Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology-Madras

Lecture-36
Hyperbolic Geodesics for the Hyperbolic Metric on the Unit Disc

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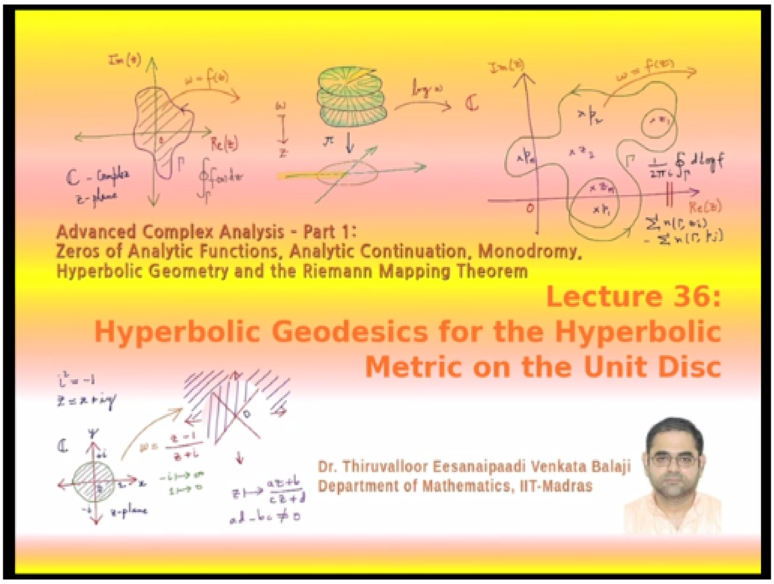

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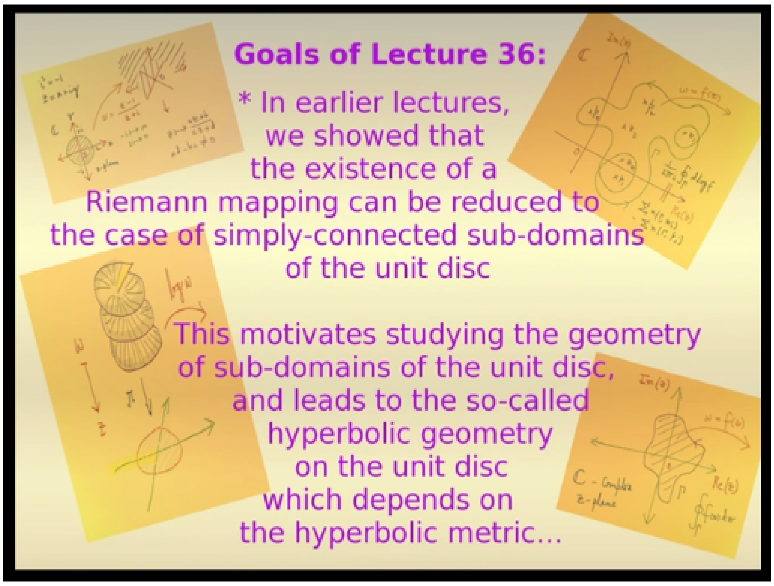

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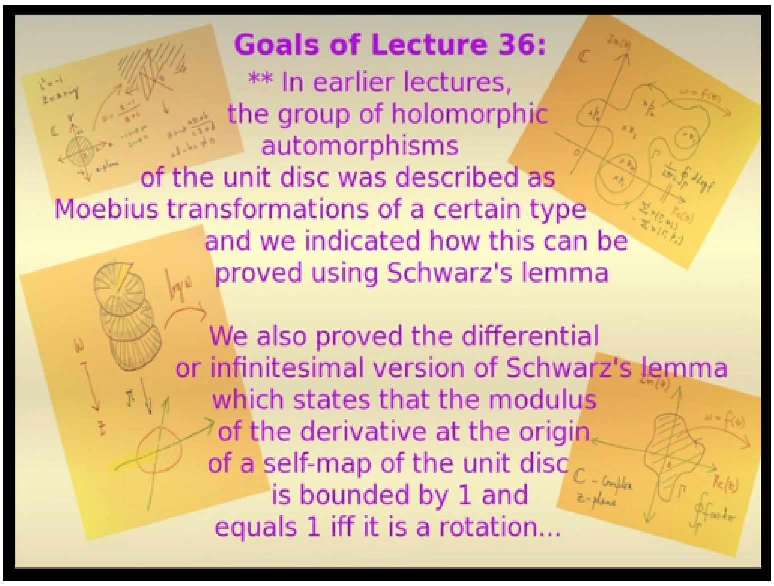

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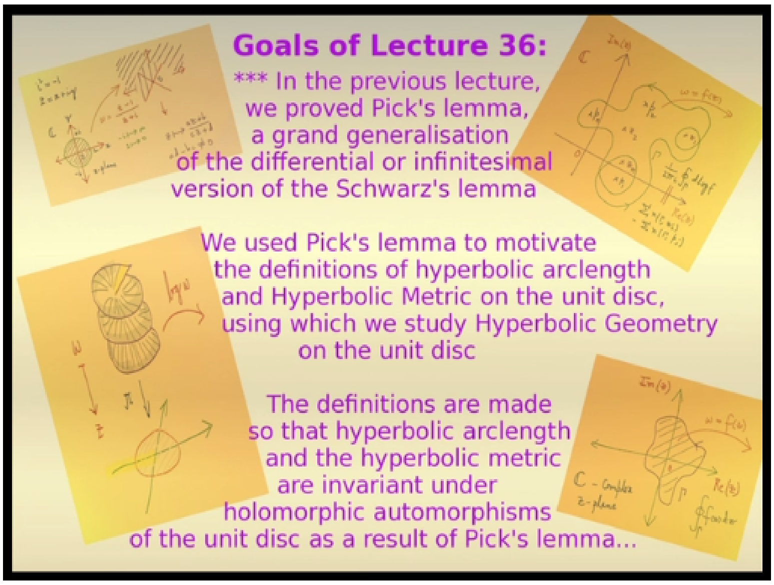

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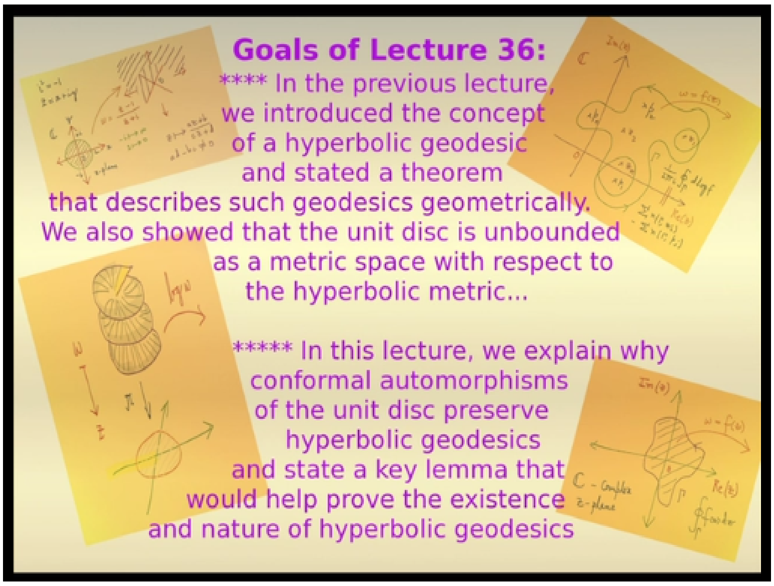

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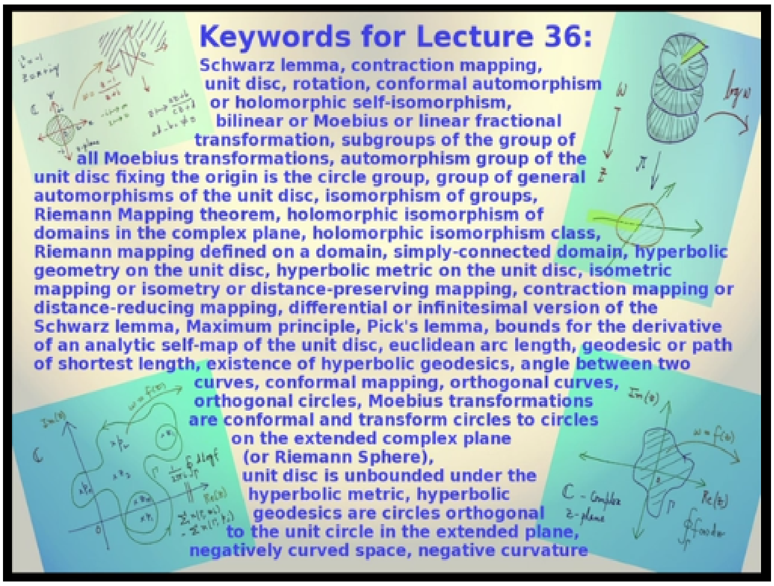

Ok so let us recall let us recall what we have been doing in for the benefit of continuity ok. So you know we are trying to prove the Riemann mapping theorem and beside the Riemann mapping theorem is started simply connected domain in the complex plane is not equal to the rho complex plane ok. So that means a domain which service at least one point in the complex plane is not in the domain.

And assume simply connected which means that any loop in the domain can be continuously showing to a point in the domain ok. Otherwise the domain does not have any hooks and you can also say this as the complement of a domain is continuously connected, it is not disconnected ok. So you take the simply connected domain which is whole complex plane the Riemann mapping theorem is that there is a holomorphic isomorphic of that unit disc ok.

So you have to find a holomorphic map the holomorphic isomorphism form that simply connected domain which is not the whole plane to be next. Of course you know these conditions in the hypothesis of Riemann mapping theorem they are necessary because you know if you the condition that the domain should not be the whole complex plane is very important because if you take the domain to a whole complex plane.

It is of course simply connected but any holomorphic function on that which maps into unit disc will be a constant by theorem because a holomorphic mapping from the complex plane into the unit disc will be bounded entire function, it will be bounded because the images you are saying it is taking values in the unit disc, so it will be bounded and it is entire because it is holomorphic on the whole plane.

Therefore it has be constant by it theorem. Therefore the condition that the domain in the Riemann mapping theorem is not the whole complex plane is important ok. That is the condition is not null should it necessary that it has to be simply connected. So any any domain that is holomorphically isomorphic to the unit disc is also topologically isomorphism to the unit disc and the unit disc is simply connected.

And anything that is topologically isomorphic that is homeomorphic to a simply connected space is simply connected ok, so if you expect a domain to be holomorphic isomorphic to the unit disc, then you should be homeomorphic to the unit disc ok. And but the unit disc should be connected and you know the motion of a simply connectedness is preserved under homeomorphism.

Therefore your domain must be interconnected. So the both the conditions in the Riemann mapping theorem namely that the domain is simply connected and that the domain is not the whole complex planes are necessary ok and the Riemann. The Riemann mapping theorem says that once these conditions are met then that domain just holomorphic isomorphic to the unit disc.

So we have to find a holomorphic map holomorphic injective holomorphic map from the from that from such a domain to unit disk ok which is isomorphism on to the unit disc. Of course injective holomorphic map is always an isomorphism on to estimate. So all you have to do is find the holomorphic map which is injective .

And whose image is the whole unit disc ok. So what we did first is the first step was we prove that we can find a holomorphic isomorphism of such a domain into a subdomain of urine disc not the domain which may not be the whole unit disc ok, that the first step that we prove and then therefore our proof of the Riemann mapping theorem reduces to studying the subdomains of the unit disc ok.

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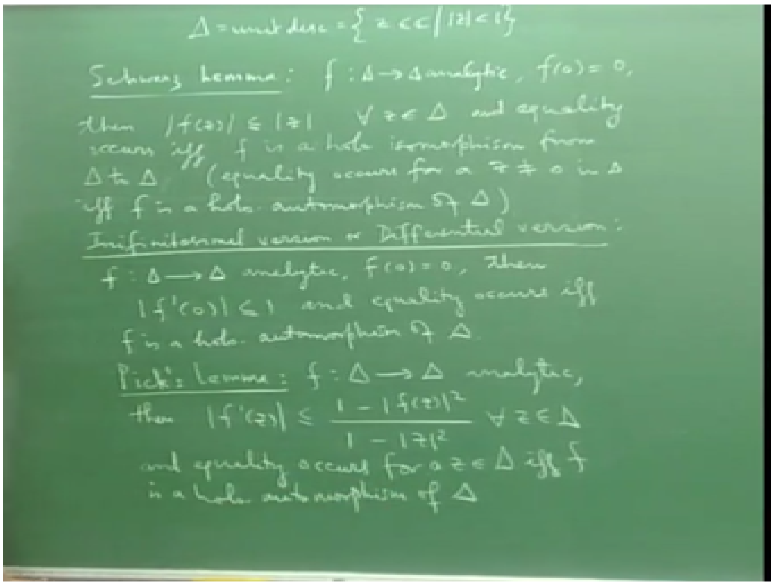

So that brings is into studying the unit disc and studying analytic maps for the unit disk to itself ok and that let us to so-called short film, so we had this so first we have this Schwarz's lemma, so this is you know if from the unit disc to unit disc analytic with f or with analytic map clicking origin to origin ok , then the modulus of the image of a point to the unit disc under f is cannot exceed the modulus of the point for all points leaving this.

And equity occurs if and only if f keys and automorphism a holomorphic isomorphism from delta to delta ok and of course equality occurs when is equality occurs you for single point, the different from origin the unit disc if and only if this f is the holomorphic isomorphism ok. So I should say equality occurs for a z not equal to 0 in delta if and only if f is a holomorphic automorphism of delta ok.

So this is the Schwarz's lemma. Now in this Schwarz's lemma you can you can actually get a version of the Schwarz's lemma and for that what you have to do is you take that z0=0 and divide by modz on both sides then you will get modfz by z is less than or equal to 1 and then you take limit as z as z tends to zero, limit z tends to 0 fz/z will be just simple fdash of 0.

Because of f(0) is 0, so if you divide by modz and and take limit Z tends to zero you will get the inequality mod fdash of zero is less than or equal to 1 and that is called the differential form for the infinity in the version of Schwarz's lemma ok. So let me write that, so infinity version or differential version of Schwarz's lemma is f from the unit disc to the unit disc analytics F0=0.

Then mod is the derivative at the origin is less than or equal to 1 and equality fs if and only if f is a holomorphic automorphism of the unit ok. This is the differential version of called infinity version of the Schwarz's lemma right. Now Pick's lemma is a generalization of this and what does it say, so Pick's lemma gives you an estimate for the modulus for the derivative at any point of unit disc.

The infinity version of Schwartz's lemma gives you an estimate for the modulus of the derivatives at the origin ok, but Pick's lemma will give it to you at any point z the unit disc. So what is Pick's lemma, so Pick's lemma is a the same conditions f from delta to delta unit disk unit to unit disc. So you know all these things of course I should mention the delta is unit disc which is set of all complex numbers such that modz less than 1 ok.

So f from delta to delta analytics ok and see here you do not put the condition f(0)=0 you relax the condition then model f(z) modulus of derivative of f point z cannot exceed 1-modf(z) the whole square/1-modz the whole square for all the point of delta, you know in particular if you put if f takes 0-0 and you put z=0 you will get mod fdash of 0 less than or equal to 1 which is the infinity version of Schwartz's lemma.

Therefore the Pick's lemma is the generalisation of the infinite version and of course you can say that also have equality condition here and equality occurs for z in delta if and only if f is holomorphic automorphism of delta ok. So this is Pick's lemma which we prove and then but the point about the important point about Pick's lemma as far as the unit disc of the geometry on the unit disc is concerned is the equality part ok, it is the it is so called it is what gives rise to hyperbolic geometry on the unit disc ok.

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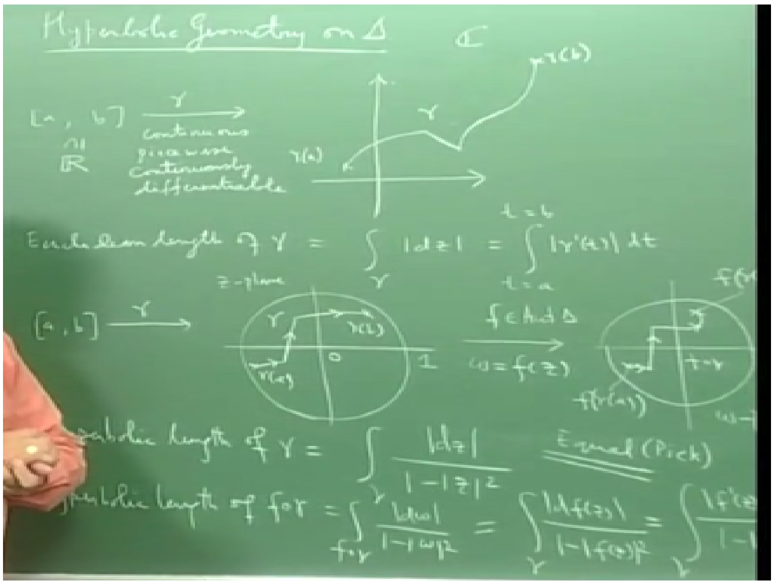

So what we do we go to first defining reboot hyperbolic geometry on the unit disc ok, so what is hyperbolic geometry, so the first thing is you know we define what is meant by the hyperbolic length of path in the unit disk ok. So you see if you have a path on the complex plane, so you have this interval in the real line bounded finite closed interval and you have this map this function gamma which is continuous piecewise differentiable.

In other words the image of this interval under gamma will give a piecewise smooth contour in the complex plane. So you know you get in get something like this an in principal it cannot be fully smoothing it could be like this, it has me piecewise smooth, it may not be points where it meet, it is not smooth. For example it will be a polygonal line ok , so of course you have the starting point which is gamma a you have the ending point is gamma b.

And we all know how to calculate the length of such a an hour from basic calculus ok length the Euclidean length of the gamma is simply given by integrating modDz you whenever you integrate modDz you will get the usual ok and what does it mean, it means that you are just you are just putting instead of you see whenever you integrate the variable of integration is on the domain of integration.

The domain of integration is this, this oriented path ok. So this is a point on gamma ok, so if you want to calculate this integral you can transform it into a real Riemann integral by substitution equal to gamma of t ok t is a variable here and gamma of t of the corresponding point here ok. So if you do that what you will get if you will get integral from t=a to t=b.

If I substitute D gamma of t then I will get modulus of gamma dash of t into Dt this what I will get and of course the modulus of gamma dash of t makes sense because gamma I told you this first of all gamma is piecewise continuously differentiable. That means I can break this interval a b into finitely subintervals over which gamma becomes differentiable alright and therefore this and derivatives are also continuous ok.

So this integral can be defined as a reminder that if you calculate this will get simply the length will get arc length ok, will get the arc length along this along this arc ok. Now this is Euclidian, but what you do is able to look at very specific case when you path gamma is a path not just in the complex plane, but the path is in the unit disc ok. So you look at this situation where your path lands inside the unit disc.

This is unit disc and your path is here, so here is your path, start at a point at unit disc and end with point in unit disc and the whole path lies inside then unit disc ok and of course for this path the usual length Euclidean length is given by this integral, but then we define a new kind of length, we define hyperbolic length. So what we do is we define hyperbolic length of gamma to be you see you do not.

Of course you integrative gamma but do not just integrate modDz but integrate modDz/1-modz the whole square, we put this factor of 1 by 1 minus modz the whole square, we put the extra factor and when you integrate like this ok you see this integrate is going to give because one because you see modz is z you know again here the variable of integration is z and that is going to lie on the domain of integration.

The domain of integration is arc, the arc is lying in unit disc therefore the denominator never vanishes alright, the integral is you know positive quantity alright, therefore this is going to give me some positive value going to and that is called hyperbolic length, now what is the importance of Pick's lemma, Pick's lemma says that you see point of pick's lemma.

Suppose I take f to be an automorphism of the unit disc suppose f is the holomorphic automorphism of the unit disc ok. Then what happen is that f will map the unit disk holomorphically isomorphically on to itself, so this is again a unit disc, unit disc so I get the mapping if I called W=Fz what will happen is it will map this path into another path and this path will well you know you can write out the starting point will be at this point is what it is just gamma followed by f.

So it will be F circle gamma, that is a path, it will be a path because it is a path composed with an automorphism ok , so again it is a path in fact a path compose by any continuous map in any differentiable map continuously differential map also be a path ok. So this is if this path is gamma then path is f should be gamma, it is gamma followed by f ok, the composition of these two maps and of course it will start at the image of this point at f which is f gamma a.

And it will end at this point which is f gamma of b but sine f is a automorphism that is the isomorphism of unit disc itself, the image of this is again going to be a path inside the unit disc alright and if you calculate the hyperbolic length of this path ok what you will get is you will get hyperbolic length of the image of the fz, the image of the path gamma and fz is gamma that what we want.

It is by the definition is going to be integral over fz gamma of modDw/1-modw the whole square, this is the definition ok. Mind you this is the a z plane and this is w plane. So here the variable a complex variable is w, the complex variable here is z and w is fz ok and well if you if you change is integral if you make a change of variable alright by plugging in W it fz you will see that you will get integral over gamma dfz mod/1-modfz the whole square.

And if you calculate this see this mod dfz is going to give you f dash of Z modfdash of z into moddz. So what I will get is I will get integrative gamma modFdashz moddz/1-modf of z the whole square, this is what I am going to get. But then what is what does picks lemma says, picks lemma says that because f is an automorphism of the unit disc we have equality in this expression which will tell you that modifies of Z/1-modfz the whole square is the same as 1/1-modz whole square.

And if you plug it in there you will see that these two are equal, if you see the these two are equal. So the moral of the story is this opposite equality is as I told you because of pick lemma. This is because of Pick's lemma ok. So the moral of the story is beautiful fact that if you define hyperbolic length by this formula then the hyperbolic length will not change and an automorphism of length.

So if I define hyperbolic length of path gamma by this formula whatever length I get will be the same if I take the image of this path and again measure the hyperbolic length of the image path under an automorphism of the unit disc. So in other words these hyperbolic length definition is adjusted in such a way that it is in variant under automorphism holomorphic automorphism of the unit disc ok.

And the and the reason for all that is equality statement in pick's lemma ok. So the pick's lemma is the kind of you know it is it is the motivation to define hyperbolic length of this, because if you define hyperbolic length like that then pick's lemma tells you that the hyperbolic length of an arc will not change if you map the arc by an automorphism unit disc you will get a new arc.

But it will still be this it will still has the same hyperbolic length ok. So that is the beginning of hyperbolic geometry from pick's lemma which is kind of generalisation of lemma ok. Now and of course we are doing all this is all this because you want to study unit disc, we want to study the domain of unit disk you want to study analytic functions from unit disc into itself ok.

So this hyperbolic length of the path gives rise to a metric ok call hyperbolic metric on the unit disc ok. Now see for example let me take your mind back to the Euclidean situation that is usual playing for exam, how you define distance between two points ok, well one is the distance all that you know pretty well ok, but suppose I ask you to define it using arcs ok.

Then how will you define the distance between two points in Euclidean space just using arc 1 natural definition will be draw any arc from this between the two given points calculate this length and take the mean ok, you draw so in other words the distance between two points in Euclidean space is given by the minimum mark length of all possible arc of this point that and you know that minimum arc is just a straight line.

It is the straight line it is a portion of the just straight line segment may be the portion of the line that passes unique line that passes to those two points of course you see these 2 points one of the same point of the distance point of 0 alright. So this is a this is eve if you do not know the distance formula ok you can define Euclidean distance ok you can defend as a minimum of arc length of all the possible arc between the 2 points.

Now you do the same thing and the same procedure to define distance function on the unit disc. So what you do is given any two points in a unit disc you take the minimum possible arc length hyperbolic arc length of all possible paths between these two points in the greatest and that gives you metric is called the hyperbolic metric on the unit ok. So this defines hyperbolic metric unit.

So let me write that down and this is what I explained yesterday and in the last lecture. So hyperbolic metric distance rho h of z-z1=infimum gamma of hyperbolic length of a path gamma from z0 and z1 in the unit disc ok. You take so in other words what you do is given in two points any disc you draw any arc piecewise smooth arc ok and you capitalize hyperbolic length using this formula right.

And then you minimize over all possible such arcs, then the fact that you get a distance function on the on the unit disc and so many ask this question you see in the usual Euclidean geometry if you give me 2 points what is the minimum distance it is a straight line distance ok. So it is the straight line segment which gives you there arc of smallest possible length ok.

In the arc in any geometry is also smallest possible length of given special names where call geodesicwe call geodesic, so in the Euclid in geometry with geodesics straight lines segments in straight lines along straight lines, segments of straight lines ok. Now you can ask the same question if you take the hyperbolic geometry namely unit disk with this new metric, this new distance function.

Given any two points in a disc you can ask what is the what is geodesic, so what is the minimum what is the part of minimum hyperbolic length connecting these two part and last lecture I ended last lecture by stating theorem which tells you what that is, it is very simple the geodesics in the case of hyperbolic metric and actual circle is perpendicular to the units circle which si the boundary of unit circle.

Circle orthogonal to the unit circle or jio D6 and their the analogue of straight lines in Euclidean geometry ok, so here is the theorem which is what I ended with yesterday setting the theorem was if z0,z1 are in delta then the unique path of shortest hyperbolic length from z0 to z1 ok is the arc of the circler through z0 and z1 which is orthogonal unit circle which is the boundary of the unit disc is S1 ok.

So so the so this theorem answer what this and what it sense, so you know so if I draw diagram it is going to be like this, so this si the unit disc and you know if you give me if you give me 2 points z0 and z1 what you have to do is you have to draw a circle passing through z0 and z1 and which is perpendicular unit disc. So you draw simple lines now you get something like this.

So at this point the angle between the tangents of the circle and unit circle with you 90 degrees and at this point also the angle between the tangents will be 90 degrees ok. And this it is this length this arc of the circle to this orthogonal to the unit disc is which is the hyperbolic z which is the shortest distance in hyperbolic matrix, so in particular you know if you join this if you take this straight line segment from z1 to z0 unit disc and measuring hyperbolic you get bigger value.

So it is rather, so you see it is a very funny geometry, the geometry in which straight line distance is not the smallest, there is some there is a curve along the distance becomes small and it is hard to imagine that you should see this with this can never happen ok with respect to the usual Euclidean geometry that you are familiar with the shortest distance between the 2 between two given points in a straight line distance.

But here just a geometry by the distance the shortest distance between two points by not by the straight line segment joining the two parts 2 points. But truth you know joining the points ok that is likely to the reflection of it something I cannot imagine right always distance should be more than the usual definition that is because your usual definition that's because you use to usual Euclidean distance ok.

But here you know our distance function is not sure twisted the length by the factor 1 by 1 minus modz the whole square that is what is creating this this page think and let me tell you one more thing is all also this also indication of another fact in differential geometry according to which you know the hyperbolic disc that is a disc to the hyperbolic a metric has negative curvature ok.

So you know it is very difficult to imagine an object with negative curve alright, so this has negative curvature alright and it is something and you can physically not very easily and all that is all because of this definition of hyperbolic ok, it has negative curvature adn because of negative curvature that the hyperbolic length through this arc of the circle is smaller than the hyperbolic length along the straight line segment.

Travelling along this curved path you see shorter length then travelling along a straight line path because your space is curved, the space is negatively curved ok, that is that is what is happening come to this is theorem, so this is the theorem so in fact you know if you draw all the to draw all you draw all these geodesics you know look like this if you take two points like this it will be like this.

On your hand in if you take two points which lie on a diameter okay the geodesic will be just any diameter actually a geodesic because if you take two points along a diameter ok if you take z0 here and z1 here this will be a geodesic because you see what is the theorem as, the theorem say that if you want geodesic you have to pass a circle through the given two points which is orthogonal which is perpendicular units itself.

But the 2 points lie on a diameter to those two points if you try to draw a circle which is perpendicular to the unit desk you will end up as a limiting case you get only the diameter ok. So you know basically though of course the diameter is not a circle it is part of a straight line, but you should think about is a circle with the third point at infinity ok, for search for determining accepting any three points.

These are two points z0 z1 there is a point at infinity, so the viewpoint is did you think of a given straight lines and circles and this is something that you should avoid I come across when you studied the Riemann theorem okay the Riemann in straight lines and circles on the complex plane on the way they correspond to just sit on the Riemann theorem okay. So it natural to think of a straight line also it is a circle.

But if you want you can even think of this as you know the limiting case of circles which are orthogonal to each pass through these two points and which are orthogonal to the unit, if you take the limiting case will get this straight line. So every diameter will be a geodesic ok alright every diameter will be geodesics. These are the only geodesic of the straight lines.

But if you go away from the diameter your geodesic will become you know they will start getting curve they become a disc geodesic will get curve ok. So this is how these are so these are this is how the geodesic look like on their hyperbolic and the geodesic is a beautiful things that geodesics play the same role in hyperbolic geometry as straight lines in Euclidean. So in fact if you take all of Euclid's axiom except the parallel axiom.

The parallel axiom says that given any point and a line not passing through that point there si a unique lien through this point is parallel to the given line, that is called the parallax or parallel line, so when you threw that out all the remaining Euclidean axiom Euclidean axiom satisfied by these geodesics. So it gives a new geometrical hyperbolic geometry ok and you got you got nice things happens.

For example what happens is that in the Euclidean geometry you know that you form a triangle by 3 lines ok and some of 3 angle of the triangle is 180 degrees you can show that in hyperbolic geometry you can form a triangle like this again using split geodesics. So you know basically can form a second form a triangle like this, so this si a hyperbolic triangle.

And of course you can also have a triangle like this you can you can have this is one geodesic, this is another geodesic alright and then you can have a third one like this. So these are two types of hyperbolic triangle, but the beautiful thing about hyperbolic triangles is that the sum of 3 angle is hyperbolic triangle will be less than 180 degrees ok. The sum of 3 angles will be not will not 1 it will less than 1 ok.

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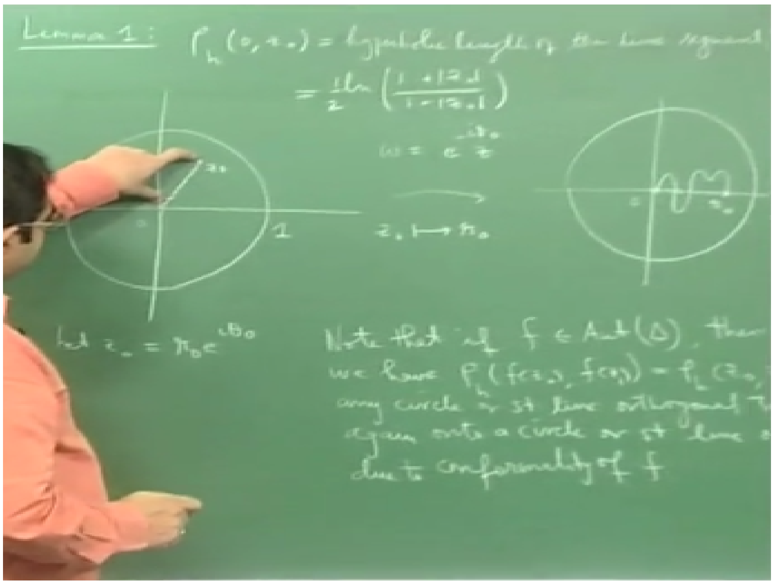

This is hyperbolic geometry, so anyway, so this si the new kind of geometry on the unit disc and beautiful thing is a disconnected is analytic isomorphism and analytic mapping with unit disc on itself ok and I have to we have to prove theorem first, what I am going to know what I am going to give you analytic lemma along so the theorem will be in two or three steps and all the steps are very easy ok.

So here is the first time, so what it tells is that if ensures the hyperbolic distance from zero of a point Z0 is actually the hyperbolic length of the line segment from zero to Z0 and it is actually equal to we calculated it addition of 1+1 1+z0/1-z0 ok. So here is so here this is a lemma, so you know you take you take 0 you take a point z0 in the unit disc ok.

And then what is the hyperbolic difference between zero and this point, I told you that has it mean if you believe the theorem in the topic I told you that all the diameters all geodesics ok. So you know I can find diameter that passes through 0 and z0 not that is radial line from zero to z0 and the clay is a this is a geodesic lemma, the lemma 1 says the geodesic from the centre of the unit disc to any point on the unit disc is just the radial line segment ok.

And so how does we prove this correct, what one does is you know so you do the following thing you put so let so let z0 be r0 e power 12 theta 0 ok, the z0 let z0 be this and what you do is I just rotate this so that I bring z0 to be really. So what will do I give this map w= z*e power-itheta, ok this is just, so e power f e power –theta is t, it is just rotation.

If I do this what will happen is of course this is the rotation of unit disc so it is an automorphism unite disc to take the origin to origin and you know we have already seen as a corollary of Schertz lemma that any automorphism of the unit disc which is the origin is rotation ok. So this is one of those, but it has been adjusted such a way that you know finally I get I get this get this situation like this.

So you know so z0 will go to r0 ok and this line segment from zero to Z0 is going to get map to the line this line segment on the real axis and I basic point about this first lemma is to show that you know this is your desk right mind you the hyperbolic length of an arc does not change under an automorphism of the unit ok, in fact that is the motivation for defining a hyperbolic length of this formula which comes from picks lemma.

So under the hyperbolic under any automorphism unit disc hyperbolic length will not change, but how is hyperbolic metric define hyperbolic metric define hyperbolic metric is also define using hyperbolic length, therefore the moral of the story is under any automorphism of the unit disc, the hyperbolic metric will be preserved ok in other words for the hyperbolic metric any automorphism unit disc will be an isometric ok.

Any automorphism of the unit disc will be an isometric which means the distance between two points will be the hyperbolic distance between two points will be the same as hyperbolic distance between their images under an automorphism unit disc ok that also just comes because of this reason that hyperbolic length invariant and an automorphism of some of the unit ok.

So you know to show that this is the to show that the straight line segment from 0 to z0 this is the geodesic hyperbolic it is enough to show that this is geodesic see what you have understand is that an automation unit disk will also preserve geodesics ok, that is because it will preserve distance ok and you know because it i an automorphism in unit disc and you know an automorphism of unit disc I just move this transformation ok.

And confirm so if the geodesic is a circle passing from the given points is automated unit circle then its image enter the mobius transformation again be a circle or a straight line which is automated unit because the mobius transformation you are considering are going to take the unit disk to the unit disc therefore they will take the boundary of the unit is to the boundary of a disc and confirm it will make sure that you know whenever to curves intersect at an angle the image will also maintain in the same angle.,

So if I take a circle which hits the unit circle orthogonally is image and mobius transformation which takes the unit disc to unit disc will again give you another circle or a straight line which is perfect with unit circle, so a geodesic will go to a geodesic ok under automorphism of the unit disc a geodesic has to go to a geodesic ok just because of the fact that any holomorphic map which is the with non zero derivative is conformal.

So it will preserve orthogonality it preserve angles between curves ok. So to prove that this is a this si of course this is a geodesic from 0-z0 it is enough to show the geodesic ok, so how do I show that this is a geodesic, I have to show that this is a path of shortest distance. So what I do is I just compare the length of this path and hyperbolic length of this path with any other path from 0 to r0 ok.

But the path could be curve and I show that this path which is a straight line segment from zero to R0 has a minimum distance, that is how I improve this lemma ok, and once I prove this lemma I am more or less I am done with more or less prove this theorem ok because I know how the automorphism of the unit disc look like right. So let me so let me let me write these two statements note that if f is an automorphism delta holomorphic automorphism delta.

Then for z0, z1 in delta he have rho hyperbolic distance between fz0 and fz1 is the hyperbolic distance between z0 and z1 and further any circle or straight line orthogonal to s1 is napped by f again on to a circle or straight line orthogonal to a unit circle due to conformity of f ok. So this is the statement that I set that any automorphism unit disc will have to preserve the hyperbolic metric.

And it has to preserve the geodesic it will map geodesic to geodesic right, so so show that this is a geodesic to 0-z0 is enough to show that this is a geodesic from zero to R0 because this map is an automorphism unit ok and to do this I have to compare this distance with the length of this hyperbolic distance along a straight line segment with hyperbolic length along some other path gamma from zero to z0 to R0 ok, so let me stop here.