Advanced Complex Analysis-Part1: 0s of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem
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Lecture-37
Schwarz-Pick Lemma for the Hyperbolic Metric on the Unit Disc

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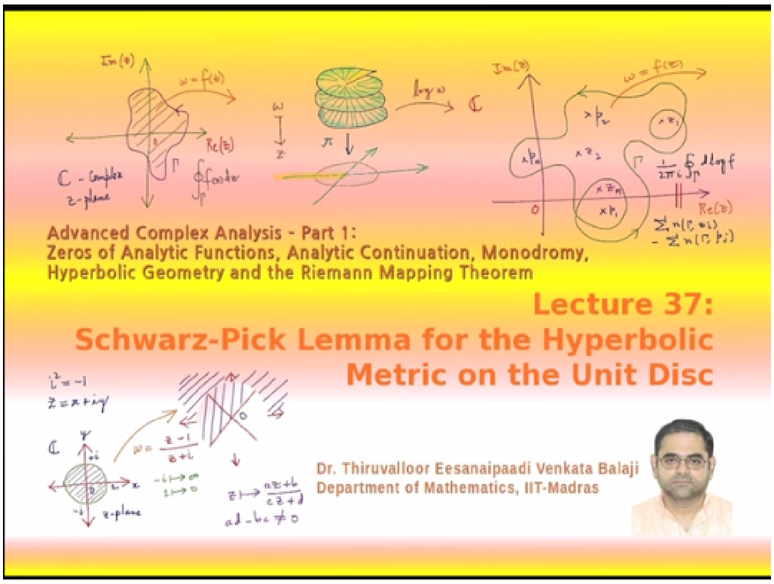

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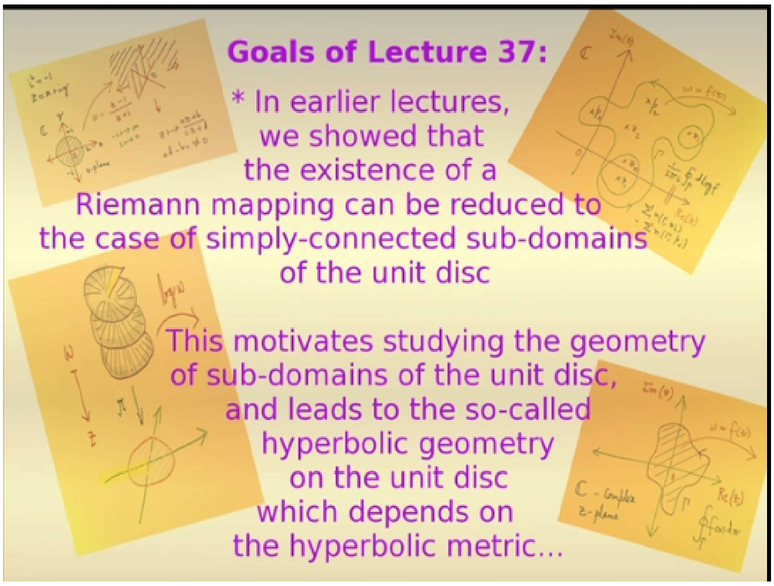

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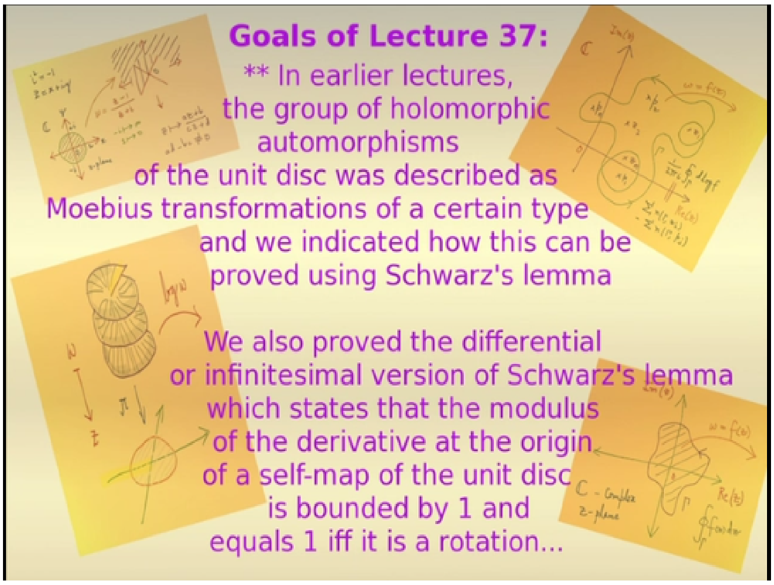

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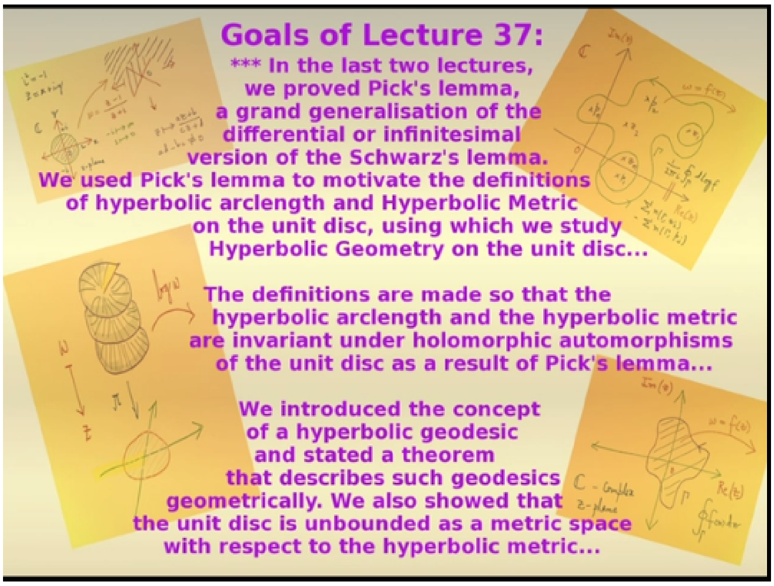

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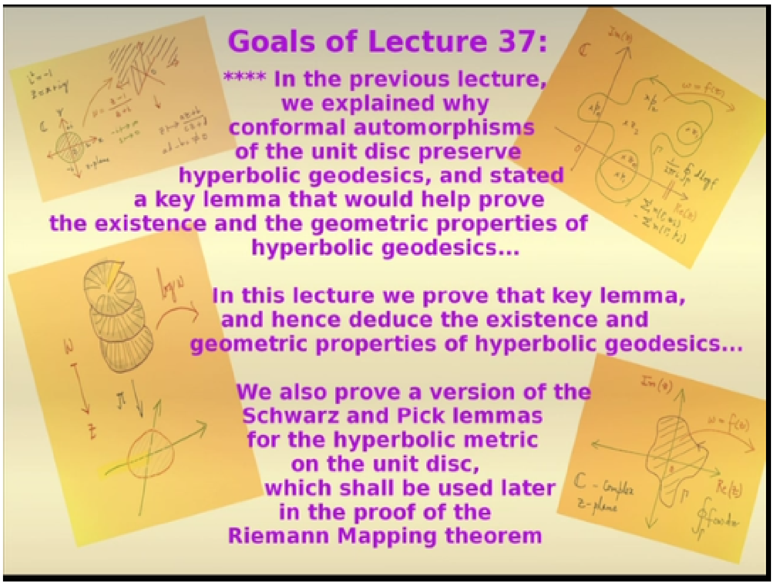

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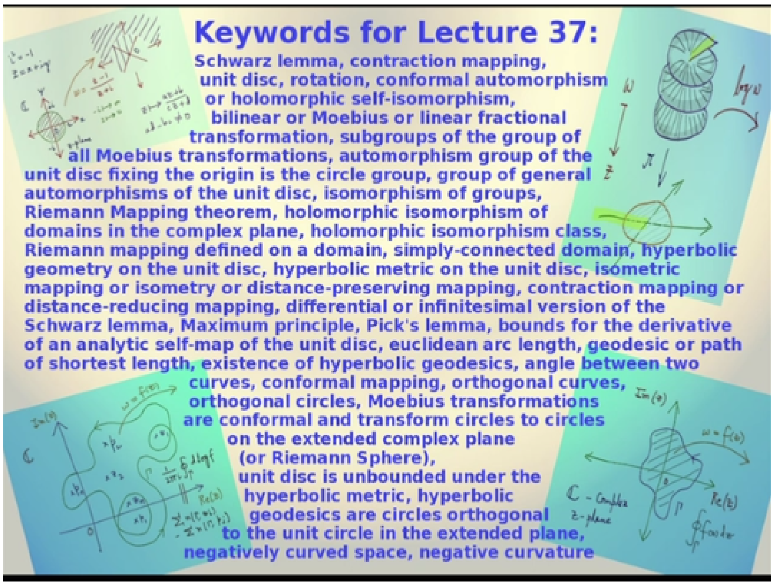

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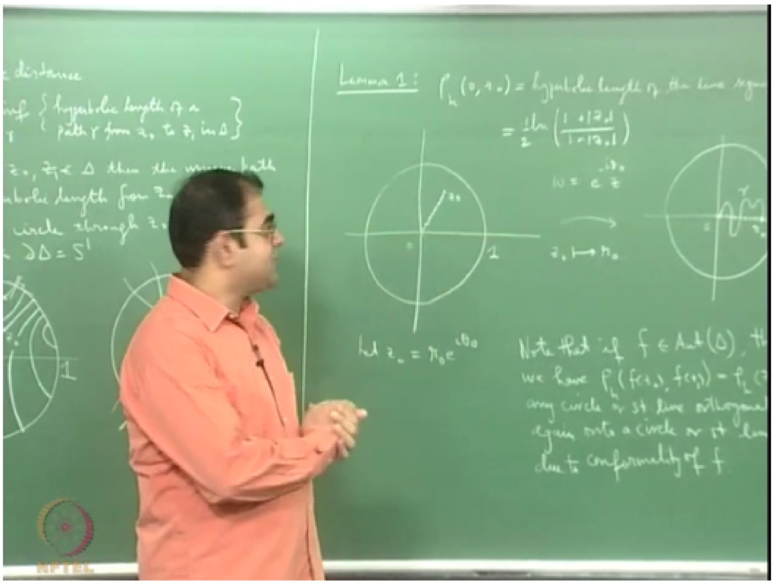

Alright, so we are here is the continuation of the previous lecture, I am trying to show that for the origin and a point z0 on the unit disc, the geodesic is simply the line joining the origin to that point okay. In other words I am trying to show that any diameter is a geodesic okay I am just this if I prove this is lemma I am just trying I am showing that any diameter is a geodesic okay.

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And my point is that the proof of this theorem will be complete just because of this because you can move any geodesic to a diameter by using a conformal automorphism the unit disc okay. So essentially the whole proof of the theorem is here okay essentially here okay so, how do I prove that that the shortest distance from 0 to z0 is given by this line segment what I will doing is I will rotate it like this by this e to the -itheta not where theta not is a argument of z0.

And I so, this map z of point z0 is a map to r0 then I will have to show that this is a geodesic okay. And so, this line segment from 0 to this point r0 on the real line that is a geodesic. I have to show and what therefore I have to show I have to show that you see it is that is a segment which has least hyperbolic length okay. And that is because hyperbolic distance is defined as the least hyperbolic length of an arc between the two given points.

So, what I am going to do to show that this is this the hyperbolic length along this line segment is the least I will simply compare it with the hyperbolic length along any path from 0 to r0. So, here is Gamma which is the path from so, let me keep this because I need it so, let me estimate the hyperbolic length among gamma is to the simple calculation.

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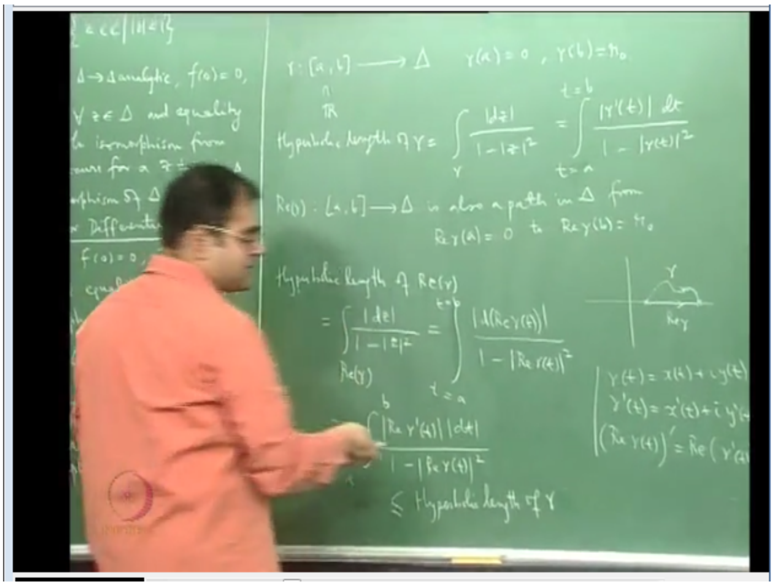


So, gamma is from sake closed finite interval on the real line uh and taking values in the unit disc and of course gamma of a is the origin gamma of d is r0 okay come my some path and what is the hyperbolic length of gamma hyperbolic length of gamma is well you know by definitions integral over gamma mod dz by 1-mod z the whole square okay.

And of course you know we calculates this, this is just integral over gamma mod gamma dash of integral from t equal to a to t equal to b mod gamma dash to t dt by 1- mod gamma t the whole square okay. So I have just plugged in m z equal to gamma of d. And I need a change variable to the real parameter t which varies from a to b okay now you know what do you must understand is that .

If you give me any path gamma like this okay if I project that path to the real axis I will get a path along the real axis mind you see this path could go out I mean there is no there is no reason it could go out like this. And here also you know it could be even more complicated it go like this that path gamma could be just any path inside the unit disc. It starts with 0 and end z ro it could twist and turn.

I mean there is no it just has to start at the origin it has to end at r0 and it has to be in the unit disc that that is only condition okay. So, but you know give me a path like this I take of that projection on the real axis. I will get a path along the real axis which starts with 0 and ends with r0. And how do I get that path you know projection on to the real axis is given by simply taking the real part okay.

So, what you must understand is that real part of gamma if I take that is also is also a path in the unit disc from real part of gamma of a which is because gamma a is a real part of gamma a is also 0 to real part of gamma of b. And gamma of b is r0 that is also real so, real part of gamma of b is also equal to r0. So, it is so, real part of gamma is also a path from 0 to r0.

It is just the projection of that path on to the real axis okay so, you know basically what I am saying is that you give me 2 points on the real axis and then you give me a path like this okay. Then as a point travels on this path okay it is projection will travel along this line segment and that will give you a path on the real axis okay. So, you know if is a path like this.

Then the real part of gamma will be it is projection the fact is the real part of gamma is also a path okay. And for this path what is the hyperbolic length the hyperbolic length of real part of gamma is what it is by this definition is integrate over real part of gamma taking mod dz by 1- mod z the whole squared alright. And now if you calculate this is going to be well here.

I have to put integral over modulus of d of real part of gamma of d divided by 1-modulus of real part of gamma of d whole square this is what I will get if I plug it in. So, when I calculate for real part of gamma I will have to again take t equal to a t equal to b but I have to plug in instead of z real part of gamma okay. And of course the what is the derivative of the real part of gamma of d it is a real part of the derivative of gamma of d because gamma of d has a real part and has an imaginary part and t is derivative with respect to t is derivative of the real part with respect to t+ i times the derivative of the imaginary part with respect to t. So, this is just going to be integral from a to b okay. Here I am going to get real part of gamma dash of t mod mod dt divided by 1-mod real part of gamma of t the whole square.

This what I am going to get right because you know if gamma of t you write it as x of t+iy of t then, gamma dash of t is going to be x dash of t+i times y dash of t. So, x dash of t which is the real part of gamma dash of t is the derivative of a x of t which is a real part of gamma of t. So, real part of gamma of t derivative is the same as real part of derivative of gamma of t that is what I am say okay.

But then if you if you know the real part of a complex number is always less than the modulus of the real part of the complex number is always less than or equal to modulus of the complex number so you know this real part of mod modulus of real part of modulus of gamma dash of t is certainly less than or equal to modulus of gamma dash 50 and the denominator 1-real part of gamma t whole square that will be greater than 1-gamma t the whole square.

So, it is reciprocal will be less than therefore for you know this quality is certainly less than this quantity okay. So, what will you get is that you will get that this is less than or equal to hyperbolic length of gamma**.** So what I have proved is you give any path gamma alright. Then it is hyperbolic length is certainly bigger than the hyperbolic length of it is projection to the real axis okay.

And therefore you know the moral of the story is that if you want the hyperbolic distance you have to take a path of minimum hyperbolic length. So, you know obviously I will first of all I have to take a path which is equal to its real part okay. Because if you have a path gamma which is not equal to its real path part it means that this integral this length will be bigger than this length.

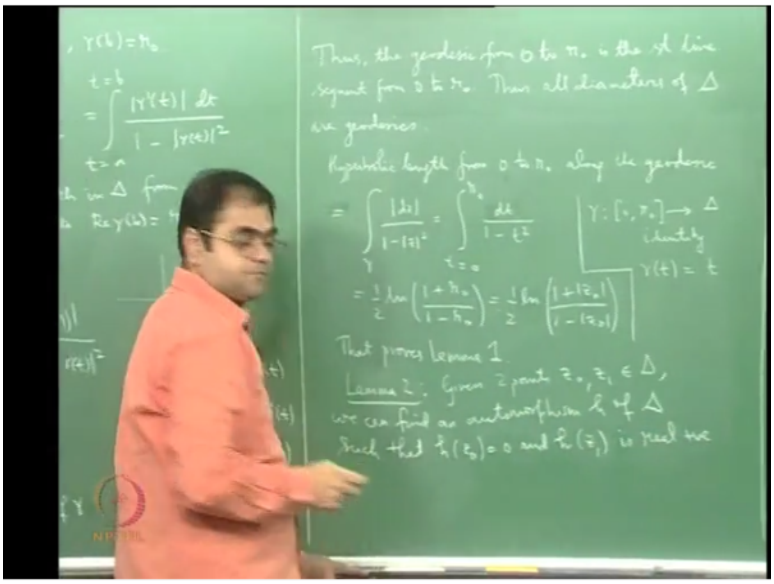
You will get equality if and only if the imaginary part of gamma is 0 okay. Therefore moral of the story is you give me any part first of all in order to minimise the hyperbolic length it is you will have to take paths which you on the real axis okay**.** So, therefore and among the pass which is real axis from 0 to r0 which is the part that we will give you minimum length minimum hyperbolic length.

It is the path for which it is the path which starts at 0 and continuously and it increases to r0 that is the only path. Because if the path decreases at some subinterval you can cut out that sub interval and get a shorter path so, for example you know if I have a path which goes like this for example if you take the projection of this path it will start from here it will go up to here.

Then it will come here and then it will slow down then it will go here it is slow down that will come here slow down a little then will go all the way there and then it will come back okay. That is all the projection of this path will be but you know if you want whenever part of the path goes the reverse direction. Then there is sub interval where it is decreasing that is sub interval.

You can cut out by simply cutting out the portion the path which the movement is start going backward and until it comes back to the same point that portion the path you can cut it out because until unnecessarily give you extra length after all this you want to minimise the length after all you want to minimise the length. So, all this shows that if you want to path of shortest length.

You just have to take this path starting at 0 and ending at the r0 it should be a increasing function okay. So, this just this calculation and common sense common mathematical sense of of course tells that the geodesic is simply the straight line segment from 0 to r0 okay. **(Refer Slide Time: 13:39)**



So, let me write that down thus the geodesic from 0 to ro is the straight lines segment from 0 to r0 okay. So, it is very clear so, the you must remember that even if gamma even if real part of gamma decreases okay then this derivative will become negative but I am mod okay therefore even if gamma decrease even if you take a path which goes which want does away which goes starts at 0 then goes back.

And then comes forward I am going to get more length okay so, if you want shortest length you have start at 0 go along this and then straight ended r0 okay. It should be a increasing function it should not decrease in sub interval okay. So, that tells you that the so, must always remember that there is a mod here. There is a mod real gamma dash of t so, the integrand is always positive quantity okay.

So, the geodesic from 0 to r0 is straight line segment from 0 to r0 and therefore from 0 to z0 the geodesic will be just this line segment. Because this is the inverse image of this under the inverse map which is rotation by e power which is rotation theta not this is rotation by -theta not this universe map is rotation by theta not and you know it is automorphism unit disc.

So, it will preserve geodesics so, if this is the geodesic then this also geodesic so, what so, the effect of this lemma is that we have proved that all diameters are geodesics okay. So, if the lemma is actually proves that all the diameters are geodesics okay thus all diameters of delta or geodesics okay. And as for as this expression is concerned this expression is something that I have calculated in the in the last the lecture before the last lecture okay.

In fact I can do that for you hyperbolic length from 0 to r0 along the geodesic which in this case of z. So, is the radial line from 0 to r0 line segment what is the hyperbolic length you have to just integrate over this line segment so, you know how do you parameterize this line segment from 0 to r0 you simply parameterize by the interval 0 to r0 itself okay.

So, you take gamma from 0 r0 to delta given by identity map gamma of t equal to t okay this is simply this is just the inclusion of the interval 0 r0 as this line segment this is after all the interval so, the parameter is just given by t okay. And if you take this as a path okay then integral over gamma what is hyperbolic length you integrate over gamma mod dz by 1-mod z the whole squared.

You simply get integral from t equal to 0 to r0 you will get gamma dash of t is 1. So, you will get mod dt so, in this case it will be just dt because t is increasing by 1-t square and if you integrate this you will get half lan 1+r0 by 1- which is this expression okay. So that finishes of proof of lemma 1 okay. This finishes and mind you this geodesic length is the same as this geodesic length.

And therefore this geodesic length is also given by the same quantity mind you this is also equal to half lan 1+mod z0 by 1-mod z. Because after all mod z0 is r0, z0 is r0 e to the itheta not okay. So **so** that proves that proves a lemma that proves lemma1 now after I have after I do this I have to now show that I still have to prove the theorem the theorem is that given any two points z0 and z1 on the unit disc.

The geodesic is given by the unique circular path of the circle which passes through z0 in z1 and which is orthogonal to the unit circle okay. So here is lemma2 so, lemma2 is is something that you know already it is more of a recalling lemma that you already seen given two points z0,z1 unit disc we can find an automorphism h of the unit disc such that h takes z0 to 0.

And h takes z1 to the real axis okay, so this is just playing around with mobius transformations of the unit disc on to itself. So you know this is something that we have already seen you see so, let me draw this diagram.

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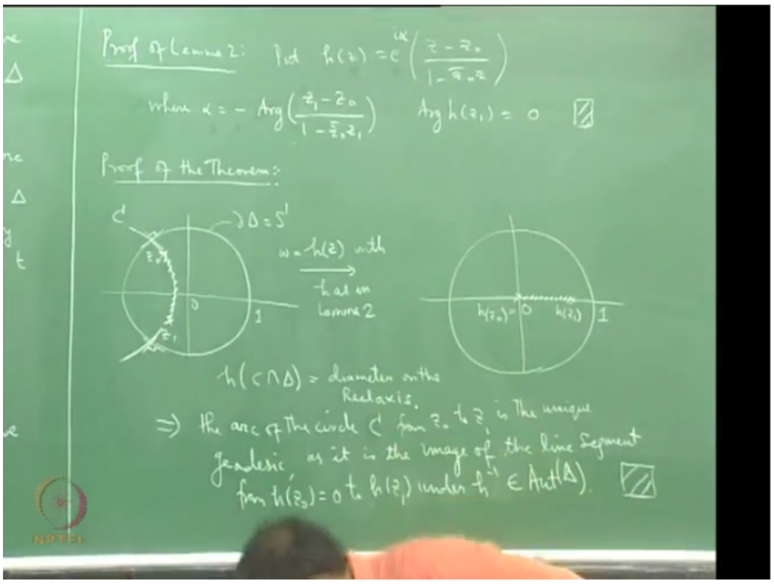

So, I am saying is well take this proof of lemma2 it is more of recalling this statement that you have already seen. You just take uh so, you know so, put h of z is equal to z-z0 by 1-z0 bar z. You know that such a map is a automorphism unit disc and it will map z0 to 0 okay to map z0 is 0. Because you have put z equal to z0 I will get 0 alright.

And what will it take z12 so, it will take z1 to h of z1 will be z1-z0 by 1-z0 bar z that will be some complex number okay. You see it will have some argument alright how will you bring the complex number to the real axis by simply rotating by the negative of that argument. So what you do is you put e to the ialpha into this where alpha is - of principle argument of h of okay.

Because if I put it like this then the argument of h of z1 okay will be argument of this which is alpha which is -argument is hz1 I should not put hz1 I should be little careful I should put not hz1 I should put h without this. So, z1-z0 is by 1-z0 bar z1 this is what I should okay you put this. Then you know then what is the argument of h of z1 the principal argument of a complete h of z1 is 0.

Because the argument of h of z1 will be argument of this which is alpha+argument of this evaluated at z1 which is –alpha. So, I will get alpha plus+-alpha you will get so okay. So if you do not want you principle argument when you ordinary argument and read model of 2pie okay. So, this is a of course the mobius transformation which takes the unit disc the unit disc it maps z0 to 0.

And it maps z1 to a point on the real axis okay that is what we wanted and in fact positive real axis right. Now I am done now I can use the proof of the theorem so, this is end of the proof of lemma2 that is nothing to prove it except that you should write this formula properly. We have already seen that any automorphism of the unit disc looks like this for some alpha for some real alpha.

And for some z0 a point inside the unit disc that we have already seen okay I am just using that fact here right. So now we count the proof of the theorem so, what is the proof of theorem so, you know what do I have to proven theorem so, here is unit disc and here are two points z0 and z1 okay what am I supposed to prove I am supposed to prove that if I take this circle.

That passes through z0 and z1 and which is orthogonal to they boundary of the unit disc this is s1 the z of union modular complex numbers. I showed that this is the this circle if you take this circle it is this arc which is a geodesic I have to show that now what I will do is well I simply map I will use this h I will put w is equal to hz with h as in lemma lemma2 okay.

Then what will happen is that you see after all this h is you know an automorphism unit disc so, it will take the unit disc to the unit disc and what it will do z it will map z0 to 0 okay it will map isz1 to hz1 but hz1 will be a point on the positive real axis. So, hz1 will be here this 0 will be h of z0 and I will get a point here which is hz1 this is what the mobius transformation will do it will map z0 to 0 okay.

And it will map z1 to a point on the real axis and because this map is a mobius transformation it is conformal therefore if you take this circle suppose I call this circle as c this circle you know mobius transformation will always map circles on to circles are straight lines okay. So, the image of this circle at least the portion of the circle inside the unit disc okay will be it has to be a circle or a straight line inside the unit disc.

But it has to pass through these two points then what else can it be that it has to be only be the diameter it has to only be the diameter. So, it is clear that h of c intersection delta is equal to the diameter on the real axis right that is just because of property of mobius transformations right. And but then what does lemma1 tell you lemma1 tells you that the geodesic from 0 from h of z0 to the z1 is this line segment okay.

Lemma1 that tells you the geodesic from h of z0 to h of z1 is just that line segment. Therefore it is inverse image under h okay which will be this arc will be the geodesic between z0 and z1 and that finishes the proof of the theorem I am just using the fact that you know mobius transformations are conformal maps. They have to map straight lines and circles on to straight lines and circles.

And their conformal therefore any circle perpendicular to the unit circle has to go to a circle or a straight line perpendicular to the unit circle which passes through these two points. And that has to be only a diameter it has to be a straight line right. Therefore the this is a geodesic it is a inverse image mind youi h inverse is also an automorphism unit disc. The image of a geodesic under an automorphism is again your geodesic.

Because geodesics are isometric I mean because automorphism unit disc they are isometries they preserve distance okay. So, therefore so, what this tells you is that it gives the proof of the theorem namely the circle the arc of the circle from see from z0 to z1 is the unit geodesic the as it is the inverse image as it is the inverse image as it is the image of h inverse.

The image of the line segment from h of z0 equal to 0 to h of z1 under h inverse which is an automorphisms of delta okay so, that tells you that that proves this theorem okay, so you know it is **it is** just a clever playoff mobius transformation is nothing I mean all this is all this geometry is pretty easy to work out. But the results you get a very nice okay so, that proves the theorem.

That tells you how geodesic actually look like for the hyperbolic geometry now I come to something more serious I come to this fact now you know now going to change the point of view.

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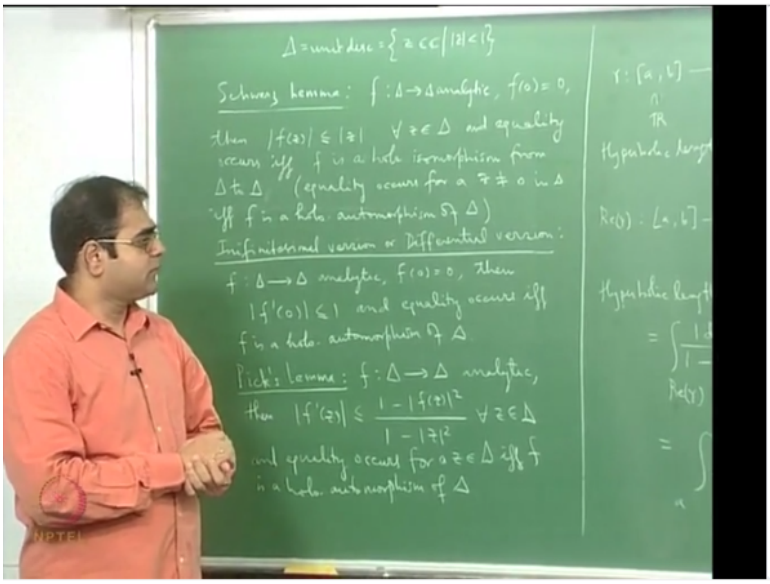

And try to prove this prove another statement which is what I want for the Riemann mapping theorem so for that you know I am going to look at short lemma again I am going to look at these three statements shorts lemma in front of the version of the short lemma and pix lemma. But I am going to look at them I am going to look at the non equality case okay.

So, look at the shorts lemma what does it say if f is analytic map for the unit disc which preserves the origin. Then mod fz is less than or equal to mod z okay for all z in the unit disc and equality happens only if f is an automorphisms. So, if f is not an automorphism you have strict inequality and what is the strict in equality the strict in equality will say mod fz is strictly less than mod z for all z in the unit disc what does it say it says the distance from the origin of the image fz.

The Euclidian distance okay it certainly is stictly less than the Euclidian distance of z from the origin. So, with respect to the Euclidian distance what is happening is it is a contraction if f is not an atom of some of the unit disc on itself the effect of f is like a contraction that is what it is mod fz will be strict less than mod z. Because even for a single z0 not equal to 0 if you get Mod fz not is equal to mod z0.

Then you have to be an automorphism we that is the that is the strong implication of the equality with that part of the short lemma. So, long as f is not an automorphism okay. And you know the condition that f is not an automorphism is equivalent to saying that f is not one to one one to one and onto f is not bijective. Because you know injective a bijective holomorphic map is certainly an injective holomorphic map.

And therefore it is an automorphism right so, the condition is either f is not injective or f is not subjective if is take such an f. Then such an f is not an automorphism and then it will surely be a contraction okay. And you get the same thing here in the Infinitesinal version also if f is not an automorphism then the derivative at the origin if you take the modulus thas to be strictly less than 1 okay.

You will get equal to 1 if and only if f is an automorphism and look at picks lemma picks lemma also says that if you take any analytic map from the unit disc the unit disc which need not preserve the origin okay. So, in these two cases you need origin to be preserved but here you do not need origin to preserve that is f of 0 need not be equal to 0 but tells you that the modulus of the derivative is bounded by this quantity.

And you will get an equality if and only if f is an automorphism if so if f is not an automorphism this is strictly it is in equality okay. Now what I want to tell you is that the same statement this in this same philosophy we can make a statement using the hyperbolic version. So, there is a version of the shorts lemma for the hyperbolic metric what it says is if you take any analytic map from the unit disc.

The unit disc then it will be either an isometry in which case it will be an automorphism of the unit disc. And if it is not an automorphism unit disc it will be a contraction with respect to the hyperbolic metric okay. So, there is only possible analytic maps from the unit disc 1 hour isometrics which are given by holomorphic automorphisms okay. The others are non-automorphisms. They will strictly be contractions okay this is the fact that we need for the Riemann preceding with the Proof of the Riemann mapping theorem okay.

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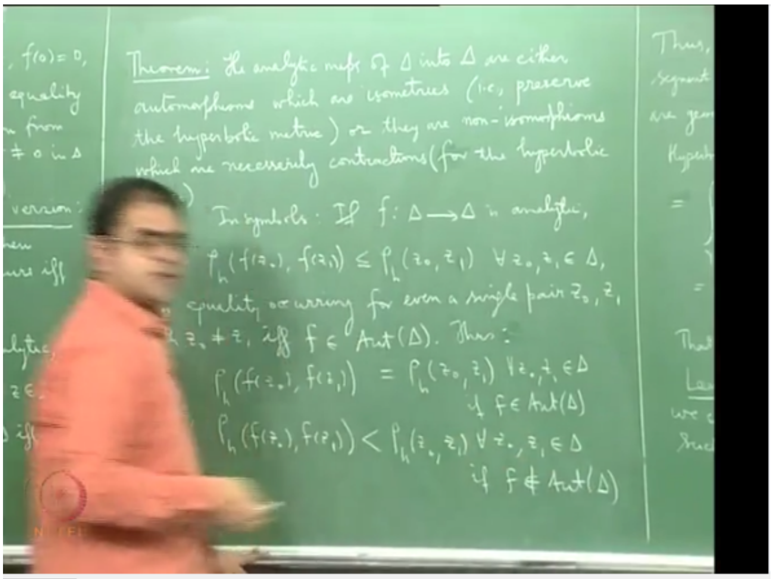

So, let me state that so, here is a theorem **so so** so, let me state it elegantly in words the **the** analytic maps of delta into delta are either automorphisms which are isometrics that is preserve the hyperbolic metric or they are non isomorphisms which are necessarily contractions for a hyperbolic metric. So, this is a theorem elegant state inverse but how do you stated in symbols in symbols.

If f from delta to delta unit disc to unit disc is analytic then the hyperbolic distance from between fz0 and fz1 which are the images of z0 and z1 the unit disc under f is less than or equal to hyperbolic distance between z0 and z1 for all z0 z1 in the unit disc with equality occurring for a single pair z0 not equal to z1 if and only if f is an automorphism. So, that is row h of fz0 fz1 is equal to row h of z0,z1 for all z0,z1.

The unit disc if f is an automorphism of delta and row h of fz0 fz1 strictly less than row h of z0,z1 for all z0 z1 in delta. If f is not an automorphism delta this is the statement in symbols

okay. So, with respect to hyperbolic distance is with respect to the hyperbolic metric a map of the unit disc onto itself either it preserves the metric in which case it
is an automorphism.

If it is not an automorphism it will just shrink it okay it will be it is effect will be like contraction okay. You have to give a proof of this proof is pretty easy I mean proof is pretty easy because we built lot of simply Riemann theorems but even otherwise all this all of these are argument is very very simple just involves mobius transformations nothing more than that so, you know so, **so** here is a proof.

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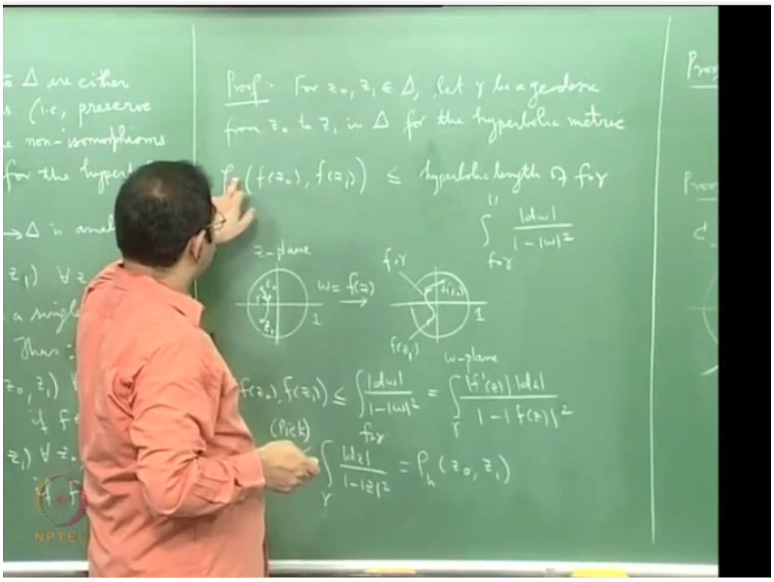

So, you know for z0 z1 in delta let gamma be a geodesic from z0 to z1 in delta for the hyperbolic metric. So, I have drawn in the diagram here z0 **the** geodesic is given by a circle which passes through z0 and z1. And which is orthogonal unit circle and the geodesic is the portion of this architecture of the circle between z0 and z1 okay. Now what is the hyperbolic distance between let us calculate the hyperbolic distance between Ffz0 and fz1 okay.

So, you know so, you know my diagram is very much like that but anyway let me draw it so, I have a diagram like this is the unit disc. So, I have this z0 I have this z1 and here is my geodesic okay this is my geodesic path gamma and I applied this map f which is an analytic map from the unit circle to the unit circle. I mean the from the unit disc is unit disc.

And so, I am going to get these points fz0 and I am going to get a point fz1. And of course the image under f of gamma is going to give me a path from fz0 fz1. So, this is a geodesic but then I will get so, let me draw something like this. So, this is the path which is first apply gamma then apply f just f circle gamma okay.

So, the image of a path from the image of any path from z0 to z1 okay drawn the arrows in the wrong direction should be from z0 to z1 is a z1. So, gamma is a geodesic from z0 to z1 bu tin any case it is a path from z0 to z1 alright. And it s image under f is going to give me a path from fz0 to fz1. Because I do not know that f is an automorphism I cannot claim that the image path is also a geodesic.

I do not I cannot say that so, but in any case you know if I measure the length of this the hyperbolic length of this arc that has to be greater than or equal to the hyperbolic distance between these two parts the these two used points. Because hyperbolic distance is a minimum length of hyperbolic path so, what I can say is a this is less than or equal to integral over f circle gamma of so, **so** I should say it is less than or equal to hyperbolic length of f circle gamma.

This is correct right because the hyperbolic metric distance is supposed to be the minimum of such hyperbolic lengths of parts from fz0 to fz1. And f circle gamma is one such one so, the hyperbolic length of f circle gamma is certain to be certainly going to be greater than or equal to this. It will be equal to this if and only if f circle gamma is a geodesic alright. So, let us calculate that what is that I mean so you know but what is hyperbolic length of gamma.

It is integral over f circle gamma of mod d so, if I call this is z and call here suppose this is a z plane. This is the w plane then my f is w equal to fz so, I will get dw by 1-mod w the whole square alright this is a hyperbolic length. And if I compute it bought like get well I get integral f circle gamma so, let me rewrite all that here. The hyperbolic distance from fz0 to fz1 is less than or equal to integral over f circle gamma of mod dw by 1-mod w the whole square.

And that is you know if I make a change of variable I will get integral over gamma f dash of z mod mod dz by 1-mod f dash f of z the whole square. I will get this, this is what I get if I put w equal to fz alright. But then by pic lemma this is less than o requal to integral over gamma mod dz by 1-mod z the whole square why this is because of pic pic lemma which tells you that m d f dash of z is less than or equal to 1-mod fz the whole square by 1-mod z the whole square.

So, if I use pic lemma I will get this but what is this, this hyperbolic length of gamma but gamma is a geodesic therefore this is the distance form z0 to z1 hyperbolic distance. So, this is rowh z0 to z1 because it is the length of the hyperbolic it is the length of the geodesic from z0 to **z** z1 because I started with gamma to be a geodesic. So, therefore I am getting the fact that the hyperbolic distance between the image points is less than or equal to the hyperbolic distance between the source points which is essentially the statement of this theorem okay.

And then I will tell you that you will get equality only if it is a automorphism and if it is not an automorphism will get strict in equality. So, for that what I want it understand is that of course you know if it is an automorphism then if f is a automorphism then of course since gamma is a geodesic f circle gamma is will also be a geodesic. So, I will get equality here okay and f is an automorphism.

I will get an equality pick lemma so I will simply get this hyperbolic distance is equal to this hyperbolic distance okay. I will get equality if f is a automorphism of delta okay but even otherwise I mean we have already noted similarly that automorphisms do preserve the hyperbolic distance. Because you are isometrics okay but the more serious accession is suppose you have a pair of points z0 and z1 which I distinct points for which this equality occurs for even 1 pair for even 1 pair if it occurs okay.

Then the accession of the lemma is very strongs it says that f has to be an automorphism okay if it preserve the distance between even a single pair of distinct points. Then it has to preserve all the distances see it is such a very powerful condition that is what you must understand okay. So how do you look at that situation see for a particular z0 and z1 if this is equal to this okay what you will get it is that.

You will get this integral is equal to this integral alright and that means you are getting equality in pick lemma for all points on the contour okay. And under but pick lemma tells you that if you get equality unit 1 point it has to be an automorphism okay. So you will get equality here if and only if f is an automorphism even for a single point. And will if you get equality for us single pair of distinct points.

You will get equality everywhere because it is an automorphism okay so, **so** that finishes the proof of this theorem so, let me write that last line. And we have done and what it also tells you is that therefore if f is not an automorphism then you have strict in equality and sit in equality means it is a contraction with respect to the hyperbolic metric okay. It is not an isometry it is a contraction with respect to the hyperbolic metric that is exactly what this theorem says.

So, we get equality for even a single pair z0,z1 of the distinct points if and only if integral over gamma mod special f dash of z dz mod dz by 1- mod fz the whole squared is equal to integral over gamma mod dz by 1-mod z the whole squared which courses equality in picks in the conclusion of pick lemma for points point gamma which means f is an automorphism delta okay see because you know if you get this integral is equal to this integral.

When you can bring it to one side and you will get and you know this is always greater than or equal to this okay. So, if you bring it to this side I will get integral of a non negative object equal to 0. And if you have real integral okay with the integrand non negative if it is 0 then the integrand has to identically vanish. So you will get equality of this integrand with this integrand at all points on the boundary.

I mean on the domain of a integration which is the path gamma which is the geodesic okay but then pick lemma tells you that you will get equality unit at one point it has been an automorphism but now you are able to get it for all points and gamma therefore it is certainly an automorphism okay. And if it is an automorphism then you always get equality towards it is an isometry okay.

So it is not an automorphism then it will be a contraction because it will be a strict in equality so, if f does not belong to automorphism of delta then f is a contraction okay. So, that finishes the proof of this theorem which we will meet to proceed with a prrof of Riemann mapping theorem okay. So, I will stop here.