Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-38
Arzela-Ascoli Theorem_Under Uniform Boundedness, Equicontinuity and Uniform Sequential Compactness are Equivalent

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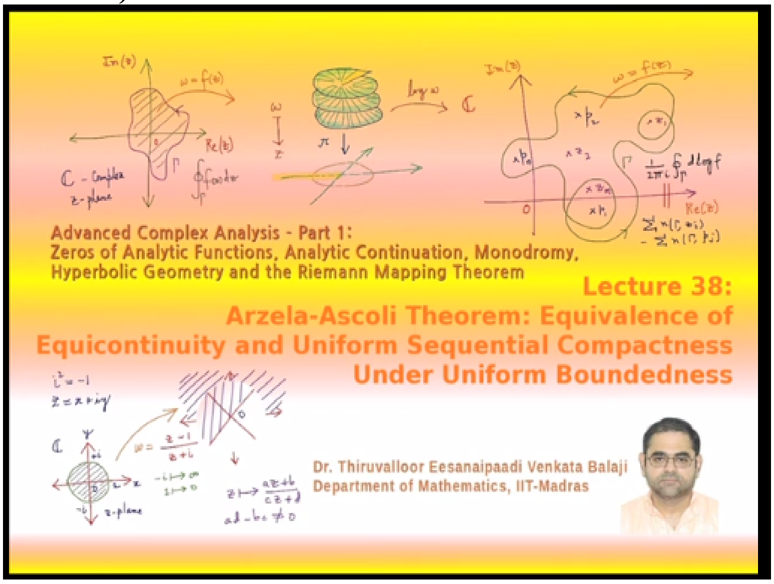

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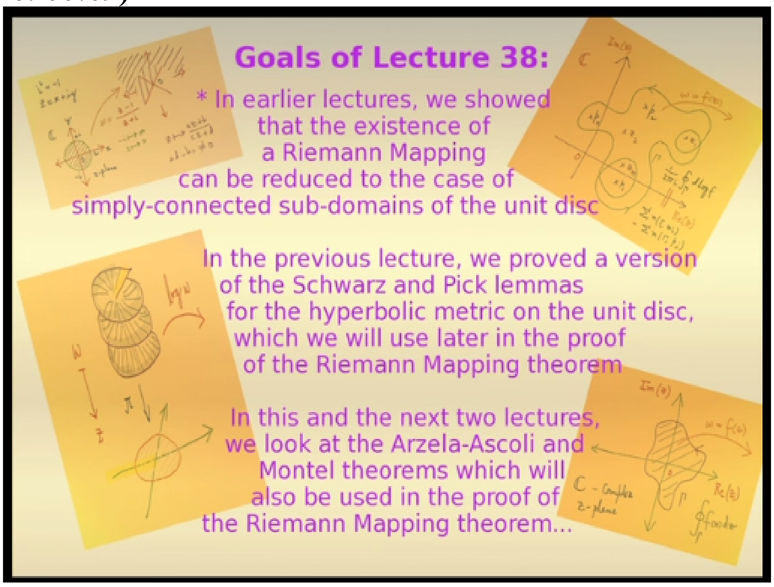

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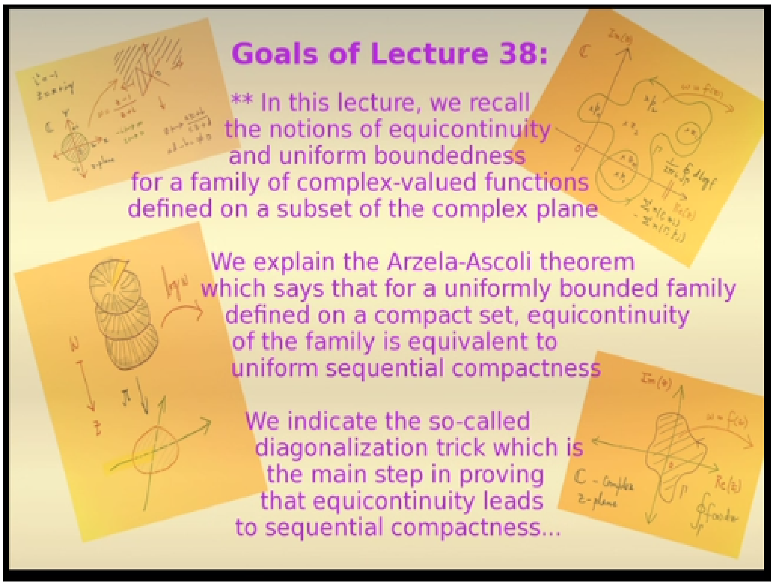

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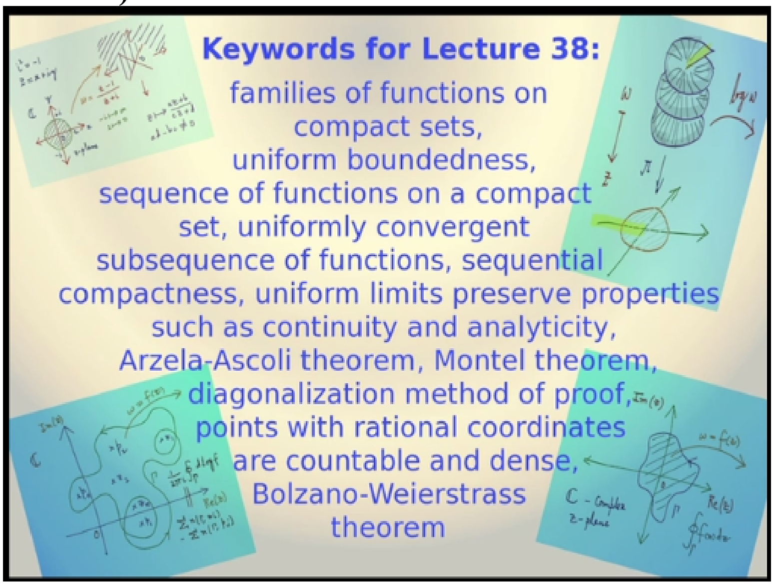

Ok so so the context is the proof of the Riemann mapping theorem ok and what you done is in the last we have looked at hyperbolic geometry ok. Now there is one more technical deto that we have to take to be able to complete the proof of Riemann mapping theorem and that is that so called complex version of Arzela Ascoli theorem and the so called Montel theorem ok.

So that is what I am going to discuss alright. So basically the Arzela Ascoli theorem and the Montel theorem, they are all theorem which you know guarantee that you are given a family of functions ok on a compact domain ok that any sequence in that family has a uniformly convergent subsequence ok. So **so** that so let me tell you the general idea, general ideas is you see I have some a family function ok a family of functions defined on a domain alright.

And let us assume that the family is defined on I mean for example if you are thinking of the simplest case of real valued function ok then you think when you assume that is a real valued functions are all define on a closed bounded interval ok which is the compact subset connected subset of real value alright or more generally if you are thing of function on the plane ok.

Then you think of functions which are define on a domain on the plane ok and in fact you assume at lease to begin to assume that you with all they also extend continuously to the boundary of the domain that the domain is bounded and you know the domain is bounded and then you add the boundary which is compact ok. So you have family of complex function complex valued functions defined on the domain.

Of course all functions here interesting setting continuous ok, then the question is if you take a sequence of functions from this family ok to expect that the sequence will converge any sequence of functions will converge too much ok, we expect that any given sequence of functions will converge is too much ok, but what you can always expect is that exactly is a subsequence which converges.

So the idea of the Arzela Ascoli theorem and Montel theorem is there under good conditions ok, you can always ensure that you give me any sequence of functions in a family satisfying of course that on a compact set ok, it will always have a subsequence which converges uniformly ok. So the general point is that you want you have family functions ok and the domain effectively where you are studying is this compact alright or in the or if you are looking at for example analytic functions.

The property that are you look at the analytic functions on close disc in your domain which are a compact closed and bounded discs ok and the result that you want to set you want to you want the given any sequence from this family there is a subsequence which converges uniformly ok. Now why is this so important this is important because you see the moment you say there is a subsequence which converges.

It tells you that at least that is the limit function for subsequent though the whole the given sequence of functions need not converge, but at least a subsequence converges, but the fact that the converges uniform they will see all the properties of the functions will also carry over to the limit. For example if you are looking at a uniform convergence and uniform limit of analytic functions uniform limit of continuous functions continues because of this.

So you just do not want you know just convergence of functions that does not help because if you just a ordinary convergence of functions you can go wrong in the sense that the limit function may not have the good properties original functions you can have a sequence of continuous functions which converts to limit functions is not contagious you may have the limit function which continuity ok.

You are not such things happen so that is when you are studying convergent the functions is natural that you at least demand uniform convergence. So that all good properties pass on to the limiting function ok. So the both the Arzela Ascoli theorem and Montel theorem they are basically situations which guarantee that you can always find a subsequence of uniformly convergent functions ok subsequence converges uniform the sequence of function that converges uniform ok, that is generally idea. Now let us get the technicalities.

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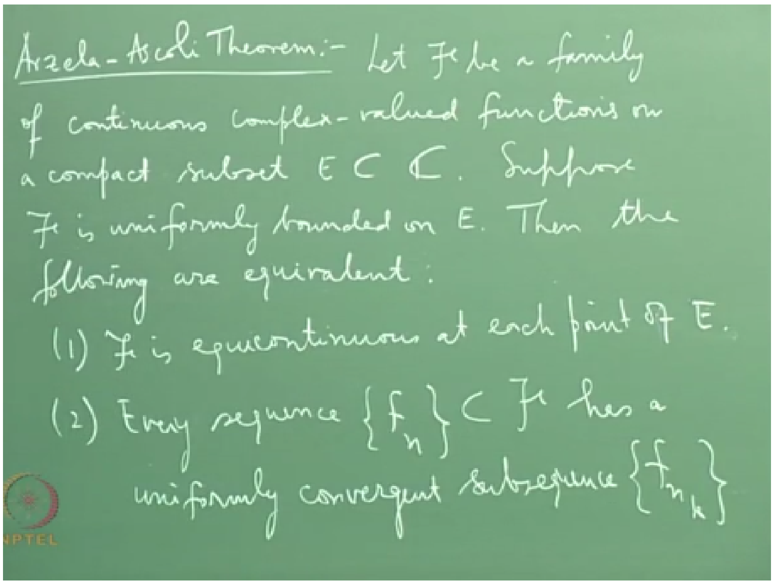

So you know so here is a so here is an situation like let F the function a family or collection so let me do the following thing let me put the title as Arzela Ascoli and Montel theorems. So these are very technical theorem but they are basically easy to prove and they are very powerful ok. So let F be the family of collection of function define on and continues and complex valued on a compact subsets D into complex plane ok.

So I have a compact subset of complex plane in which both close and bounded and I have collection of function defined on points of this capital E taking values in complex numbers that we meet I am assuming that all continues ok and they are complex plane aright, now you we all know what the definition of continuity at a point, the definition of continuity at a point is that you know given epsilon ok .

The function values at the point the function value enable the point can be drop with an Epsilon which is epsilon distance for the function value at that point. If you choose points a delta neighbourhood of the given point that is the continuity of the given point and so let me just write that down recall that f the family is continuous at point the z if given epsilon=0 or x is delta greater than zero sets that whenever the distance between z and z0 is less than delta.

And course z is point at e then distance between Fz and fz0 can be less than 0. SO this is an ordinary Epsilon delta definition of continuity of the functions small f in this family at the point z0 ok. Now what you should notice is that you know this delta this delta depends on of course is delta depends on epsilon ok and this delta also depends on the point z0 and it also depends on the point f.

It also depends on the function f ok, so delta is actually delta of f, f,z0, epsilon, then if you change for you know if you keep the function f the same, if you keep the epsilon the same but if you change z0 the delta will change ok that is the dependence of delta on z0 ok and of course if you change epsilon of f also the delta will chose ok. So this delta depends on these 3 things alright.

Now you see suppose that you know you are able to find a delta that is independent of this f okj suppose you find delta that does not depend on f ok that means the same delta for the given epsilon the same delta will work for every f ok for a given epsilon and given z0. The same delta will work for a every f small if in the family script f, if that happens we say that the family is equicontinuous at the point z ok.

If delta is independent ok if a delta if a delta independent of f and depending only on z0 and epsilon can be found for every epsilon greater than zero we will say f the family f is equicontinuous at z0. So this is the most of equicontinuity ok. So for all the functions in the family you know you are saying that the function values can be made the function values near the function the function values at points near to the point Z0 can be made to within epsilon distance of a function value at z0.

If you choose a sufficiently small neighbourhood of the point z0, but the same neighbourhood works for all functions ok, it works uniformly for all functions right, so you know so the point is that you are able to find there is no dependence on the particular member of the smallest f of the family script f that is the whole point. Of course that is clear that if a family is if a family is if your family is equicontinuity at the point.

Then it should be continuous at that point ok, because equicontinuity is stronger than ordinary continuity alright and the about this equicontinuity is that you know this is one of the ingredients one of the hypothesis that in the context of the Arzela Ascoli theorem or the Montel's theorems, it will ensure that you know it always extracts as a sequence of functions which converges uniformly ok.

So this is equicontinuity and so that is one of the technical ingredients. So you know if you take any two points if you take any 2 points in this disc ok then the distance between them is certainly a 2 delta aright and the distance between the function values by a triangle inequality is less than 2 epsilon alright. So what this also tells you is that it tells you that it tells you that each f will be kind of uniformly continuous on each of this discs ok.

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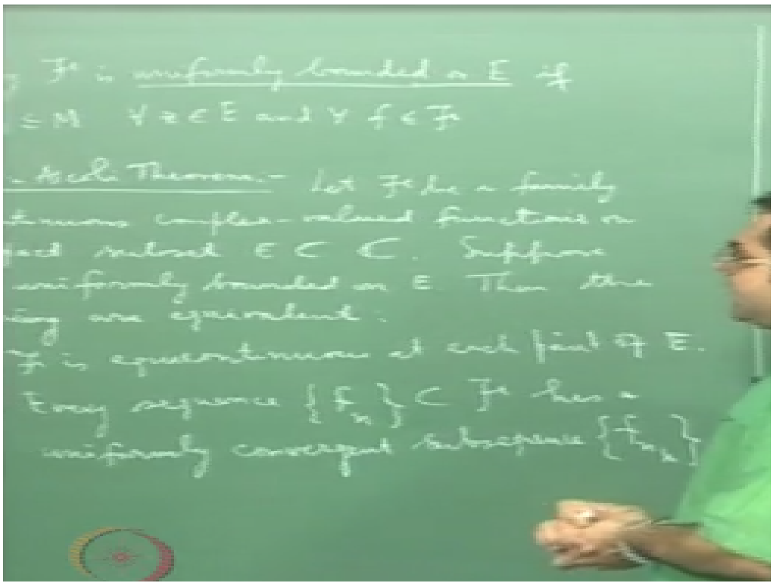

So but anyway see the other thing that one wants to worry about is so uniform boundedness, so we say script f is uniformly bounded on e if of course you know in all these in this argument I have not have still not use the compactness of the subsidy right, but I could have defined it for any subsidy of the complex plane right is definition makes sense for any subsidy of complex plane.

But the point is that the compactness is one of the ingredients for theorem ok. So of course in all these things I do not have assumed these compact but I am keeping e compact in view of these terms right. So so we say f is uniformly bounded on e if modfz is less than equal to m or all z in e and for all small f, so this is uniform bounded ok, so of course boundedness of a function composite function values means that is modulus bounded ok.

So for all values of the functions you take the modulus all this module i they do they are bounded above by some positive real number ok, you are able to find some positive real number m such fact that modz is always less than equal to m, so this m is a bound for f or modf ok and you want the same bound to work for every smallest f encrypted, if that happens you say the family is uniformly bounded on e ok.

So you have these 2 facts and now comes the now I can say this Arzela Ascoli theorem, so I am stating only one version of the theorem which is the version that we need but there are versions of the theorem for defined on compact whole work spaces with functions taking values in metric spaces and so on so for and they are very general questions ok.

But this is the version that period that is the version that I am going that I am define that is the version I am going to state and improve, so so here is with let script f the family of continuous complex valued functions on the compact subset e of the complex plane ok suppose f is uniformly bounded on e ok then the following are equivalent number 1 f is equicontinuous at each point of e.

Number 2 every sequence of f has uniformly convergent subsequence. So this is the version of the Arzela Ascoli theorem that you know ok. So you have so again let me explain you have this compact subset e in the complex plane, so the compactness is very very important alright and you have script f in the family of continuous functions defined on this complex set e and taking complex values alright.

And you put the condition that this family is uniformly bounded on e ok, so there is an there is a positive M which is an upper bound for the modulus of Fz for all ze and for all small f in script f ok. Then the Arzela Ascoli theorem actually tells you that the condition for being able to extract a uniformly convergent subsequence from any sequence in the family is equivalent just demanding that the family is equal continuous at every point of e ok.

So each pic for a family of when you have uniformly when you have uniform boundedness ok, then equicontinuity is equivalent to be able to extract a uniformly convergent subsequence ok, this is all you can state a elegant, if you are having functions defined on a compact set ok which are uniformly bounded, we have family of functions which are defined on a compact set and suppose a family is uniformly bounded.

Then what is the condition that is equivalent to being able to extract a uniformly convergent subsequence from any given sequence of functions the condition is simply the equicontinuity of the family at each point of the compact set ok. So this is the Arzela Ascoli theorem right and what I am going to do is and what I am going do is I am going to go I am going to next one to the Montel theorem which I need to which I need to use in the proof of Riemann mapping theorem.

But the Montel theorem is but the true the Montel theorem I need only the implication that one implies to I do not need the other part of the currency is 2 implies 1. So what I will do is I will just indicate how to one implies 2 and 2 implies 1 is a reasonably easy exercise ok and in fact even the proof that one implies two just parallel the proof that you would have seen in the real case in a first course in real analysis ok.

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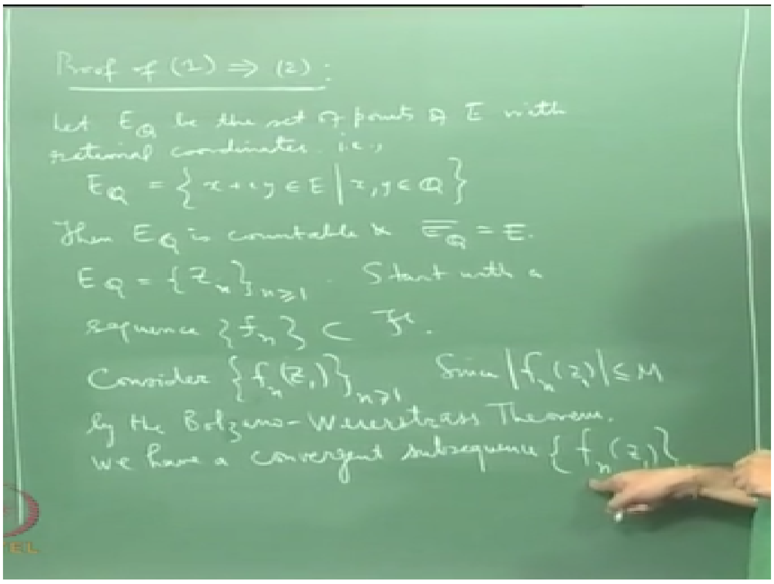

So **so** let me do that, so proof of 1 implies 2 so you see so you know this is a this is a standard technique of diagonalization that is used to prove this implication Arzela Ascoli theorem even the even for real value function defined on you know a close bounded interval on the real line ok the same proof we got ok. So how does I begin so what is given you are given a compact subset e in the complex plane.

You have given a family script f after a composite value functions on e and you are given that this family uniformly bounded so that is this constant m which bounce the modulus of the function values at each of e and for every function the family uniformly is the same constant works regardless the point and regardless of the function ok and what is given to me is it is equicontinuity is given to me ok.

So so what you do is you make for the fact you make use of the fact that if you know on the real line if you take the rational numbers take the points a rational that countable and the distance ok. So similarly if you take the plane which is it ok to ok if you take all the points with rational coordinates then that is accountable ok, and it is also dense ok, so this existence of a countable dense subset is what is used ok.

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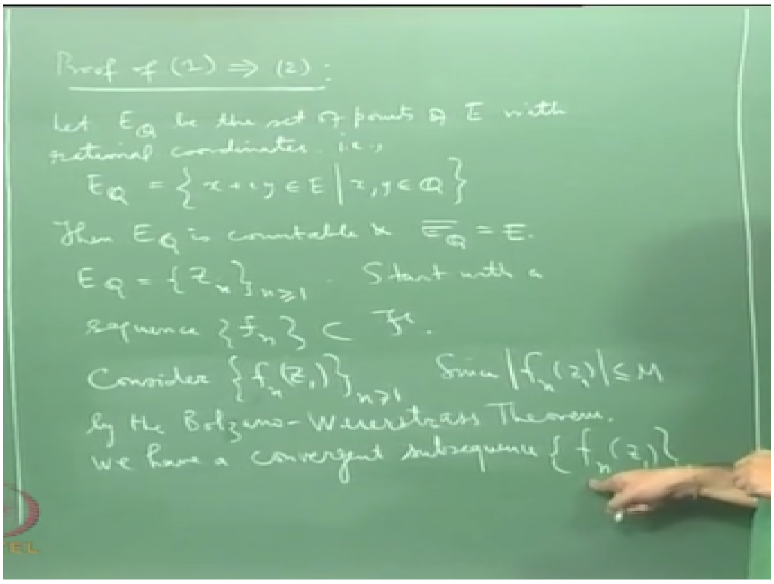

So what we do the you do the following thing like E subQ D the set of points of D with rational points ok, so ah of course you know here in other words I am looking at points that complex plane as points on points of r2 and I am going to rational I mean both the real and imaginary parts are rational ok. So that is each of Q is actually set of all x+iy which so that xiy ok.

Then of course you know that then you know that piece of Q is countable ok and it is and it is dense in EQ E because its closure will be its closure will be equal ok then E sub Q is countable and E sub Q closure will be triumph, because E sub Q is just E sub Q is just you know E intersection with q cross q, Q crossed you if you think of CS r cross r the points in the complex plane every complex numbers as rational coordinate should be Q cross Q.

And how do you take EQ is you will just EQ by intersecting E with Q cross and you know the subset up a countable set count you know Q is countable therefore Q cross Q is also countable alright and therefore a subset of countable that worries of Q also count and rational numbers are dense and therefore E subQ will be dense E and E is the closure of E sub Q and the complex plane and you get back Q ok.

So read this first right now the whole you see this is something of a mystery for example you know when this all facts that you keep using all the time but if you really want road the marital deeply in a certain way when you are only perfect. For example you know rational numbers countable ok which means that you know all the rational numbers can be put in a significant sequence ok.

So I can write rational numbers as a sequence xn ok and that is very that something that you cannot imagine okay because given any rational number your enumerating rational numbers in some order alright, but then the usual order that you know of the usual order that you know of on the real line you can tell what is the immediate next rational number two given rational number simply because how worth goes you can always find the rational number close to it ok.

So you cannot see what is the next version number but here is very or using some you know very abstract settle to say that the accountability allows you to index and you know enumerate all the rationales ok, so this si the high set a abstracting that you use alright that something that in practice only we really cannot do it, you will got expect to do it right, so in some sense that will be connected to the axiom of choice ok which is as you know see it well ordering principle and Jones Lamar.

And these are all and Jones Lamar is not a limit it is it is actually a result that is actually an axiom which you accept and you cannot do it only you can prove it only if you are seeing each other equal informs memory Johns Avenue reaction of choice as well ordering alright, so this is the depth of section that is involved but if you say its bread and butter when you do analysis ok.

So you therefore you know I had written E sub Q as zi and ok this is the reading complex subset, you write all that all points in E you any write them alright and you all you write this completely existential you really do not know what is it one is rz it to it is all you know is that you can write all this points like this that is used and I will use that as you say. So so you write like this, so this corresponds to what this corresponds to the fact that this is countable ok.

Or it also corresponds the fact that this is countable and therefore we can order it using real numbers in using natural numbers therefore if you choose some ordering with respect to write the natural numbers you get a secret and that' is the sequence I am writing ok, so this si something we obstruct right but what you do is now what you do is you do the following thing, you take.

And of course you know what I am supposed to do if I am supposed to take a sequence in the family and I am suppose that produces subsequent which is converges uniformly ok not just converges but I wanted to convert uniformly right. So so start with start with sequence fn in this family ok consider so take the first point ok and apply fn quit considers and what will happen is since modules of all this fellows is less than equal to M which by the theorem.

We have a convergent subsequence which we write as fn1 of z1 ok, so this is fn 0f z1 is a sequence of all numbers which is bounded, so it convergence of sequences and called converges sequences f n1 of z ok. So this fn1 is the subsequent fn fn1 is the subsequence from fn right, now what you do if you repeat this process you repeat this process with z2 and with the functions in the subsequent ok.

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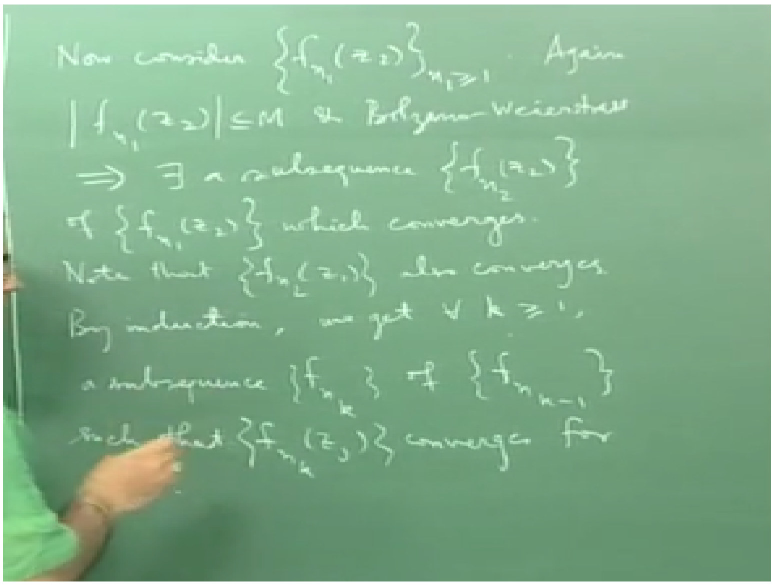

Now consider what you do is you take this fn1 you take applied to set and let anyone verify and wanted, so I get sequence of function values of z2 ok and again the same argument works again modulus of fn1 of z2 is less than or equal to M will tell you that theorem will tell you there exists a subsequence fn2 of z2 of fn of z2 which converges ok. So you see in the first step I am applying all the members of the field sequence to the point z1.

And from that I get a subsequence, in the second step what I do if I forget that the points and once but I only look at this subsequence fn1 and apply z2 to do it and again apply theorem and give the uniform boundedness to show that subsequence of a subsequence ok which converges at z2 ok and notice see fn1 already converges Z1 and fn2 is a subsequence of a fn1. Therefore fn2 will not only converge z2 it is also converge z1 ok.

So note that fn12 of z2 also converge ok, so now you go by induction ok by induction the we get for every n greater than equal to 1 a subsequence. So let use M or k a subsequence fnk of fnk-1 such that fnk of z0 converges for j less than or equal to j ok. So you know so this is what is happening right. So you know if you write it in a pictorial , so you have n1 which is with the property that fn1 fn1 converges at Z1 alright.

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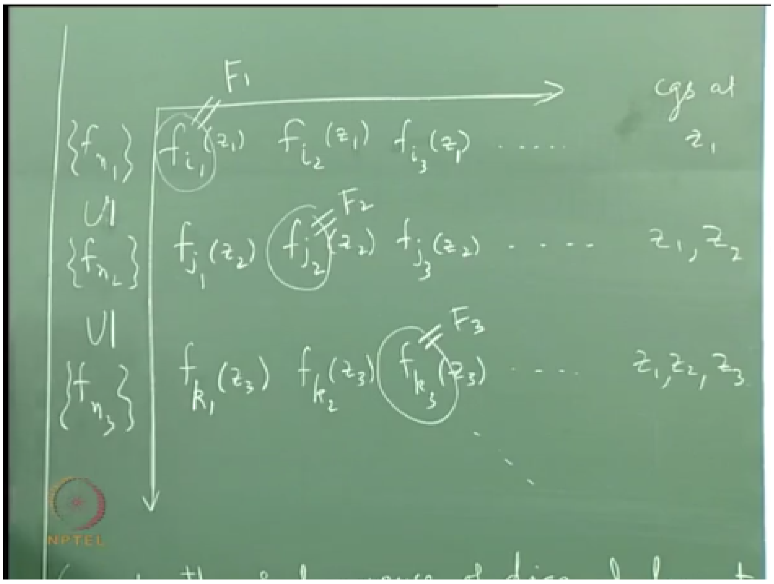

Therefore if I write it I will get f I will get fi z1 fi2 of z2 and let me so let me write fi3 of z3 again z1 and so on. So this is converges at z0 alright then I will get so this is sequence fn1 then for this I get subsequence fn2 ok, this fn2 is subsequence of a fn1 that means all the integers that occurs here they are among the indices**,** but still I write it only in this order.

So now I will write it as fj1(z2) fj2(z2) fj3(z2) and so on and this will converge at both z1 and z2 ok and mind j1 and j2 j3 is subsequence of i1, i2, i3 and i1, i2, i3 is subsequence of rational numbers ok. Then if I again repeat the process once more I get a fn3 I get this subsequence which is a further subsequence of this and if I write the indices as fk1 of z3 fk2 of z3 and fk3 of z3 and so on.

Then I get the subsequence of this subsequence which converges at z1, z2 and z3 ok, I get this situation like this aright and now what you do you know you take the diagonal sequence of functions and you take this you take this you take this you take this is called diagnologist, you take the diagonals of subsequence ok. So consider the subsequence so you know I will give it up.

I give it a special symbol I called capital F FL is actually Lf member of fnl, this is how it bigin. So F1 is first member of fn1, F2 is second member of a fn2, F3 is third member of fn3 that is your different ok. So this is so this lie here is F1 this lie here is F2, this lie here is F3 and and so on, so this diagonal sequence of little ok. Now the beautiful thing about design a sequence is that converge at every point of EQ ok.

So that is the and that is the power of the diagonalization process, you are able to extract this, after all you want a sequence which converges on all of E ok, but then you know that because everything is continuous ok if you can get uniform continuity and dense open subset of E ok, then you will get everything alright and what is the dense of open subset E is this open subset and what you are what helps in the diagonalization process is the fact that this count ok.

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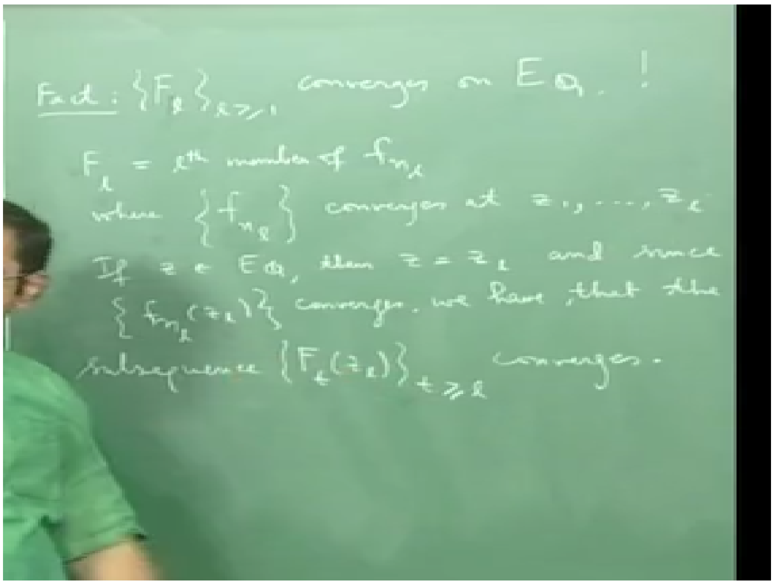

That is what allows you to enumerate and then expect this diagram alright. So what is the point. So the the fact that lies is that fn Converges are F I think are useless L l greater han 1 converges on EQ, so this is the bit of it, why because you see why is that true because you see F, what is fl, (FL) is actually lth member of Fnl where if you take the sequence fn l converges at z1 etc of zl ok.

See if this si the lf member of fnl alright and the but if you look at fnl that sequence converges at all points up to zl alright and therefore you know if you give me any point of EQ that point will be because of this is a numeration ok which is as far as I told you any point of

Eq is some zl any point of Eq some zl and but you know fnl the sequence fnl will converge zl alright.

Therefore all fl is greater than l which apart come from subsequence of this, you will also converge that and therefore this itself will converge at zl ok. So let me write that if z belongs to Eq then z is zl and since fnl of zlconverges we have that the subsequence ft of zl t greate than equal to l converges ok and you for a sequence of function it converge a point it is enough to converge belongs of the state.

So what I am saying is that the sequence fl will you know converge at the point zl at least after zl but and this is, so this will prove this one alright. So the moral of the story is you are able to extract here subsequence which converges point wise on this set of point ok. Now from this and equicontinuity we can show that you have that this that this sequence is equals the diagonal equal you extract it converge is actually uniform and that will give to proof of it. So I will continue with that in next video.