Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-04
Hurwitz's Theorem and Normal Limits of Univalent Functions

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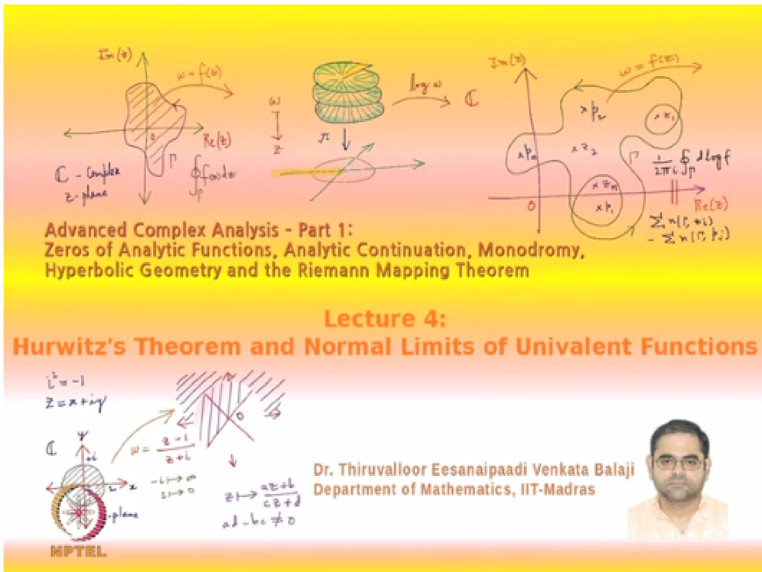

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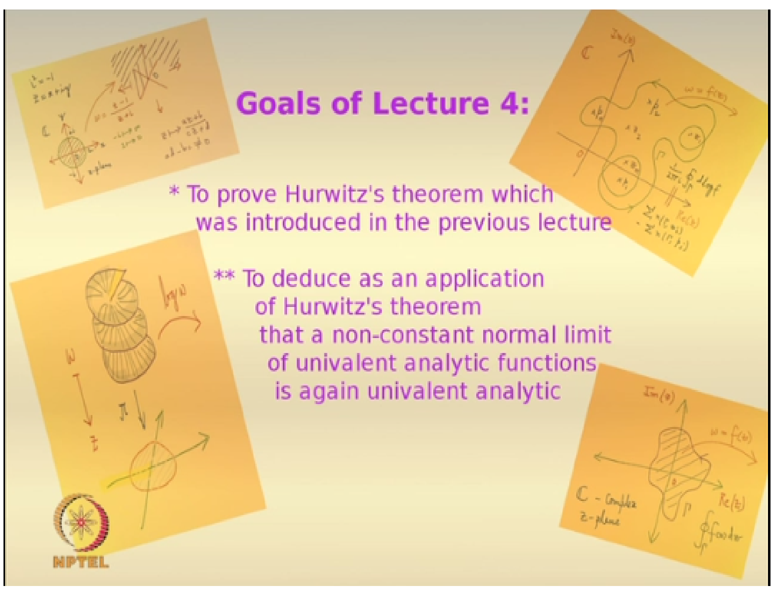

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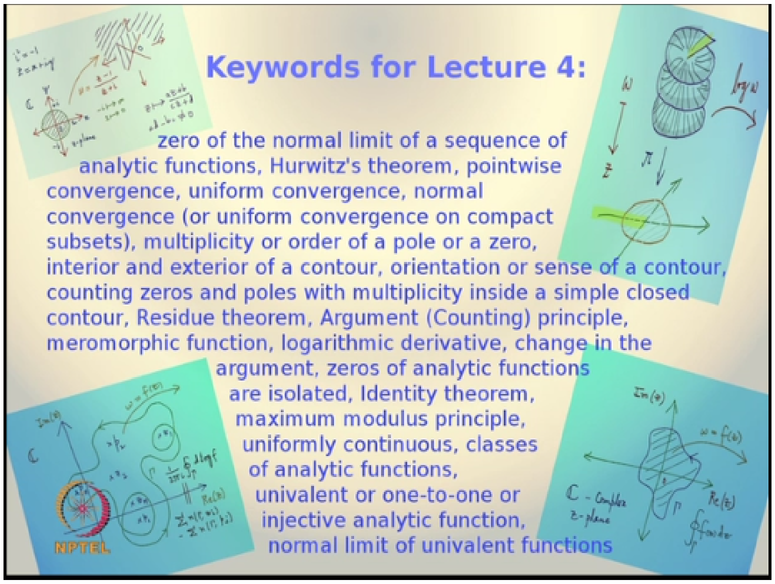

Okay, so alright so what we are going to do now is look at a proof of try to look at a Hurwitz's theorem.

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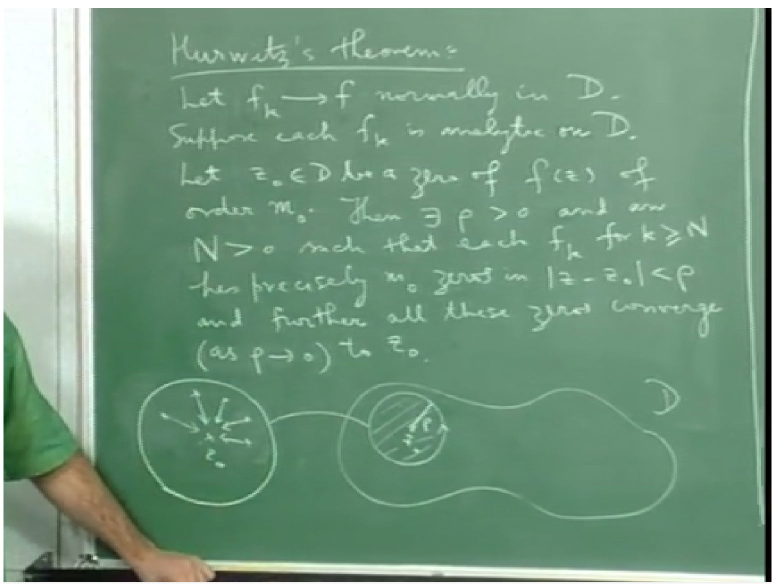

And try to look at a probably an application of Hurwitz's theorem , so let me again recall what Hurwitz's theorem says I am in shorter it says that if you have a sequence of analytic function that converge to the function normally in a domain. Then 0 of the limit function is obtained by I obtained as an accumulation point of the zeros of the functions in the sequence beyond certain stage okay.

But if you want it more precisely this is the statement, if fk is th sequence of analytic functions that converges to the function f normally in D mind you normally means that the convergence is uniform not on the whole of D by this uniform on compact subsets of D. and suppose of course suppose each of case is analytic and if z0 is a 0 of the limit function f, then there is a small disc surrounding z0.

Such that beyond a certain stage N all the f sub k for k greater than or equal to N they all have the same number of zeros counted with multiplicity in this disc as the multiplicity of z0 of the 0 z0 namely the order of the 0 z0 of f and as you make rho smaller all these zeros of the various fk's they converge to z0 okay, so this is Hurwitz's theorem. So, I mean I for example you may ask what is the use of such a theorem okay.

So, I will tell you 1 use, 1 application of this theorem just in words, this Hurwitz's theorem you can prove that if each fk is 11 1 to 1 as a map, then the limit function f if it is not constant it is also going to be 1 to 1 okay. So, you see normal convergence is already something very strong because it is actually uniform convergence on compact subsets and you know uniform convergence is a very powerful thing okay, you know we just proved in the last lecture that since if fk is analytic f is also analytic.

That was because of the uniform convergence on compact subsets due to Morera's theorem an application of Morera's theorem. But what you get is also is that if each fk is 1 to 1 then you get f is also 1 to 1 okay, that is an application for example. Of course you assumed f the limit functional non constant. So, if you have a sequence of 1 to 1 analytic functions injective analytic functions, analytic functions are injective as maps okay, function is do not take 2 different points of the same value.

Then the same property of injectivity is passed on to the limit function, provided the limit function is not there not a constant function okay, that is an application of this okay. And you know why this is important it is important in study of classes of analytic functions. For example people often study the class of functions analytic on the open unit disc and they also normalize it by putting conditions like function has the value 0 at the origin.

And it is derivative at the origin is 1 okay and then they study special classes of functions which are for example 1 to 1 on the unit disc. And this leads to lot of geometry and all those for examples in all these under such conditions you can show that if you have a class of functions to the certain property then any sequence of such functions which converges normally to a limit function.

Then the limit function also has a same property, so it again belongs to that class and that is true for the example for the class of all univalent functions that is 1 to 1 functions on defined on the unit disc which are normalized to taking the value 0 at the origin and whose derivative at 0 is 1 that is for example an application of this okay, so it is very important for function at the purposes .

So, what I am going to do now is you know try to give the proof of this theorem, so you know so you see the proof of the theorem is basically mind you it is of course you know the general theme is we are looking of zeros of analytic functions, this is the general theme which you are now discussing about and you already know how to count the number of zeros of analytic function inside a closed curve okay.

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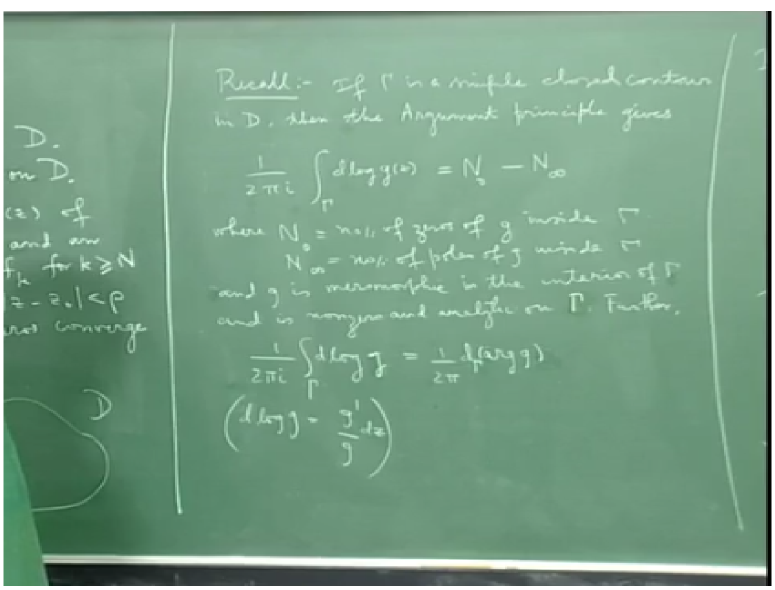

That is precisely given by the residue theorem in fact more precisely by the argument principle okay. So, you know. So, let me so if you recall if gamma is a simple closed contour in D, then 1 by then 1 then the so called the argument principle gives 1 by 2 pie i integral over gamma d log g z dz or d log g is number of zeros-number of poles okay where N is of 0 is number of zeros of g N infinity inside gamma when I say inside gamma, it is in the region enclosed by gamma.

And gamma is given and what do I mean by region enclosed by gamma it is region that lies to the left that lies to your left, if you walk alone gamma in the prescribed direction mind you whenever we write integral over gamma I know you have write a path integral already there is an orientation okay, usually we take the anti-clock wise orientation. So, that the region inside gamma is actually the region that enclosed by gamma okay.

And N infinity is number of poles of g inside gamma and where of course g is geomorphic in the interior of gamma and is a non 0 and analytic on gamma on the on gamma okay, this is what you will get, this is the argument principle okay. So, the logarithmic derivative if you take the integral and divide by 2 pie i you will get the number of zeros-number of poles.

And of course mind you you can also recall that 1 by 2 pie i integral over so further 1 by 2 pie i integral over gamma d log gz d log g is actually 1 by 2 pie into the change in the argument of gz change in the argument of g over gamma okay. So, the so this is another interpretation which is actually the reason why it is called argument principle.

If you compute this integral what is get is 1 by 2 pie into the change in the argument and you know since gamma is a closed curve the change in argument will always be a multiple of 2 pie, so dividing by 2 pie will give you an integer and what is that integer, the integer is exactly number of zeros-number of poles that is exactly the argument principle says okay, so this is the reason why it is argument principle okay.

Now what is and of course you know how do you get this is basically from the residue theorem, we apply the residue theorem to the logarithmic derivative g which is the function g dash of z by gz. So, this so d log g is actually is g dash by g, this is what it is okay. So, maybe I should also add a dz will be subsets, d log g is g dash by g dz okay or in other words d by dz log g is g dash by g okay .

So, fine so now why this is so important because suppose you are function is not melomorphic but it is actually analytic then there is no poles. So, what you get is you get the number of zeros okay, so it helps you to count number of zeros inside a closed curve okay, that is why this is important and we are worried about numbers of zeros okay, you want to count numbers of zeros.

So, this is the starting point and you see what this will tell you is that you can now apply this to calculate what is the number of zeros of of the limit function f and what is the number of zeros of each fk.

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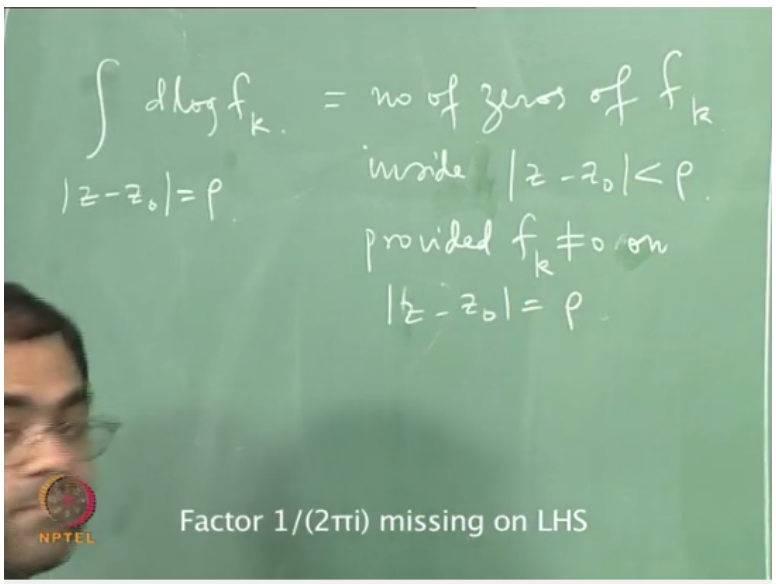

So, you know you will get that you know integral over mod z-z0=rho, so you know I am not told you what this rho is you but take a disc centered at z0 okay mind you z0 is a point of D where f is a 0 the limit function f has a 0 and the 0 **0** is a border m0 okay that is how the function. So take this point z0 and take a circle centered at z0 radius rho.

And orient this circle in the anti-clock wise sense. So that the region inside the is actually the region inside the circle as we could normally think of it and what you do is over this is if you integrate d log fk what you will get is number of zeros of fk inside the circle mod **z** z-z0 less than rho. Of course the assumption is that fk should not vanish on the boundary okay provided fk not equal to 0 on the boundary mod z-z0 okay. Because you know to apply the argument principle the function should not vanish on the boundary alright.

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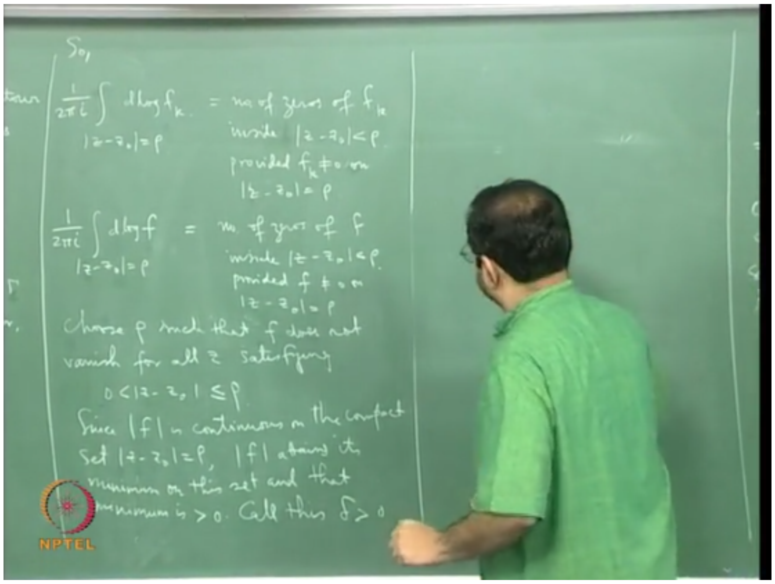

And I can write a similar statement for the for f what if I integrate mod value z-z0 it over equal to rho, of course when I write this it is with this positive orientation anti-clock wise orientation. If I integrate d log I think I forgot to put a 1 by 2 pie i okay, so let me do that yeah it is very common to forget a 1 by 2 pie or 1 by i or i or something like that, so you have to keep track of this.

So, d log f of course I am saying it is common but I am not saying it is correct, so you should not be done on purpose, so this the number of zeros of f inside mod z-z0 strictly less than rho provided f is mod0 on boundary mod-mod of z-z0=rho okay, this is what you have. Now let us go back and so I will have to tell you what this rho is I have to find a rho such that mod z-z0=rho on that circle fk should not vanish, f should not vanish I need a rho like that.

Now the point is see here is where a technicality comes in because fk converges to f normally and since each fk is analytic f is analytic that is what we have already seen. Now since f is analytic you know the zeros of an analytic function are isolated okay. So, this z0 is an isolated 0 okay, it is an isolated 0 that means you can find a disc surrounding z0 where there is no other 0 of f okay, this is where technically I am using the fact that f is analytic.

And the zeros of an analytic function are isolated okay and you know if you go back to your first quotient complex analysis that is actually due to what is called the identity theorem, the identity theorem says that if an analytic function has is has 0 at a set at every point of the set which has an accumulation point okay and the accumulation point is in the domain of analyticity.

Then the analytic function has to be throughout 0 it has to be identically 0 okay, in other words if you have a sequence of if you have a set of zeros of an analytic function which converges to a point where the analytic function is defined and analytic then obviously that point is also be a 0 because of continuity and therefore for that point every neighborhood of that point will contain a 0, that 0 will not be isolated.

The limit point which is a 0 will be such that every nice neighborhood of that limit point how was a disc you take there is some other 0 and that cannot happen for an analytic function unless it is completely 0 identically 0 okay. So, of course you know in all these things I am not assuming that the function f is is a constant function because of course you know if you assume f is a constant function then that means and if you assume f is 0 of order m0 at z0.

Then you know it really does not make sense because if it is it will tell you that f is 0 at z0 and then it will also tell you that since it is constant it is 0 everywhere. So, I am certainly not looking at the case where f is constant okay, so f is certainly not constant, so the zeros are isolated okay and I am using this fact. Now because zeros are isolated I can find a rho such that f does not vanish not only in the interior but also on the boundary circle.

So, that is how I choose my rho okay, so choose rho such that f does not vanish for all z satisfying 0 strictly less than mod z-z0 less than or equal to rho okay. So, this is the fact that means I have put 0 strictly less than because I do not want to include z=z0 because I have z=z0 it does vanish, it is a 0, z0 is a 0 of f . But I do not want apart from the center of the disc I do not want any zeros for f, I do not want any zeros even on the boundary.

And such a disc I can get that is because the limit function f is analytic and the zeros of analytic function were isolated okay, fine so this is the one part of the story. Then the other point of the story is of course you know I need to also worry see the movement I assume this, this integral is well defined because for this integral to be well defined mind you d log f is just f dash by f dz and I am dividing by f.

And I do not and I am integrating on the circle okay, so what I am dividing by should not be 0 on the circle this is true okay. So, this integral is defined and I have to worry about why this integral is defined okay, now the answer to that is that the following okay, so f does not vanish on this boundary okay. That means that mind you if you look at mod f, mod f is a continuous function okay.

And this boundary is a compact set you know a continuous real valued function on a compact set is uniformly continuous and attains it is maximum and minimum value. So, if you look at mod f, mod f will attain a minimum value and a maximum value on this compact set which is the circle. And since it is never 0 the minimum value cannot be 0 okay, so that is exactly what I want I saw, I just want to say that the f is bounded away from 0 on the boundary circle.

So, let me write that since mod f is continuous on the compact set mod z-z0=rho, mod f attains it is maximum, it is minimum on this set and that minimum is as positive okay, see a continuous so this is a a you know want it is a theorem from real analysis, first quotient real analysis it is the if you want in the simplest form you take a real valued function, a continuous real valued function of real variable.

If you take the image of the closed interval, the image will again be a closed interval topologically what you are saying is that since first of all since the you are taking the image of a interval and which is connected and since the function is continuous the image will also be a connected subset. So, that means the image will also be an interval and since you are started with a compact set, because an closed interval is both closed and bounded, so it is compact.

And the continuous image of a compact set is compact, so what you will get is a image of a closed interval is again an interval which is compact which means it is an interval which is closed and bounded which means it has to be a closed interval. So, what it means that if you take mod f if you restricted to this compact set mind you the circle is both closed and bounded as a subset of R2 okay.

If you want you can re-parameterize it as a interval on the real line after all you have to just take the parameterization z=z0+rho e power I theta where theta lies from 0 to pie okay. So, then you will see that mod f will the image of this circle under mod f will be a closed interval okay and that closed interval will have a minimum and maximum value left hand point and the right point, the left hand point is this minimum and that will be positive.

Because it is a value taking by f on the boundary and f does not vanish on the boundary that is our assumption it **it** vanishes only at the center okay. So, call this minimum delta call this value as delta okay. So, what you get is the following you get that, so what you get is.

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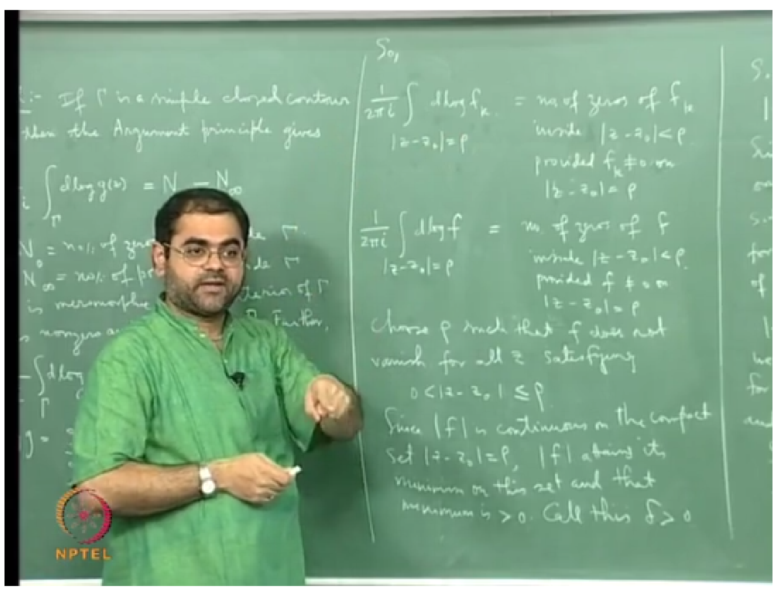

So, for mod z-z0 strictly less than rho less than I am sorry equal to rho mod f of z is greater than or equal to delta okay, yes. So, you see so you have this so I was thinking for moment because I have to justify why this integral make sense, so you see the fact is that this integral will make sense for k sufficiently large okay . So, let me come back to this, so you have this right.

Now you see since fk tend is converges to f okay fk converges to f and it is uniformly on mod z-z0=rho, in fact less than or equal rho. Because it is the convergence I have told you is normal that means the convergence is uniform and compact subsets, so in fact it converges uniformly on the closure of the discs that is the open disc along with the boundary circle. So, since is a uniform convergence okay what will happen there exist an N there exist a N1 .

Such that k greater than or equal to N1 implies that the distance between fk of z and f of z can be made I should say less than epsilon for a given epsilon greater than for any given epsilon greater than 0 independent of z of z on independent of z okay, this is what uniform convergence says, uniform convergence means that the value of if you take any point z with lies in the domain the in the set you have considering**.**

In this case is the closed disc namely a point z such satisfying mod z-z0 less than or equal to rho then mod fk of z-fz the distance which is the distance between the f of z and fk of z that can be made lesser than epsilon okay, you are just saying that fk comes to within a distance of epsilon from f, if k is sufficiently large and this does not depend on what z you choose and this non-dependence of z is the uniform, so the convergence okay.

If it is non-uniform then you know this N would change along with z the fact that this N you are able to given an epsilon you are able to get an N1 which does not depend on z is the uniformness of the convergence okay. Now you see you know what this actually tells you see so in fact see in fact you see you also in fact since all since mod fk also converges to mod f okay, see if fk converges to f okay.

Then mod fk will converges mod f will converge to mod f, so what will tell you is that the values of the distance between mod fk and mod f can also be made as small as you want independent of what z is okay. We can make given epsilon greater than 0 we can make we can get mod fk of z-mod f z lesser than epsilon for k greater than or equal to N2 depending only on epsilon and not on z okay right.

So, what this you know it is a lot to write down but for the moment just do not look at all of this but just think of it like this fk converges to f, so mod fk converges to mod f and this is this convergence can be it is uniform but you see mod f is greater than or equal to delta on the boundary. So, what it will what this tells you is that since mod fk's the values of mod fk's approach mod f.

And since mod f is greater than or equal to delta you can make mod fk's greater than delta by 2 if you want okay, if you choose k sufficiently large that is what I want. So, there exist N3, so I am writing N1, N2, N3 so that you know you do not get confused so the exist N3 such that k greater than or equal to N3 implies mod of fk of z can be made greater than or equal to delta by 2 for all z such that mod z-z0=rho, this what I want.

And why I want this is just to tell you that you see this integral is also well defined, see what is this integral this is 1 by 2 pie i integral over this circle of fk prime over fk dz is the logarithmic derivative of and this logarithmic derivative always has the function in the denominator okay, it is a derivative of the function divided by the function. And therefore for it to make sense that function cannot vanish, so I need to make sense I need to make sure to make sense of this integral I need to make sure fk does not vanish.

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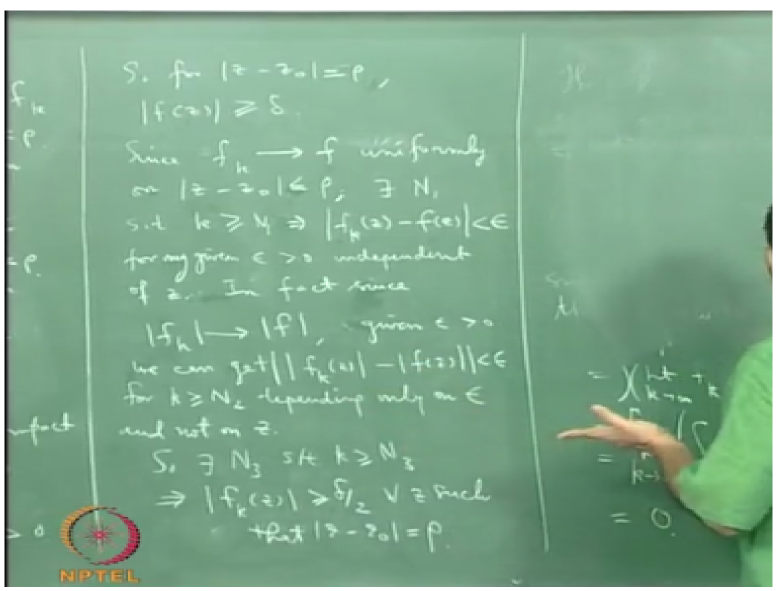

And how do I ensure by fk does not vanish by ensuring that fk is greater than or equal to some positive quantity on the boundary and for that I am saying I can choose a case of simply large and that is comes out of this, that is all I want. So, the moral of the story is that therefore you see for the rho I started with not only is this integral well defined also this integral is well defined okay and therefore both integrals make sense okay.

And mind you this integral the what is the value of this integral, the value of this integral is now be m0 because the value of this integral is equal to the number of zeros of f in this circle inside this circle. But I have chosen the disc in such a way there no other zeros because of the isolation of zeros, so the only 0 is at the center at z0 and that 0 is of order m. so, you have to come mind you whenever you count zeros or poles is always count them with multiplicities.

So, this integral will be m0 okay and you know what I have to actually prove is that as k tends to infinity this also tends to m0 that is essentially what I have to prove okay. So, that should more or less tell you how the proof is going to end.

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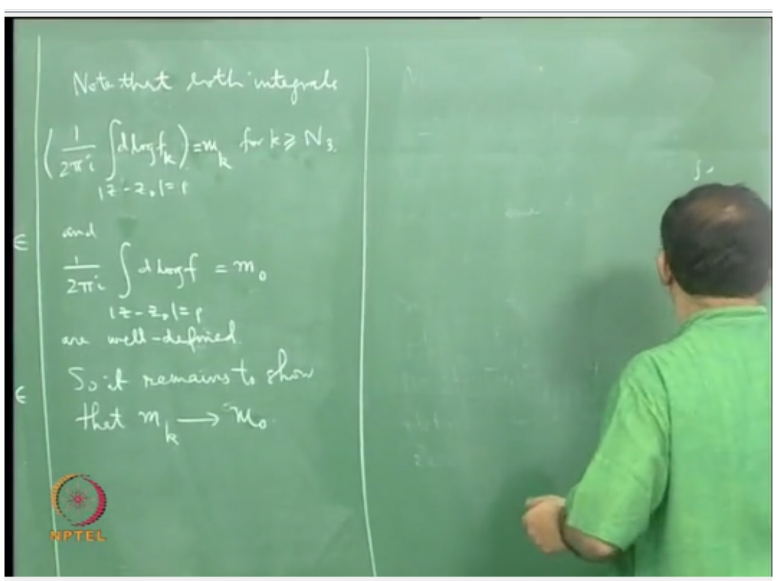

Note that both integrals 1 by 2 pie i integral over mod z-z0=rho d log fk for k greater than or equal to I think did I use N3 yeah N3 and 1 and you know , so let me call this as let me call this as mk okay. And 1 by 2 pie i integral over mod z-z0=rho d log f=m0 are well defined okay you see now what I want to tell you now you know what I have to prove I am claiming that I just have to prove that.

You see I get this sequence mk okay, mk is what mk is the number of zeros of fk inside that circle inside that disc okay. And you know you can expect that since fk converges to f this integral will converge to this, so you can expect that mk converges to m0 okay, that is natural to expect and that is exactly what happens mk converges to m0 but then mk converges to m0 means that beyond a such certain stage mk is exactly m0, see if you have a sequence of integers which converges to an integer.

Then the sequence of integers must beyond a certain stage equal to constant sequence okay, see when you say a sequence converges to a value what it means is that beyond a certain stage the values of the sequence come very close to the given value. If you now if it is a sequence of integers which converges to an integer what you are saying is beyond a certain stage all the integers are very close to this integer.

But 1 an integer very close to be the integer has to be the same as that integer okay, so if you just prove that mk tends to m0 then you are done what it will tell you is that maybe you will have to choose the bigger value of N such that for k greater than or equal to N, mk converges to I mean if you prove mk converges m0 then you can choose a index N such that for k greater than or equal to N, mk is actually equal to m0 okay, what will that tell you, that will tell you that fk has as many m0 zeros in that disc which is the assertion of Hurwitz's theorem okay.

That is part of the assertion of Hurwitz's theorem then what you must understand is whatever you have done so far will not only work for rho but I can replace rho by half rho then I can replace it by one third rho, then I can replace it by one fourth rho, so you know that means I can shrink this disc smaller and smaller and smaller and everything works the same argument works, so what it will tell you is that for that value of N beyond which mk's are equal to m0 all the zeros of fk okay as I shrink rho they are going to come closer and closer and closer to z0.

So, they are all going to converge to z0, z0 will be an accumulation point and that is exactly what Hurwitz's theorem says all the an 0 of order m of the limit function comes from zeros of order m they comes from m zeros of the functions in the limit beyond a certain stage and these m zeros they actually coreless together to give you the 0 order m in when you take the limit that is exactly what Hurwitz's theorem says okay.

So, I will do that so it remains to show that mk tends to m0 okay this is what I will have to show and the answer is to that is again as you will as you can expect it is again uniform convergence okay. So, the answer to that is again uniform convergence and how does 1 prove to it, it is pretty easy you see you know basically I am trying to show that this converge to this, so which means I will have to first of all look at the you know the integrant. I have to show that this-this is eventually 0 okay which means I will have to show that the integrant-this integrant is essentially 0 okay. Now so you see now I am going to make you such a following thing.

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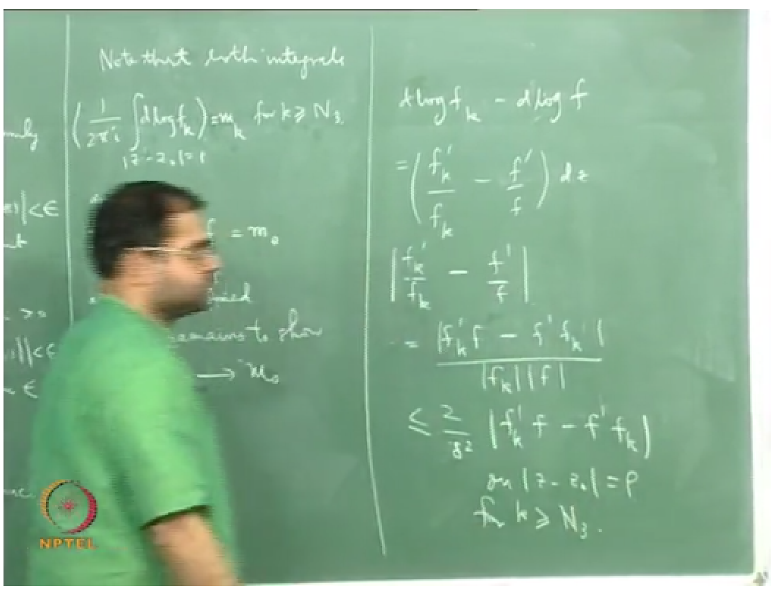

If you look at d log fk-d log f this is just fk dash by fk-f dash by f dz this is what it is okay and if you look at the modulus of this quantity modulus of fk dash by fk-f dash f, if I look at the modulus of this quantity okay, mind you when I am writing such an expression I am assuming that whatever values of z I plug in fk and f cannot be 0, so for example this makes sense on this boundary circle mod z- z0=rho because I have chosen it like that okay.

And of course I have chosen k, k has to chosen sufficiently large namely k has to be greater than or equal to N3 okay. So, if you look at this what you will get this well you will get fk prime this is simple calculation the kind of calculation that if you should have been already use to in course in first course in any first course in analysis. And you know well you see this is less than or equal to you see mod fk mod f is greater than or equal to delta.

So, 1 by mod f is less than or equal to 1 by delta mod fk is greater than or equal to delta by 2, so 1 by mod fk is greater than or equal to 2 by delta. So, this less than or equal to 2 by delta square okay into this quantity modulus of fk prime f-f prime fk and you know how to handle this okay, it is a very simple trick that you always use to for example prove you know very simply how you prove product rule for differentiation, that kind of trick very simple trick.

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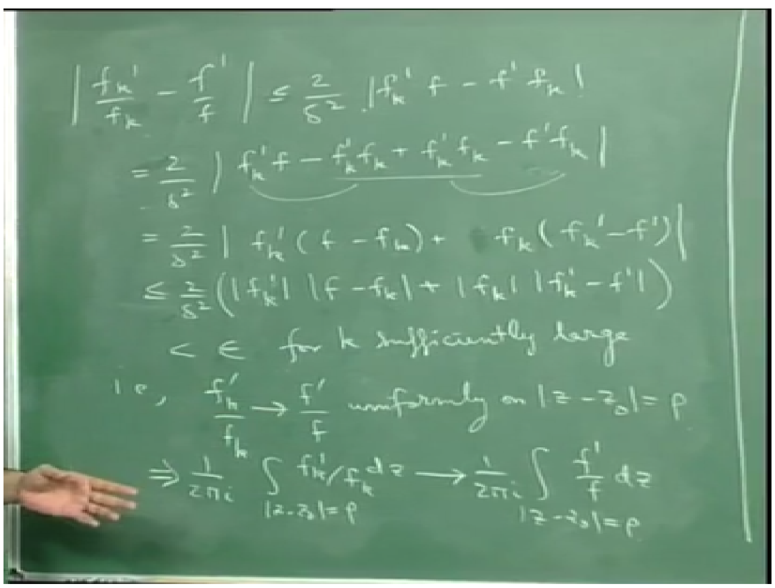

So, let me write this here mod of fk prime-by fk-f prime by f is **is** less than or equal to 2 by delta square so of course I should write here on mod z-z0=rho for k greater than or equal to N3 okay. So, if I continue that here, so let me rewrite what I have written there I get if k prime, f-f prime k and of course you know the trick is this is 2 by delta square you add and subtract an obvious term fk prime f- fk prime fk+fk prime fk-f prime fk.

So, you see I am adding and subtracting this term okay and then I group these 2 and these 2, so this becomes 2 by delta square modulus of k prime*f-fk+modulus of fk sorry fk* fk prime- f prime I get this and now this is less than or equal to 2 by delta square mod fk prime into mod of f-fk this is by an triangle inequality mod fk, mod of fk prime-f prime this is what I am get okay.

And now what I want to say is that this can be made less than epsilon for k sufficiently large okay and why is so that is because you see mod fk prime see fk converges to f okay and the convergence is uniform. Therefore fk prime will converge to f prime because under uniform convergence the derivative the if a sequence of functions converges uniformly to a given function okay.

And if of course the you can actually check again by if you want use of Morera's theorem that fk prime will also converge to f prime you see fk's are all already analytic okay we and since fk it since the fk's converge to f normally you prove that f is analytic but then you see f is analytic means f is infinitely differentiable and each fk is analytic, so it infinitely differentiable what is the relationship is between the derivatives.

The derivatives of the fk's namely the fk primes they will converge to f prime okay this will this can again be check by a Morera theorem kind of argument if you want okay. Basically it is uniform convergence, so fk prime converges to f prime, so this bounded you know convergent sequence is bounded and since the convergence is uniform, this uniformly bounded, so I can simply make this less than some constant, the uniformness I am using.

Because I do not care what the z is so long as z lives on that boundary circle okay, similarly I can make this less than or equal to some constant because of uniform boundedness okay and this also can be made as small as I want this I can make as small as I want. Because this is also uniform convergence, this is also uniform convergence, see the fact is if fk converges normally to f then the derivative of fk is converges normally to derivative of f.

The second derivatives of the fk converge to normally to the second derivatives of f and so on and so forth this will happen this will go on and on and on. So, because of the convergence I can make this as small into some constant and this again something small into some constant and if I choose the index large enough I can make this whole quantity very small I can make the since this is already a constant I can make the whole thing less than epsilon k sufficiently large.

So, what this tells you is that fk prime over fk converges to f prime by f uniformly, so that is fk prime by fk converges to f prime by f uniformly on the boundary okay this converges is uniform alright but then you know if you have uniform convergence if a sequence of functions converges uniformly to a function then the sequence of integrals will also converge, because uniform convergence allows you to interchange limit and integral.

So, this will tell you that 1 by 2 pie i integral over mod z-z0=rho of fk prime over fk dz will converge to 1 by 2 pie i integral over mod z-z0=rho f dash by f dz that is because if a sequence of functions converges uniformly to a limit function then the sequence of integrals over a path will also converge to the integral of the limit, the integral and limit can be interchanged, so that is why you get this. But this is exactly the statement that mk converges to m0 okay, so **so** that tells you that mk converges to m0 that means mk is actually equal to m0 be for k is sufficiently large okay.

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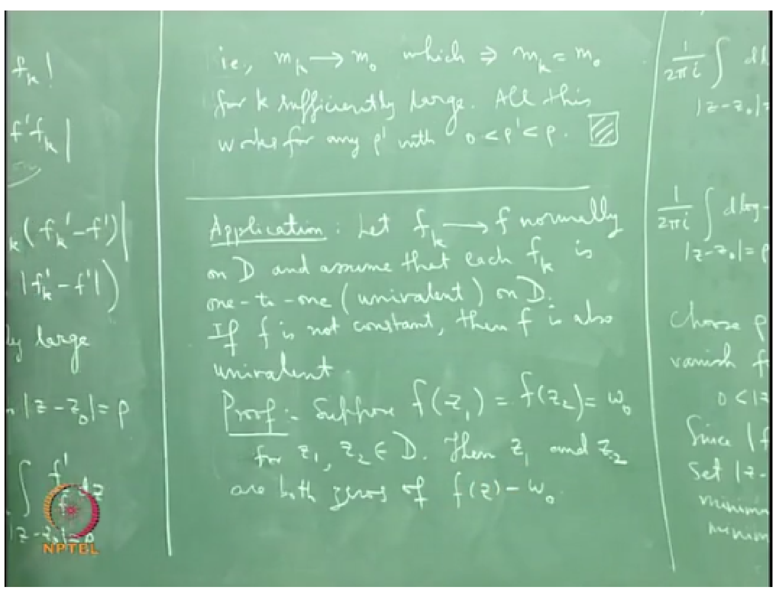

That is mk converges to m0 which implies mk=m0 for k sufficiently large okay, so that finishes the proof of theorem, so yeah I so let me add all this works for any rho prime with 0 lesser than greater than rho prime less than rho and that is the end of the proof of the theorem okay, that is the end of the proof of the theorem. So, let me explain and quickly an application of this theorem.

So, here is an application, so that we will appreciate this power of this theorem application let fk converge f normally on D and assume that each fk is 1 to1. So there is a very special name for such function they called univalent functions univalent on d okay. Then if f is not constant then f is also univalent that is an application a normal limit of univalent functions is univalent so long as the limit is not constant that is a beautiful application of Hurwitz's theorem.

And how does 1 prove this I mean you have to show f is not constant you have to show f is univalent, you have to show f is injective, so you have to show that if f takes 2 values to the same value then those 2 values of the same that is what you will have to show. And how does 1 prove it, it just 2 lines suppose f of z1=f of z2 =omega0 for z1 and z2 in D, so I should think in the following way saying that f of z1=w0 should be thought of you have to think in terms of zeros.

So, you should think of z1 as 0 of f of z-w0 and you must also think as z2as another 0 of f of z-w0. So, then z1 and z2 are both zeros of f of z-w0 right and but then what does Hurwitz's theorem tell you, Hurwitz's theorem tells you that there is a sequence of zeros of fk of z-w0 which converges to z1 and there is another sequence of zeros of fk of z-w0 which converges to z2 okay.

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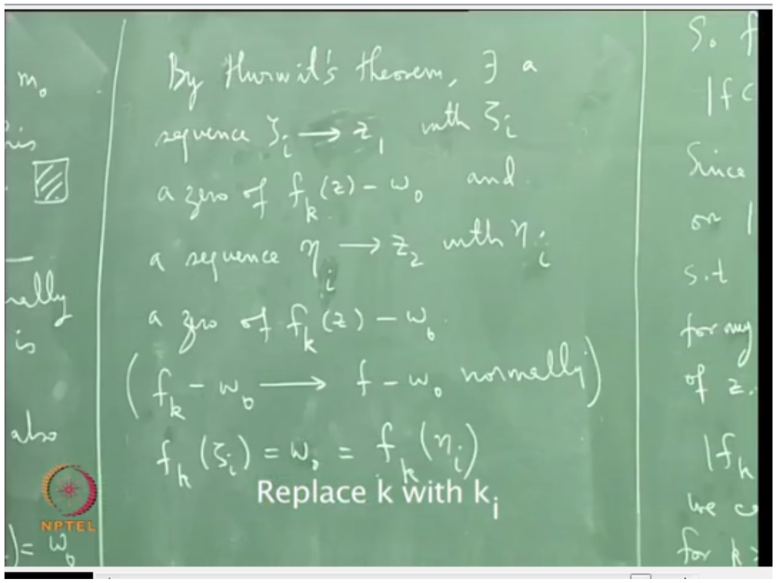

But that should tell you that z1 has to be equal to z2 by Hurwitz's theorem there exist a sequence so let me give it some name let me call this as zeta i of zeros zeta i tending to z1 with zeta i a 0 of fk of z-w0 and a sequence eta i tending to z2 with eta let me call it as neta j if neta j a 0 of fk of z-w0, this is what Hurwitz's theorem says**.**

Because you know fk-w0 will tend to will converge normally to fk f-w0, fk-w0 tend converges to f-w0 normally that is a reason okay and you are applying the Hurwitz's theorem to this sequence fk-w0 okay. So, you see I have to say that limit of the zeta i's is a same as limit of the eta j's okay and why is that obvious that is because the fk's are univalent, see that is where I have to use the fact that fk's are univalent, see fk.

So, maybe I will use the same index okay I will use the same index because this zeta i and neta i are zeros of fk's okay. So, well what I will get is basically fk of zeta i is 0 is fk of zeta i is zeta i is a 0 of fk of z-w0 means fk of zeta is w0 okay and that will also equal to fk of eta because eta i is also a 0 of fk-w0.

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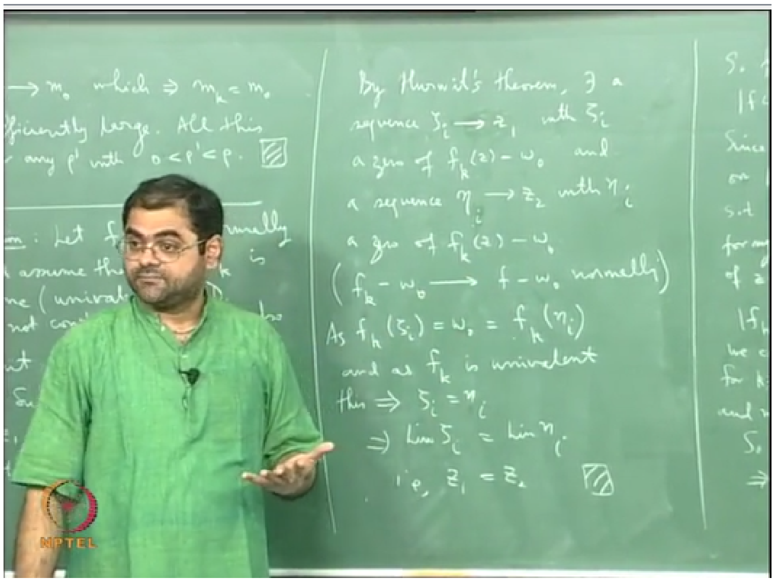

But then this is true and as fk is univalent okay mind you I am taking sufficiently large, since fk is univalent okay what this will tell you this implies that zeta i=eta i. Now if you take the limit as i tends to infinity on the limit of zeta is z1, the limit of the neta i is z2. So, you get z1equal to z2 okay, so this implies limit eta i=limit eta i that is z1=z2, so that tells you that if f of z1=f of z2, then z1=z2 that is the proof that f is analytic is univalent and that is a proof okay.

So, if you have sequence of univalent functions that converge normally to a limit function and the limit function is not constant then the limit function is also univalent okay, so that is know and of course you know in all these things where I have use the non constant is because Hurwitz's theorem does not apply when the limit function the limit function is constant okay.

So, Hurwitz's theorem does not apply when the limit function is constant and here the limit function is f-w0, if f-w0 is constant that means is the same as saying f is constant and that is not allowed because I have already assumed f is not constant okay. So, I can apply Hurwitz's theorem because f is not constant okay alright, so we will stop here.