small quantity alright and what Cauchy's integral formula, Cauchy's integral formula of the derivative will tell that for any analytic function on this close disc alright the derivative of the function at the centre of the disc is given by 1/2piei integral over this boundary circle modz-z0=rho-epsilon will get g(zeta) ok.

This is Cauchy's integral formula for the derivative right, this is Cauchy's integral formula right for for g analytic in in this disc ok epsilon there ok so this is I am just writing Cauchy's integral formula I am not doing anything else right. Now in particular so this si you get a bound you get a bound by you know putting by parameterization this you will get z-z0=1/2piei integral from 0-2pie.

So you know the points in the z on this disc can be parameterized as zeta=z0+rho-Epsilon*e power i thetha, this is how I can parameterized that z ok. So the theta vary from 0-pie. So if I transform this integral to a real based on a real parameter. So what I will get is I will get mod I will get I will get g gzeta is going to be rho-epsilon ie power i theta t theta/this isa going to be zeta-z0 is rho-epsilon the whole square e power 2i ok.

This is what I am going to do right and now what I am going to do is I am going to take modulus and note that the modulus of the integral is less than the integral of the modulus ok I am going to set inequality which is always used whenever your estimating integrals is otherwise known as the ML formula. So so modg-of z0 is going to be mod of this thing on right but that will be less than or equal to you know if I take if I take mod.

You must outside I will get i/2pie alright and then model of the integral of the modulus, so will get 1/2pie integral 0-2pie and I will get L so let me put m into rho-epsilon this mod of i i e power i theta is going to be 1 and modb thetha is just b theta because theta is increasing along this interval and here I am going to get rho-epsilon squared adn this is again going to be 1.

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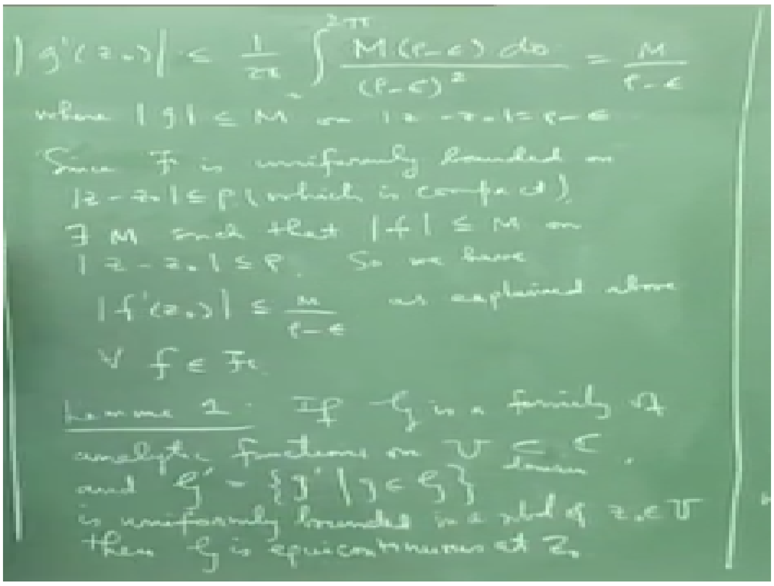

And this what I will get and what is this m, m is bound for the modulus of g on o**n** this boundary set ok where modg is less than equal to m on modz-z0=rho- ok. So here this is just equal to n. You know if I calculate this so integrate 0-2piethetha will get 2 pie and 2 pie will cancel this 2pie, I will simply get m/rho-epsilon, this is the boundary ok. This for an analytic function on this which analytic on this close disc.

Now what I am going to do is I am going to apply this to all the functions in my family script f to all the functions in this family, see you should take all the functions in this family alright their personality the analytic integer therefore they are analytic on such discs alright and the point is they all uniformly bounded. So I can find a single m which will work for all the functions ok.

And therefore I will get this uniform bound for all the derivatives ok and that is good enough to tell me that the family is equicontinuity ok. So let me make the statement for any since f is uniformly bounded on modz-z0 less than or equal to rho, you see you have the uniform boundedness and compact subsets of D ok. Therefore you I am applying this uniform boundedness.

I have uniform boundedness on this close disc and this close disc is compact because its closed and bounded alright. So this is a compact subset of B that this family is inform bounded ok. So this is compact there exist m suspect modf is less than or equal to m on modz-z0 you have this, so if you so we have modf-of z0 is less than or equal to m/rho-epsilon as explained above ok.

So this inequality that this is estimate I have got for the model of a derivative I applied F alright, so I get this right and now what I want to say so what this tells you that this is this is for what this is for every F in the family, so what you have got is you have got that mrho-epsilon is a uniform bound for all the derivatives ok. So what we have so what you have got is that all the derivatives are uniformly bounded at that point.

Now you see now it is a fact that if derivatives if you know if you have family functions derivatives are uniformly bounded then that family is equicontinuity ok. So I I will so let me write g is family of analytic functions on you which is domain and you see and g and t and g dash which is equal to set of all F small g dash where g belong to g the derivatives is uniformly bounded in a neighbourhood of z0 in you.

Then g is equicontinuity to z0 ok, so I am setting this fact that the uniform boundedness of the derivative imply uniform boundedness of the derivatives in a family implies equicontinuity of the family. uniform boundedness of the derivatives at a neighbourhood of the point in a family implies equicontinuity at that point ok and this proof is m of b it is just given by simple I mean if you want to prove it for real functions it will follow.

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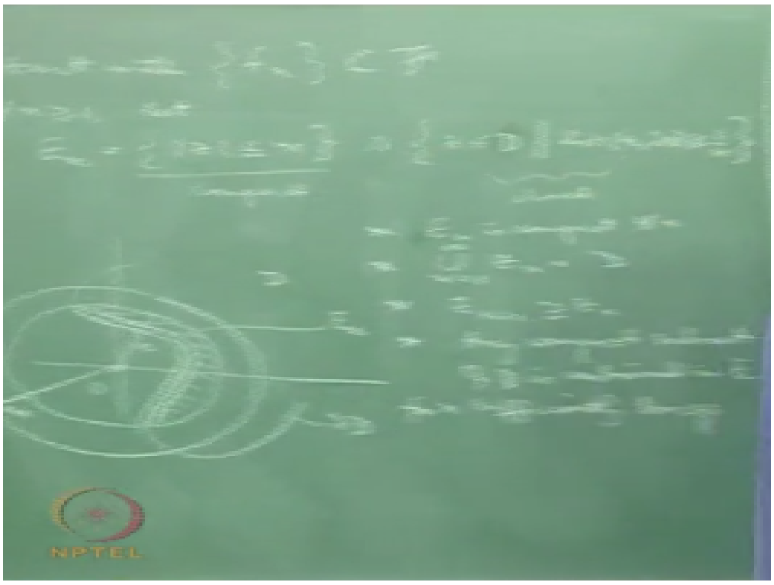

I mean if you are working with real value function on on a close boundary interval or an interval ok then the proof will come by applying the mean value theorem ok. But if you are working with complex value functions you have to use integration. So what you do is see you know the situation is that so you know I have this new and I have this point z0 and you know I have this disc centred is z0 radius some arc ok.

Modz-z0 less than r r is greater than zero ok this is this is inside you I can find such an r because after all z0 is interior point of you and you is open set and what I am giving is that all the derivatives is uniformly bounded in the neighbourhood therefore there exist an M such that mod modulus of z dash is less than or equal to M in modz-z0 less than r, this is given to me all the bounded derivatives are abounded alright.

And now how do you show that the family is equicontinuous at z0, so what you do is you know you calculate g(z-gz0) modulus ok. So of course this is for all g in g ok. So small g is in script g so small g dash in scrip g dash is given the script g dash is bounded uniformly bounded in the neighbourhood of the point ok and now what is MODGZ-Z0 You SEE a what you can to see this is integral which is internal along a straight line path from z0 to z of g-dz edash beta bzeta.

Of course you know if you integrate g dash you will get g because after all derivative of g is g dash you can you can integrate mind you g is analytic z dash is also analytic ok and therefore the integral is actually independent of the path chosen in a simply connected neighbourhood of z0 and of course we are always considering this this surrounding z0 is simply connected ok.

So for example you can take z to be any point here and you can take the you can simply take this straight line segment from z0-z and you can integrate alright, but then again use the fact the modulus of integral is not the integral of s0 is integral of the modulus. So what you get is that this less than or equal to the integral from z0-z modzdash of z mod d zeta ok and what is but modzeta is you know uniformly bounded by this M.

So I will get this is equal to this is less or equal to m times and integral from z0-zmod these will give you the length of the arc from z0-z which I am considering and that is I am considering that to be a line segment or simpler get modz-z, this is this is the value of integral form z0-z modd ok, normally when you integrate moddzeta along the path you will get arc length, but now I am integrating along the straight line path.

So I will simply get the same length of that straight fit line segment which is the modz-z0, but you see so this true for all g in script g ok. So so what this tells you is so given epsilon is greater than 0 there exist delta which is equal to epsilon/m such that modz-z0 lesser than delta which is epsilon/M implies modgz-gz0 is less than epsilon ok for all g this is what you get given an epsilon I am able to find the delta ok.

The delta only depends on this of course this delta I fix this z0, the delta depends only on epsilon ok and this is z0 but it is independent of the small g in script g because M is uniform for all g dash ok, but what is this tell you this actually means that the family script is equal continuous at the point z, that is the definition of equality, definition of equality is Epsilon delta definition for continuity should hold at a point for an epsilon you get a delta which work for all the which work simultaneously from the functions in the family.

So I have got a delta which depends only on epsilon this delta does not depend on g means that the family is equicontinuous at z0 ok, this implies g is simply continuous at z0. So that that uses the Lemma, so what the lemma tells you is that whenever you have uniform boundedness in the neighbourhood of a point ok, then you will have equicontinuity at that point ok.

Now if you now look at what we have written here you have uniform you have uniform boundedness of the derivative derivatives at that point ok and therefore it will work also in a small neighbourhood of this point alright and therefore you will have by this lemma you will have equicontinuity at at every point to the domain. So you have you have this bound at z0 ok and it is uniform for all f alright.

So what will happen is the same a you can get a bound for all ah points in a small neighbourhood of this z0 ok and therefore the derivatives are all uniformly bounded in small neighbourhood of zo and if you apply the Riemann you will get that the family is equicontinuity at z0. So this si a statement that one is always 2 ok that the family will always be equicontinuity ok.

So for analytic functions the derivatives will be bounded just because of Cauchy's formula ok and since derivatives are bounded equicontinuity will come automatically because that was the lemma sets, that whenever derivatives are uniformly bounded you will get equicontinuity right. So so by the Riemann f is equicontinuous at z0 at each z0 in the near domain ok. So so the point of those story is that you know in equicontinuity is automatic is just because the derivatives are all automatically bounded.

Because of caches integral formula right, so everything comes from just uniform bounded uniform boundedness of your family of analytic functions compact subset and will automatically give uniform boundedness on compact subsets of derivatives you will get uniform boundedness of derivatives in compact neighbourhood of each point and that is give you equicontinuity at that point.

And in this week you can cover all the points, so you get equicontinuity to everyone right. So so this part 1 this is always true ok and so I have to I will have to do this, I will have to show that so this is always to therefore this always true. So the only thing I have to do is I will have to show that give me a sequence here I will have to show that I can put up a subsequence which converges uniformly and compact subset ok.

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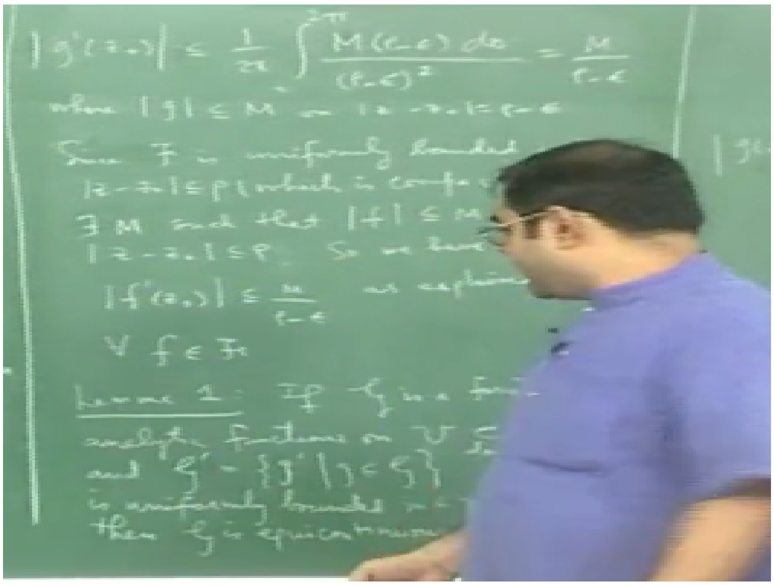

So what I will do is so let me retain this, so start with a sequence in f ok I already know that the whole family script f is equicontinuous at each point of the domain alright, start with the sequence. Now what you do is so we do if we do favour construction ok, so the construction is you know see I want to basically suppler Arzela Ascoli theorem but you know Arzela

Ascoli theorem division of Arzela Ascoli theorem that I want apply will only work on complex subset.

Therefore you know what I have to do is have to find I have to chop down this domain into union and increasing union of compact sets and apply repeatedly Arzela Ascoli theorem on each member of the union and then apply organisation of argument ok. So what you do you you do the following thing, so this is a trick, this is the trick of chopping up a non compact set ok into union of compact sets which cover it.

So what you do is for every n greater than equal to 1 let en then you know you look at modz less than or equal to n this is the disc this is closely centre at the origin radius n ok. So you know if so if I so let me draw diagram, so you can think little bit just for motivation, so you know you have suppose my domain is like this ok of course the way the way I have drawn the domain is already the closure of the domain which is already compact ok.

But let me do the following thing, let me just remove this, so that you know you can think of the domain has been probably unbounded, this is part of the boundary of the domain alright. So this is my domain d and this is the boundary of the ok now what you do is you look at modz less than or equal to n alright. So that is going to be so you know if I take m I am going to get disc like this ok.

And of I take n+1 I will get a bigger disc, so this is so this is n and this is n+1 ok as n increases this close discs they cover the whole complex plane ok, the union of all these this just n goes from 1 to infinity of whole complex set alright and now what I will do is you see I intersect it with the set of all points in the boundary the set of all points in the domain D such that the distance of that point to the boundary si greater than or equal to 1/n.

Look at this rather funny condition, so the condition is I am looking at all the points in the domain ok which lie inside this disc and whose distance from the boundary by distance of course I mean perpendicular distance, the shortest distance from the boundary is at least one by ok that means I am avoiding points whose I am avoiding points in the domain whose distance on the boundary is less than 1/1 ok.

So if you think of it like this then what will happen you know see here is this is a portion of the boundary alright and what I am doing is I am am avoiding all the points whose distance is so if I take the smaller disc ok if I intersect the smaller disc with which is modz less than or equal to n with the domain alright.

What I will get is I will get this this is what I get alright, this si the intersection of the smaller disc modz less than or equal to n with the domain alright and of course this is the portion of the boundary that intersects the disc mod less than or equal to M modz less than or equal to n right, now what you will do in this boundary you throughout all those points in the domain ok.

You take do not take all the do not take all of the shaded region but true at all the points in the domain whose distance is less than 1/n ok so it means that you know I am throwing out all points here I am just avoiding all points close enough whose distance is so you know this is this is what I am throwing out ok I am throwing out this because all points here the distance with the boundary is less than 1/n I am throwing that out.

I am just throwing that a piece of the domain which is close to the boundary ok and this si here it sets which is e1 and this is en this here is set this is en ok if you take en+1 what will happen is that I will get this whole intersection-I thorough out all points whose distance from the boundary is less than 1/n+1 which is smaller distance than this ok. So you see as n become larger see you can see something that happen n becomes larger.

I am covering more and more of the domain because after all as n goes to infinity these desk will cover the whole complex plane, therefore as n becomes larger and covering more and more of the domain and I am what I am throwing out is lesser and lesser I am throwing out points very very close to the boundary of a domain ok. Therefore in this way I will cover the whole domain.

So what you must understand is that and of course you know this is a compact set ok this is a compact set and this is a close set, this si the close set alright, this si a compact set and this is a close set and therefore the intersection these continues to the compact, so the moral of the story is that en is compact for every n and union n=1 to infinity en is your domain, your domain has been chopped up into compact sets.

So this is and this is the clever trick that one need to make to be able to apply yourself ok, so en is compact for everyone for n union of all the en is d and of course you know en you can see that you know en+1 will contain en+1 will contain en alright en+1 will contain n, so this increasing, so this si increasing sequence alright and another beautiful thing is you take any compact subset of D any compact of subset D will be contained in a sufficiently large en ok.

Any compact of set of any compact of set subset of D is contain in a insufficiently large in en for n sufficiently large. So you see these are the these are properties of these of ens, the ens are all compact, they are union their union is D, they are increasing and any compact subset of D is contained D en for n sufficiently large ok. And this is essential you can think of this as chopping the domain out into .

I think you are saying chopping I should say you know your it is more about filling out the domain in terms of an increasing sequence of compact subsets, the en fill out the domain, the union is the domain ok. Now since en is compact ok I can apply Arzela theorem because you know each en is compact and each en is compact subset of D but on compact subsets D have uniform boundedness because it is normally uniformly bounded ok.

And of course equicontinuity is already there ok so I can apply Arzela Ascoli theorem on each compact subset ok. So what I do is I do the following things, I again I cleverly use again a argument and if you recall and doing that Arzela Ascoli theorem you use the diagonization argument ok, we again use the diagonization argument apply Arzela Ascoli theorem to e1 to a sequence on e1 ok.

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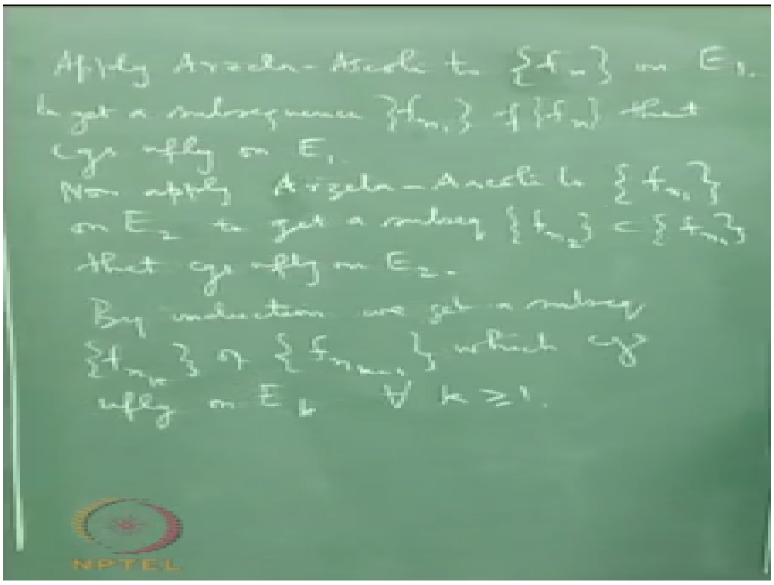

E1 is a compact subset of D and I have a sequence of functions on e1 that sequence is uniformly bounded why because the sequence is part of the family script F which is uniformly bounded on compact subsets therefore if you have also uniformly bounded on e1 alright. Now Arzela Ascoli theorem will tell you the there is subsequence which will converge uniformly on e1 ok.

So to get subsequent fm1 of fn that converges uniformly on e1 ok, now what I do is I take this subsequence and apply Arzela Ascoli theorem to it on e2 ok, so I go next bigger set ok, now apply Arzela Ascoli to this fn1 on e2 ok, to get fn2 for further subsequence which converges on e2 uniformly ok. Now what I will do is I will take this fn2 and apply Arzela Ascoli theorem to it on e3.

And I proceed like this ok by induction we get a subsequence that n k of fnk-1 which converges uniformly on e k for all k greater than equal ok, alright and now comes the big deal, now what you do if you from this ah see from all these you take the diagnosis sequence that will give you the subsequent to the original sequence that you converge uniformly on compact subsets of D.

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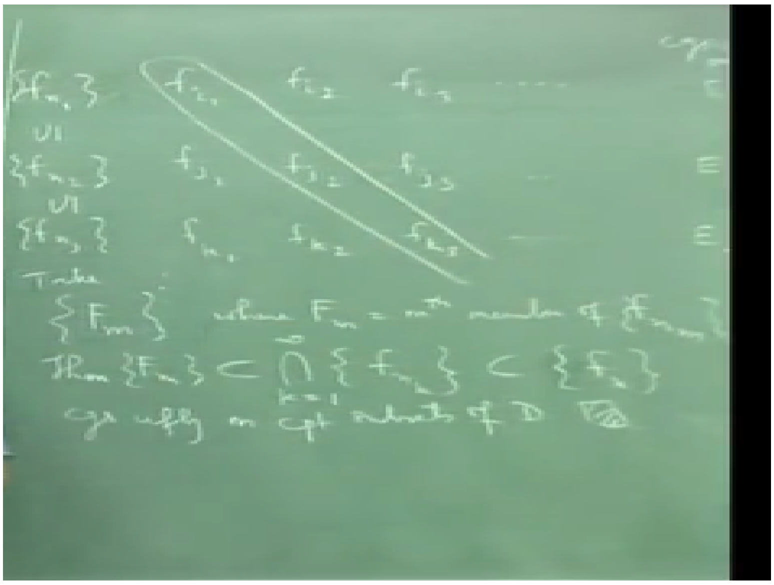

So again you know I again draw that the same kind of diagram that we will drew for the diagnosis and argument in the Arzela Ascoli theorem see you know you have the situation you have you know fn so you have fn1 and then I have a subsequent fn2, then I have the subsequent fn3 and so on and so this is you know this is some Fi1 Fi2, Fi3 and this is the sequence it converges uniformly on e1 ok.

Then here I have FJ1 FJ2 Fj3 that converges uniformly on e2, then I have fk1 fK2 fk3 that converges uniformly on E2 ok and it goes on like this and now what I am going to do is I am going to take this diagnosis ok you define Fm where fm=m member of fnm ok, then fm is contained the intersection of all these subsequence which is in of course everything is a subsequence of fn converges uniformly on compact subsets of D.

And that finish and why does it converts uniform and compact subset of D because you take any compact subset of D any compact subset of D is in some en and ok and si take any compact subset of D it will be in some eK, but on ek Fk+1 Fk+2 extra they all converges uniformly on Ek ok. So this sequence of function eventually converges uniformly on every compact subset of D.

Therefore it converges after all converges itself is eventually it has to happen only beyond the certain finite stage alright, therefore you get the last statement that this sequence converges uniformly on complex subset of D and that gives a proof of 1 implies 2 ok where 1 is already, so you prove to 2 is 2 ok and that gives the proof of Montel's theorem ok.

So you see it is a so the clever thing is to chop the domain up into the increasing sequence of compact subsets. And then repeatedly apply Arzela Ascoli theorem and again get a apply diagnosis o, so I will stop here and we will continue to the proof of Riemann mapping theorem in the next lecture.