Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-35
Differential or Infinitesimal Schwarzs Lemma, Picks Lemma, Hyperbolic Arclengths, Metric and Geodesics on the Unit Disc

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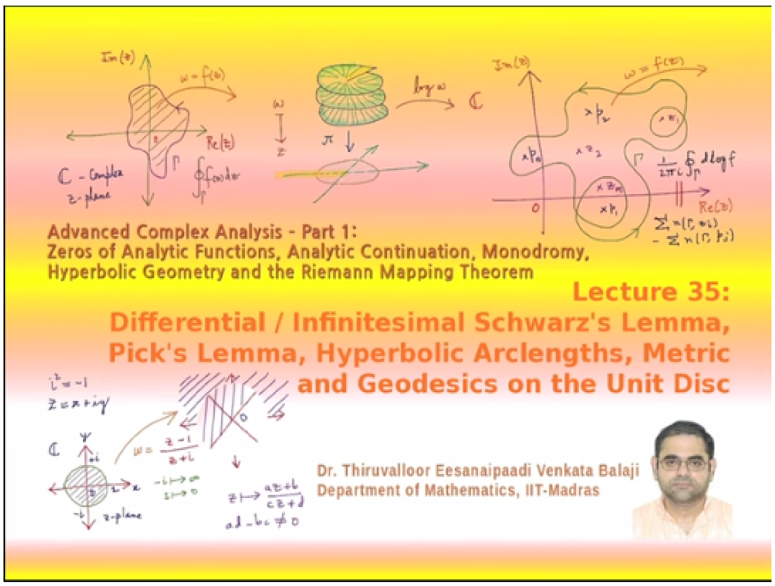

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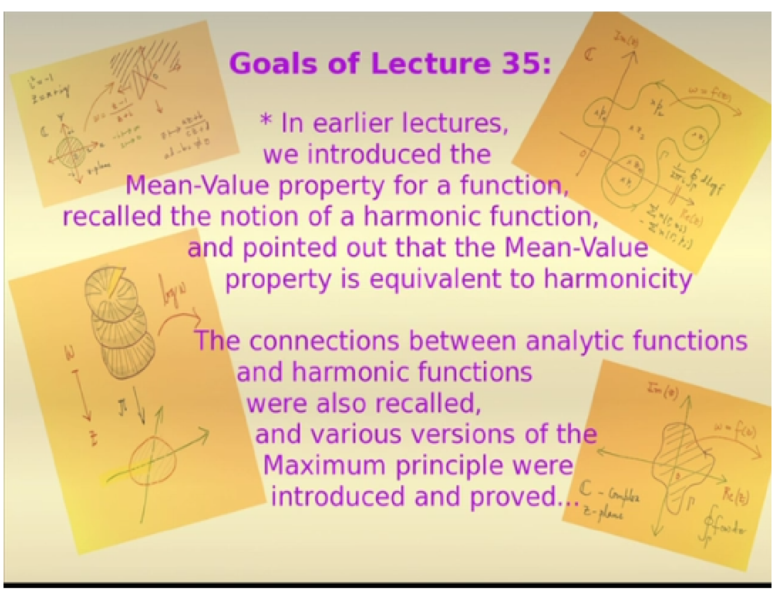

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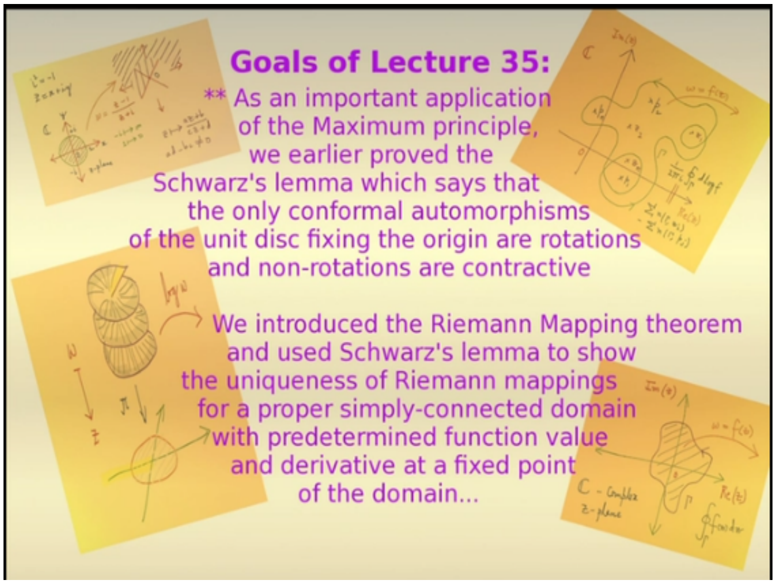

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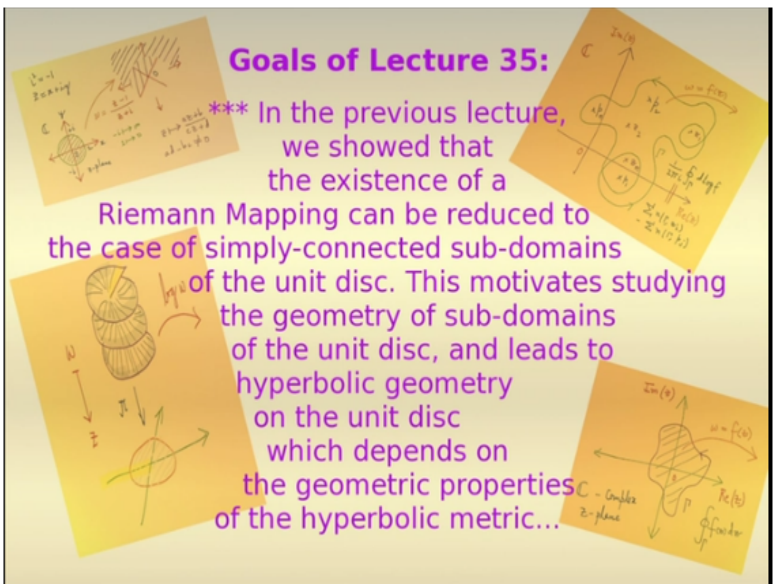

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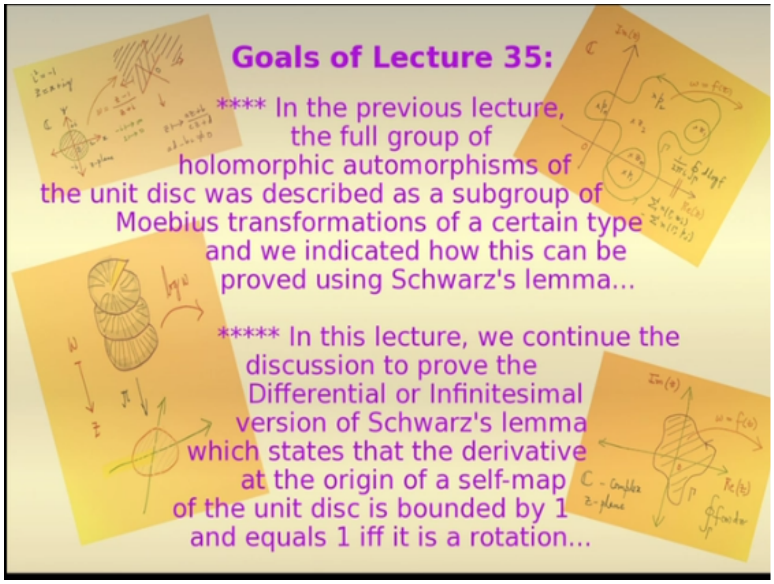

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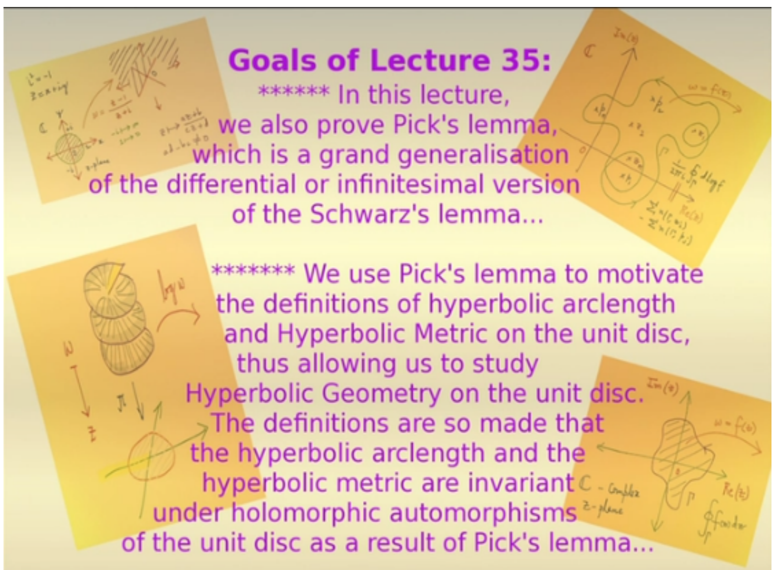

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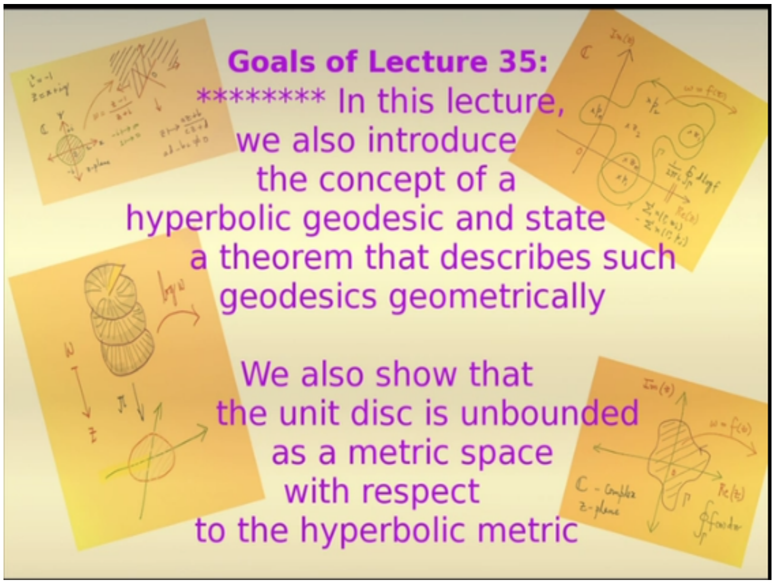

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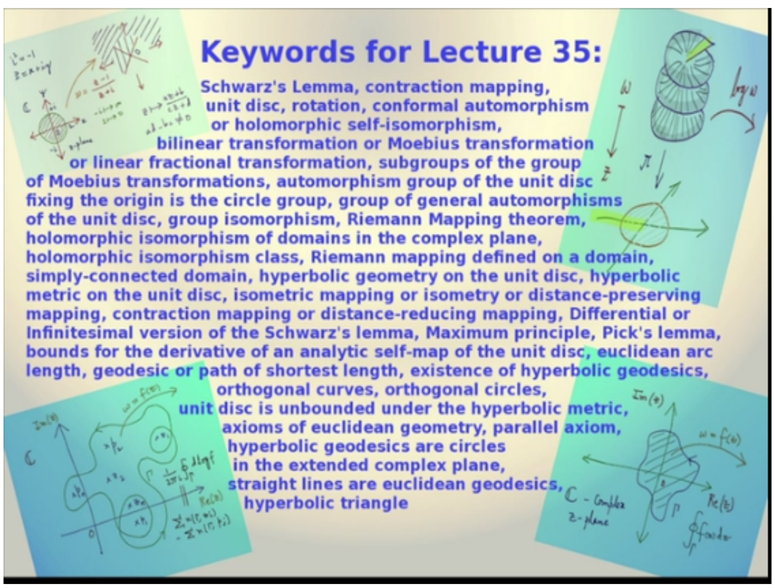

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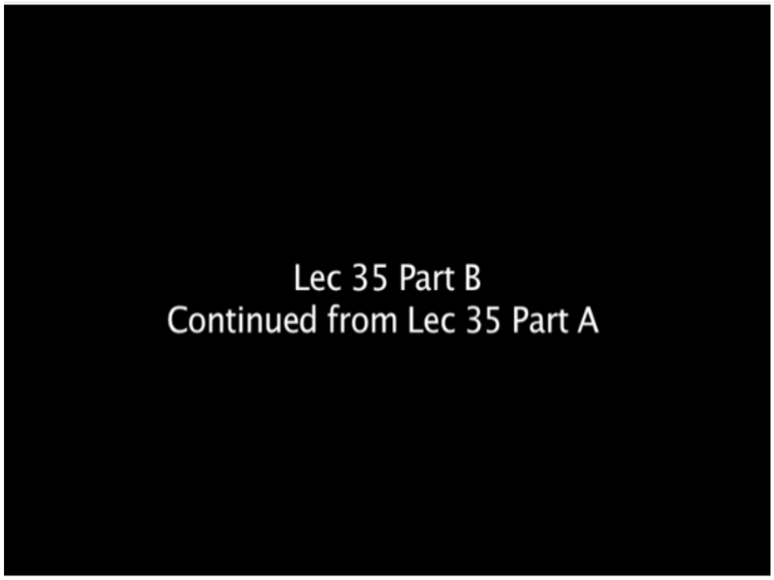

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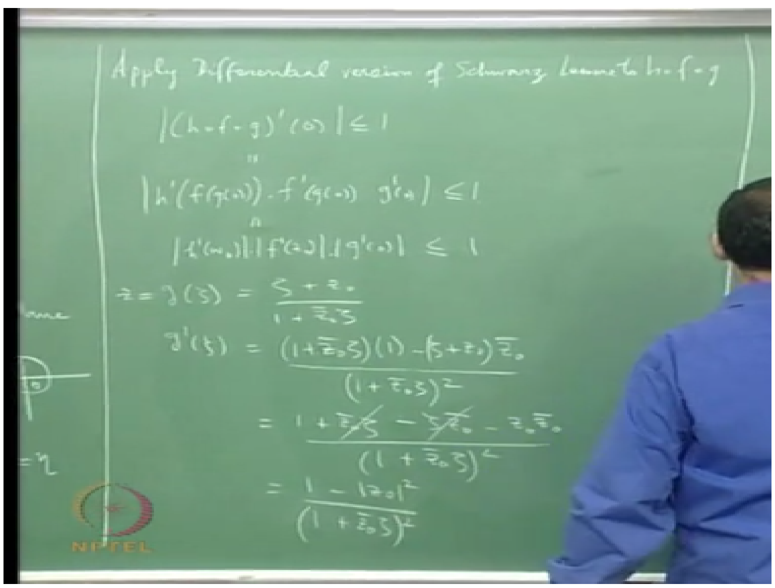

So, the part I want to make is that if you compute this, if you compute it and you will get this inequality okay.

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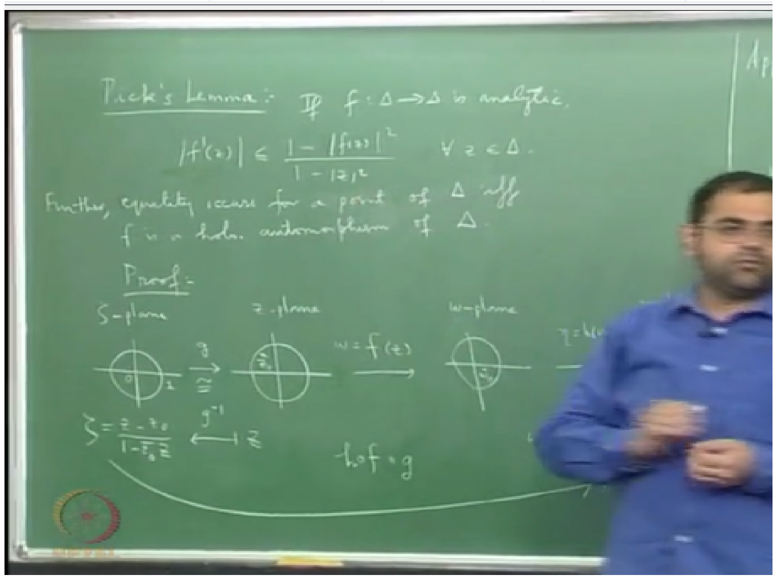

So, let us compute it see what is so what is the expression for g, so what is g of so g is like this g is a function of zeta and z is g of zeta alright and what is g of zeta, g of zeta is just zeta+z0 by 1+ z0 bar zeta, this is the inverse of g okay, you can check that this is the inverse to this alright. And if you now just differentiate calculate the derivate using the quotient rule.

So, you will get differentiate with respect to zeta you will get this is 1+z0 bar zeta whole square 1+zo bar zeta into I will differentiate this with respect to zeta I will get 1-I keep the enumerator constant, I differentiate this with respect to zeta, I get z0 bar. And so fine so what I get is, I get 1+z0 bar zeta – zeta z0 bar –z0 z0 bar alright divided by 1+z0 bar zeta the whole square. And of course these 2 cancel out, so I get 1- z0 z0 bar is mod z0 the whole square by 1+z0 bar zeta the whole square.

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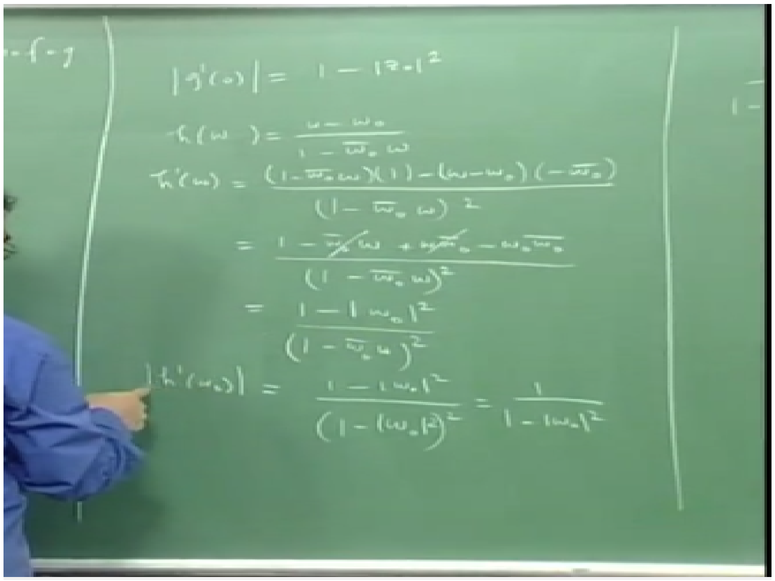

So, you know you calculate mod g dash of 0 which is what we want mod g dash of 0 you put zeta=0 you will get 1-mod z0 square okay, that is what you will get for mod g dash of 0 that is the expression. And then I will have to calculate what h dash of w0 is now what is h of w0, h of w0 is this function , so what is h of w, h of w is neta it is just this expression, so h of w is just w-w0 by 1-w0 bar w.

So, if you calculate the derivative again by the quotient rule I will get the following denominator square, denominator constant derivative of the numerator with respect to w it is going to be 1-numerator constant derivative of the denominator with respect to w is going to give me –w0 bar okay. And so next point out I will get 1-w0 bar w- so I will get + w w0 bar –w0 w0 bar divided by 1-w0 bar w in the whole square.

And these 2 will cancel I will implicate 1- mod w0 the whole square because w0 w0 bar is mod w0 the whole square by 1-w0 bar w the whole square. So, mod h dash of 0 h dash of w0 (()) (04:36-04:47) and I let us write I will **I** should get this 1-mod w0 square. So, you know now you plug in these values in this equation in this equation.

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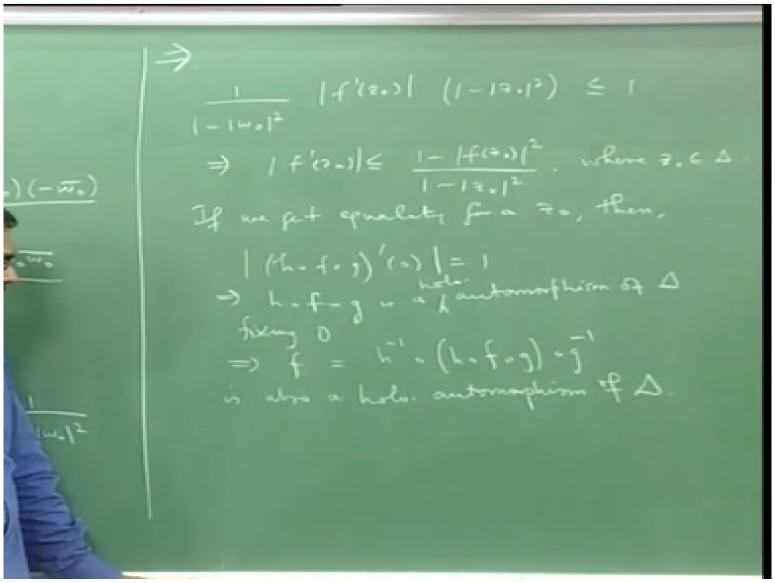

And what you will get is I will get 1 by 1-mod w0 the whole square that is the value of the derivative modulus of the derivative of h at w0 and then I write mod f dash of z0 as it is and I have to plug in mod g dash of 0, mod g of 0 is 1-mod z0 the whole square, this is less than or equal to 1. So, that gives me what I want I get mod f dash of 0 is less than or equal to 1-mind you w0 is fz0 by 1-mod z0 the whole square okay.

And that is the statement, that is the inequality of Pick's lemma okay. So, and what you should understand is that now the of course this in when I do this calculation I have fixed I have simply taken z0 to be any point in the unit disc and I have taken w0 to be it is image okay, z0 is an arbitrary point, so that that inequality holds for any z0 in delta and therefore I can replace instead of z0 I can put z where z belongs to delta.

And therefore I get the inequality of Pick's lemma okay. Now , so the only thing that I have to tell you is that you get equality if and only if f is a holomorphic automorphism of the unit disc. So, and that to for a single z0 if we get equality for a z0 then if you get equality here that means you are actually getting equality here okay.

And but then you know in Schwarz's lemma both the differential version and the original version of Schwarz's lemma you always get equality only if the automorphism is an automorphism only if the analytic function is an automorphism okay. So, you get mod so if I write that out h circle f circle g derivative of the 0=1 implies h circle f circle g is an automorphism of delta fixing 0 using the origin.

And this implies that because you know h and g are also automorphism okay you will get that f which is just h inverse composition h circle f circle g composition g inverse okay is also an automorphism of of course when I see automorphism holomorphic also a holomorphic automorphism of delta okay. So, you see the what is the both the usual version of Schwarz's lemma from the unit disc to the unit disc and the differential version.

Both are statements about inequalities okay both give you inequalities and they tell that you can get an equality only in the case when the function that you are considering from the unit disc to the unit disc is an automorphism you get equality only when it is an automorphism if it is not an automorphism by that if it is not an automorphism is supposed to be self map okay, map from a given set back to itself.

So, if you have an analytic function from the unit disc to the unit disc which is not an isomorphism namely not an automorphism. Then you will get only a strict inequality at every point in the Schwarz's lemma's statement okay and the Schwarz's lemma itself says both the differential form and the usual form of the Schwarz's lemma says that if you get equality even at 1 point okay which is at 1 point is this in the differential version if you get equality of the derivative at the origin with 1 modulus of the derivative at the origin with 1.

Then the function has to be an automorphism okay, so and also the earlier version the usual version of the Schwarz's lemma also says that. That whenever you get an equality for point which is different from the origin okay then the analytic function has to be an automorphism. So, that condition will tell you that this h circle f circle g which is the function which we applied the differential version the Schwarz's lemma.

That this will be an automorphism but then you can get f from this function by pre-composing with h inverse on the left and post-composing with g inverse on the right okay. And that is possible because h and g are of course Moebius transformations they have inverses and therefore so this is an isomorphism this is an isomorphism this central thing is an isomorphism, this also an isomorphism and composition of isomorphism is again an isomorphism, so you will get f is an isomorphism.

So, that proves Pick's lemma okay, so pick's lemma tells you that you will get equality here for a single z0 if and only if you get equality there for all z for all z0, for every z okay. And that will happen if and only if the function is an automorphism, so if you so in other words if f is not an automorphism of unit disc then there will be strict inequality here, it will mod f dash z0 is strictly less than this quantity on the right side okay.

So, this is Pick's lemma and as you can see it is just a generalization of the differential version of Schwarz's lemma. Because in this if I put z0=0 I get the differential version of the Schwarz's lemma which says that modulus of the derivative at the origin cannot exceed you know it cannot exceed 1 okay. If I put z0=0 and assume that f takes 0 to 0 that is if I put z0=0 and assume that f of z0 is also 0.

Then if you put that here you will get mod f dash of 0 is less than or equal to 1, that is the differential version of Schwarz's lemma. So, Pick's lemma is just a generalization of the differential version of Schwarz's lemma okay. But the point is that it is the key to so called hyperbolic geometry on the unit disc which is what we have to study see. So, let me again remind you what we did earlier was that you know.

We were we are on our way to prove the Riemann mapping theorem okay and the Riemann mapping theorem what we have suppose to do is we are suppose to start with a simply connected domain which is not the whole complex plane. And you are suppose to map it holomorphically isomorphically on to the unit disc, the first step that we achieved was to map it holomorphically isomorphically onto a sub-domain of the unit disc okay.

So, this was possible because the domain was simply connected and it was not the entire complex plane okay. So, we reduced the mapping problem to a sub-domain of a simply connected sub-domain of the unit disc okay alright. So, you have to now we are reduce to proving that given any simply connected sub-domain of the unit disc, you can map it conformably on to the unit disc okay.

So, our problem is completely reduced to studying sub-domains of the unit disc and so in other words you have to study the unit disc carefully. And how we are going to do it or the way we are going to do it which will help us is study hyperbolic geometry on the unit disc and the hyperbolic geometry depends on so called hyperbolic metric and the hyperbolic metric the key to the hyperbolic metric is the Pick's lemma.

So, which is a nice generalisations of Schwarz's lemma okay, the differential form of Schwarz's lemma. So, you know that is how it is enters into the discussion of the proof that we are looking at the Riemann mapping theorem alright. So, now we go onto study hyperbolic geometry, so **so** let me do that.

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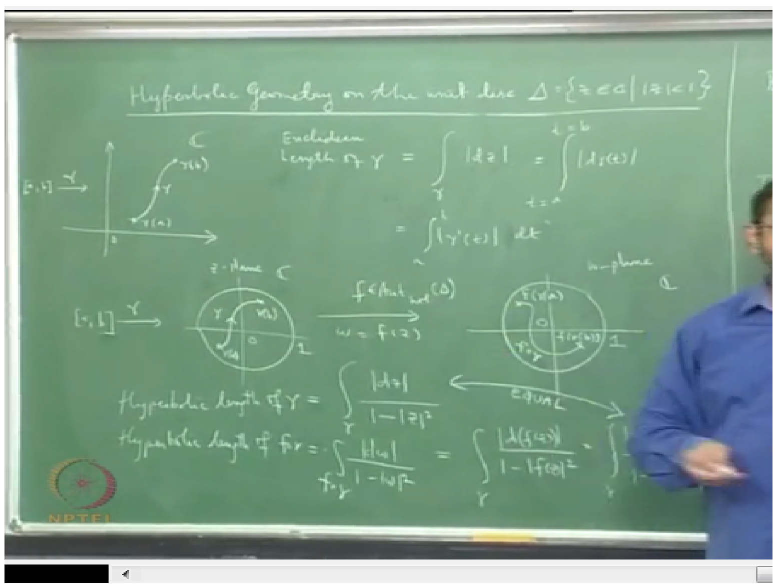

So, this is hyperbolic geometry on the unit disc delta, so this is open unit disc centred at the origin radius 1. So, you know so you see so let us begin by recalling certain facts you know if you have so suppose this is a complex plane and suppose you have an arc suppose you have a piece wise smooth arc or a contour to which is just image of the unit interval or **or** a any closed interval on the real line by a function gamma which is piece wise differentiable okay and which is continuous okay.

And such that the derivatives are also piece wise the derivatives are also continuous okay, so if you take a contour from this point which is gamma of a starting point to gamma of b this is my path gamma or contour then you know how to get the length of gamma and I will stress it I will see it I will put in I will just prefix it by Euclidean okay.

Because this is the length in the usual sense arc length, you know what is the formula for the Euclidean length of gamma all you have to do is you simply have to integrate over gamma mod dz, this will give you the length of gamma alright. And what is that integral I mean it is this integral is well you can also substitute you can this is from you know t=a to t=b modulus of d gamma t.

And that will be just integral from a to b mod gamma dash of t into mod dt okay and of course t is increasing so you do not have put this mod here. So, this is the Euclidean length of a **of** an arc okay. Now what we are going to do is we are going to take a special case we are going to look at this arc we are going to look at such arcs or contours inside the unit disc okay.

So, you are going to have a situation like this you have this closed interval a, b finite closed interval on the real line and you are going to have this path or contour gamma and see the point is that this gamma lands inside the unit disc. So, you know it is something like this, so this is the unit disc delta and this is the complex plane again alright and now I am going to again this is gamma of a, this is gamma of b and this is my path gamma.

And what I am going to do I am going to something new instead of defining the Euclidean length of gamma which you know is this you integrate mod dz over gamma I am going to define the hyperbolic length of gamma okay that is only in the special case when the path is a path in the unit disc okay. So, **so** here is the definition hyperbolic length of gamma is you see what you do is it is also an integral over gamma okay, see if I put integral over gamma and put mod dz I will get the Euclidean length okay.

If I put integral over gamma and if I integrate mod dz I will get the Euclidean length but I will do is I will integrate mod I will integrate 1 by 1-mod z the whole square into d mod dz. So, I am adding this factor 1 by 1-mod z the whole square which is the hyperbolic factor okay. So, this is called the hyperbolic length of gamma okay and what is so special about this expression.

This special thing about this expression is you see if I now take f to be an automorphism holomorphic automorphism of the unit disc that is it is a map from the unit disc to the unit disc which is holomorphic injective bijective holomorphic, so it is inverses also holomorphic. So, it is a holomorphic automorphism of unit disc to unit disc and then you know so what will happen is this you know this f will map .

So, this is my so this is a map w=fz okay and so this is the z plane and here I have the w plane alright. And what is going to happen is that because f is an automorphism 1 to 1 onto inverses also holomorphic what is going to happen is that the image of this path is also going to be a simple path is also going to be a path inside the unit disc. So, what I am going to get is I am going to get another path like this starting point will be f of gamma of a.

And the ending point will be f of gamma of b and I will get this path which is gamma followed by f okay. So, it is this you first apply gamma then you apply f then you get a path from a from this closed interval a, b into the unit disc and what is that path that is that path is just gamma circle f yeah it should be f **f** composition gamma f circle gamma that is right, it should be f circle gamma right.

So, I get this so the path gamma is map by f isomorphically onto the path f circle gamma, now you see the beautiful thing you calculate the hyperbolic length of f circle gamma okay what is the hyperbolic length of f circle gamma, well it is by definition integral over f circle gamma of mod dw by 1-mod w the whole square is this the definition of hyperbolic length where I am using the fact that my variable here is w and the variable here is z.

So, I am using the correct variable alright but you see watch carefully what is so now comes if now comes the you know importance of Pick's lemma okay, now comes the importance of Pick's lemma, you see what does Pick's lemma say.

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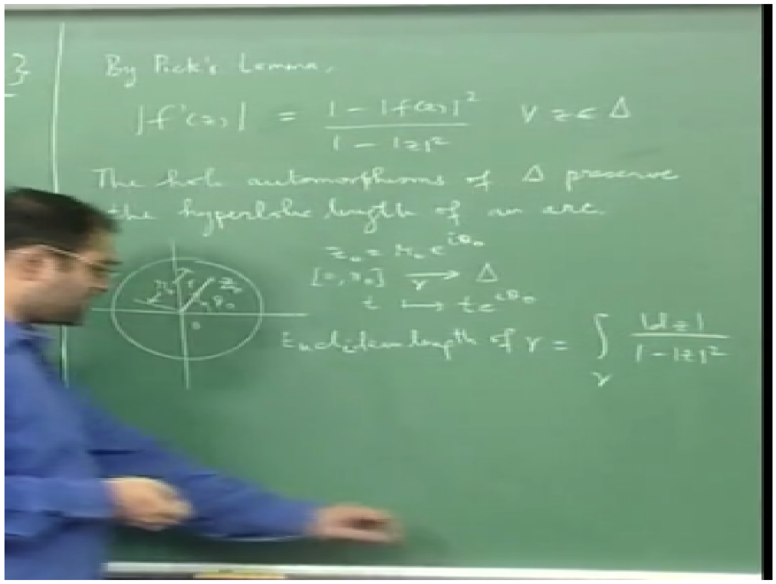

See by Pick's lemma what you will get is mod f dash of z is is equal to 1-mod fz the whole square by 1-mod z the whole square for all z in the unit disc, you get this, you get equality in Pick's lemma because f is an automorphism of the unit disc, Pick's lemma says that you get the inequality of Pick's lemma will become an equality if they map is if and only if the map f is a automorphism okay you get this. But then you see in this integral you know I can make change of variable by putting w=fz.

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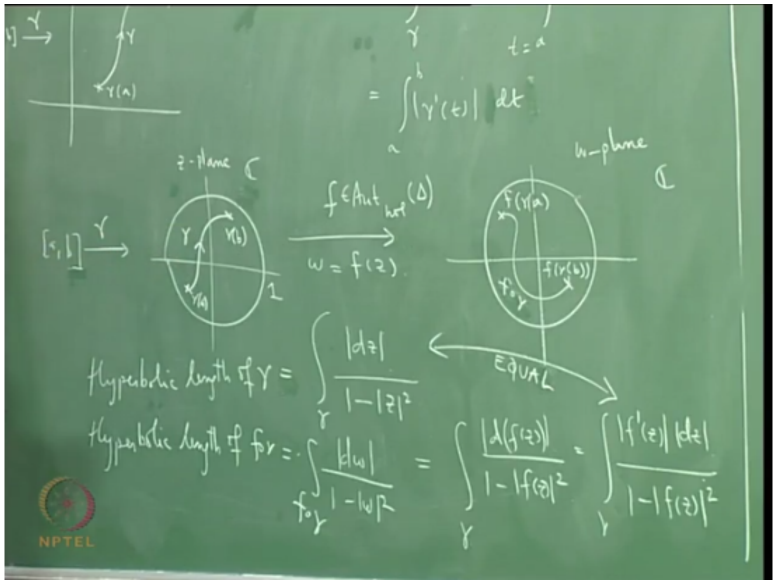

If I make a change of variable then this integral is the same as integral over gamma okay mod d fz by 1-mod fz the whole square okay. But what is this, this is the integral over gamma d of z is f dash of z dz, so I will get mod f dash of z mod dz by 1-mod fz the whole square, this is what I will get okay. If I make change of variable from w to z using w=fz I will get this.

But what is this equal to by Pick's lemma mod fz of z by 1-fz te whole square is simply 1 by 1-mod z the whole square. Therefore what you get is this is the same as this, these 2 are equal these 2 are this expression is same as this expression because of Pick's lemma because of the equality in Pick's lemma which comes because f is an automorphism of the unit disc alright, so what is the moral of the story, the moral of the story is the following.

The moral of the story is if you define the hyperbolic length of an arc or contour in the unit disc, the hyperbolic length will not change if you apply an automorphism of the unit disc, you whether you take the hyperbolic length of gamma or whether you take the hyperbolic length of it is image under automorphism of the unit disc, you will continue to get the same hyperbolic length.

Therefore the so we expresses by saying that Pick's lemma the equality in Pick's lemma actually asserts that for the automorphism of the unit disc preserve the hyperbolic length. The equality in Pick's lemma assert that you know automorphism of unit disc preserve the hyperbolic length okay. So, let me write that down the automorphism, the holomorphic automorphism of delta reserve the hyperbolic length of an arc.

So, this is the geometric you know statement concerning the equality in Pick's lemma okay and that is the importance of this expression. Instead of just integrating over mod dz which will give you the Euclidean length you integrate over mod dz by 1- you integrate mod dz by 1-mod z the whole square okay, that is the that gives the hyperbolic length right.

Now you know it for the hyperbolic length is I mean the unit disc is anyway as for as Euclidean at spaces concern are the that is the plane is concerned unit disc is bounded and you know if you take the ordinary length. The ordinary length between any 2 points is going to be finite of course it cannot exceed 2 which is the diameter of the unit disc alright if you take a straight line segment the length is less than 2 okay.

But what about hyperbolic distance, so you can it is rather curious we can make a computation, you know if you take the unit disc and you know take a point 0 take the point z okay, take this straight line to 0 and z okay. And then try to calculate what this length is what is the Euclidean length, Euclidean length, so you know to calculate the Euclidean length first of all I need a parameterisation I must think of this is a path.

So, you know what I do is I just map 0, so if z suppose I call this point as z0 and zo is r0*e power i theta 0 okay where r0 is this length which is actually the Euclidean length. And theta0 is this angle, so this angle is theta0 and this length is r0 from here to here okay, then you know how to parameterize this path. So, this path can be parameterized as 0, r0 map 2 which is map 0, r0 to t to the unit disc by simply sending t to t e power I theta0 okay when you put t=0 you will get the origin, when you put t=r0 you will get z0.

And you will get this straight line segment joining the origin z0, so this is your path gamma now, the straight line segment from 0 to z0 where z0 is a point of the unit disc what is the Euclidean distance, Euclidean length of is going to be integral over gamma mod dz by 1-mod z the whole square, this is the sorry this is I should not 1 by I should not put this it just mod dz.

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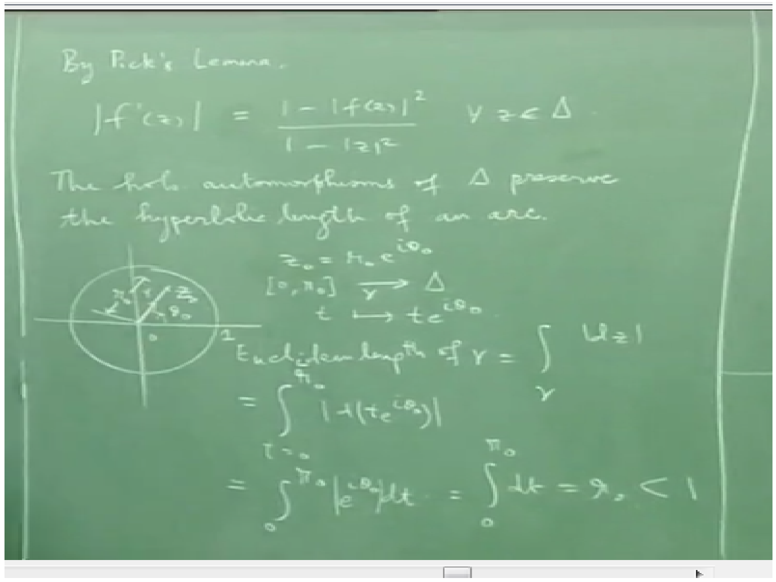

So, what I will get is well if I substitute for the x-for gamma of t it is a z is gamma of t, so I will get t=0 to r0 and I will get here mod d d of gamma of t but gamma of t is t power i theta0 okay. And if I simplify this I will get integral 0 to r0 I have to differentiate this I take differentiate and the variable of integration is t, so I will get I differentiate this respect to t which will give me e power i theta0 .

And then I will get dt and then I will have to put a mod and of course mod e power i theta0 is 1. So, I will simply get integral dt from 0 to r0 and I am going to get r0 which is what I expect the length of this straight lines even from 0 to z0 is r0 which is Euclidean length okay. Now let us calculate the hyperbolic length what is a hyperbolic length and of course I should tell you that of course r0 is strictly less than 1 because z0 is a point of the unit disc, r0 is strictly less than 1 alright.

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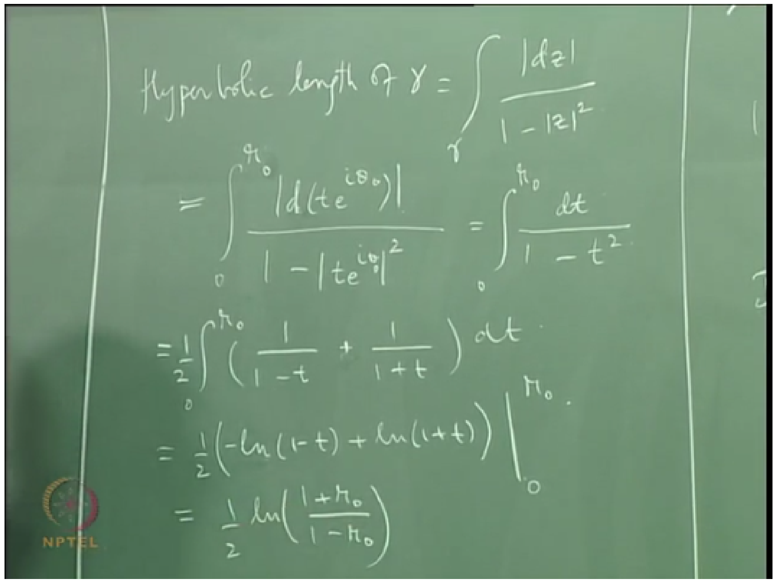

But what is the hyperbolic length of gamma, hyperbolic length of gamma well it is integral over gamma mod dz by 1-mod z the whole square, see this is the formula for the hyperbolic length okay. So, if you calculate that so you will get et is that, so well I will get integral from 0 to r0 so again substitute this I will get mod d d of d e power i theta0 by 1-mod d e power I theta0 the whole square, this is what I will get.

So, I will get integral from 0 to r0 as usual this is going to give me dt alright, so and the denominator I am going to just get 1-t square okay and you know how to integrate this, you split as partial fractions you know it is 1 by 1 –t-1 by 1t if I am not wrong maybe I should a + here let me I will get 1+t+1-t is 2 by 1-t square, so I will have to divide by 2, so this is what I will get dt.

And you know what this is going to be this is going to just give me , so this is half 1 by 1-t is log 1-t so this is lan if you want lan of course t is less t is positive, so this lan 1-t into -1 and here I am going to get +lan of 1+t and I am going to take limits from 0 to r0. So, this is just lan 1+t by 1-t alright, so I am going to get half lan 1+r0 by 1-r0, this is the hyperbolic length.

So, the hyperbolic length, the Euclidean length is r0 okay which is the modulus of z0 whereas hyperbolic length is you have a fine funny expression it is half lan 1+r0 by 1-r0 and you can see something the Euclidean length is finite okay it is bounded by 1. But you know if z0 tends close to the border of the unit disc okay if z0 gets close to the boundary of the unit disc, then r0 gets close to 1 and as r0 gets close to 1 this approaches infinitely okay I mean this denominator approach is 0 alright.

And therefore this quantity approach is infinity because r0 is going to tend to 1 from the left okay, so this is going to approach 0+. So, this quantity is going to approach infinity +infinity and lan of that is going to go into +infinity, so the moral of the story is the hyperbolic length will tend to infinitely as the point z0 moves to the edge of the unit disc as it goes to the units circle okay.

So, you know the so this is the beautiful fact about the hyperbolic length, the hyperbolic length makes this in as far as the hyperbolic distances concerned this is not a bounded thing the unit disc is not bounded okay, the disc even the straight even the even a segment if you take straight line segment if you compute radial segment it is length tends to infinity in the hyperbolic distance.

If the end point goes closer and closer to the unit circle, so this is the point about the hyperbolic metric it makes the hyperbolic distance it makes this in the sense of the hyperbolic distance it makes unit disc unbounded okay. So, that is one fact that you have to notice alright, so well now you know I have to , so I will tell you that what is that we need actually we are looking for a statement like this.

We are looking for a statement which says that if you take any analytic function from the unit disc to the unit disc, then if it is not an automorphism of the unit disc then it acts like a contraction okay. So, what I have defined so you know to define the notion of a contraction I have to tell you what a metric is because you know a contraction map is defined between metric spaces, it is defined from 1 metric space to another.

And a mapping set to be a contraction map if you know it decreases distances you take 2 points in the source space they have certain distance but you take their images and then you measure the distance becomes smaller. So, if this happens for a map it is called a contraction map and essentially the statement that I need for the proof of the Riemann mapping theorem is a statement that if you take a analytic map of unit disc to itself which is not an automorphism.

Then it will necessarily be a contraction map but contraction with respect to what metric it is with respect to the hyperbolic metric which I am going to define now okay, what I have defined so for is just hyperbolic length of an arc in the unit disc I am going to define the hyperbolic distance between 2 points in the unit disc okay, so let me do that next.

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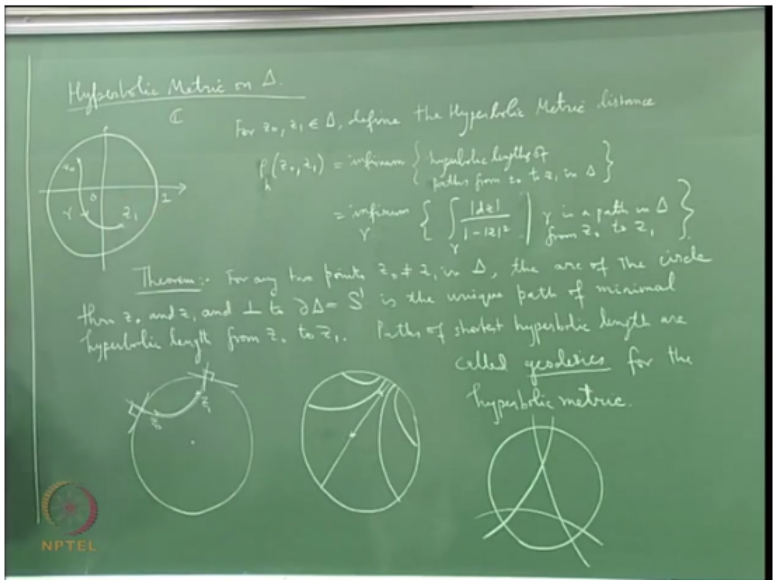

So, hyperbolic metric on the unit disc, so here is the definition of hyperbolic metric, so here is my unit disc okay and of course here also I should have mark this is 1, here I should have marked it as 1 as well, this is the origin, this is the origin alright. So, **so** you take 2 points in the unit disc alright take 2 points of unit disc and what you do is the following for z0, z1 it delta define the hyperbolic metric, rho .

So, hyperbolic metric or hyperbolic distance, distance function, the distance from z0 to z1 I am using the single rho okay to be you know maybe I will use rhos of h to insist that this hyperbolic and you know what I do, I do the following thind simply join z from z0 to z1 you choose any path contour gamma measure it is hyperbolic length and then minimise over all such possible paths.

So, the hyperbolic distance from z0 to z1 is the least is the least of hyperbolic lengths of paths from z0 to z1 various paths from z0 to z1 inside the unit disc you measure they are hyperbolic lengths and then you take the minimum, you take the infimum okay. So, here is the definition this is equal to infimum of hyperbolic length of paths from z0 to z1 in the unit disc okay.

So, this is the hyperbolic metric, so in other words this is infimum over all gamma such that the hyperbolic length is given by integral over gamma mod dz by 1-mod z the whole square, this is the hyperbolic metric where of course gamma of such that gamma is a path in the unit disc from z0 to z1. So, this is the hyperbolic length okay**,** so you know I am taking of course all lens all these integrals are non negative.

And you know so this infimum does exist alright but what is it what is this infimum and you know does the infimum value does it correspond to actually length of a particular path that is the question and the answer is yes even any to so here is the important statement about hyperbolic geometry. So, even any 2 points in the unit disc there is a special path from passing through from this between these 2 points which is called a hyperbolic it is the path of shortest hyperbolic length from .

I mean between the 2 given points and what is that path the answer to that is the that path is a circle passing through those 2 points which is orthogonal to the unit circle okay, so that is a theorem. So, the theorem is here is the very important theorem, the theorem is for any 2 **2** points z0 not equal to z1 in delta the arc of the circle through z0 and z1 and orthogonal to the unit circle okay is the unique path of minimal hyperbolic length from z0 to z1.

So, this is the theorem, so the theorem is that what is this hyperbolic distance it gives you see the hyperbolic distance is defined by some minimisation it is the minimum you are suppose to take all possible path inside the unit disc from z0 to z1 measure their hyperbolic lengths and take the minimum okay which seems a very it is not a definition that will help you to make calculations because you have to find minimum.

But the theorem makes it clearer it tells you what is that path which will give you the minimum hyperbolic length that path of minimum hyperbolic length is nothing but the arc of a circle passing through these 2 points and which is orthogonal to the unit circle namely it is where it hits the unit circle it will hit at 90 degrees okay, you know 2 you can also talk about the angle between 2 curves at a point at an intersecting point, it is bit of by definition the angle between their tangents at that point.

So, we circle are orthogonal if they intersect at say 2 points and at each point of intersection the tangent to the 2 circles are perpendicular to each other alright. So, you know so the picture is like this you know if I take the unit circle if I take the unit disc you know if I take a point if I take a 2 points like this okay then you know my hyperbolic will be something like this, it will be this will be the from here to here.

And that is because this is the circle which passes through these 2 points okay and which hits the unit circle at 90 degrees. So, you know so this is z0, this is z1 and **at** at this point if I draw the tangent to the given to this circle and the unit circle this will be 90 degrees. Similarly here if I draw the tangent from here and here this will be another 90 degrees, so this will be the hyperbolic path.

And the beautiful thing is that you know if you draw all these hyperbolic paths of numeral minimal hyperbolic lengths which are called you will get things like this see you know if you take 2 points along a diameter okay. Then the hyperbolic the geodesic will be the diameter itself any diameter is a geodesic**.**

Because if you take 2 points if you try to find the circle passing through 2 points which lying on a line in principle there is no circle but you think of it as a circle with you think also straight lines as circles with the third point at infinity okay. So, if so the point is that the geodesic will look like this you know this will be 1 geodesic any diameter will be a geodesic then you know if you go the little to the left the geodesic will become like this.

And you know if you get smaller the geodesic will become smaller, this is how the geodesic will look like okay and the fact is that any diameter will be a geodesic okay and the theorem says that these are the paths of shortest length, that is how you get the path of shortest length. So, this the theorem that we will have to give a proof of and we will do that in the next lecture.

So, **so** let me write let me add here paths of shortest hyperbolic length are called geodesic for the hyperbolic metric and so let me finish with one important statement you see if you are looking at the Euclidean distance Euclidean metric okay. Then the geodesic are all straight lines okay, if you take any 2 points in Euclidean space what is the shortest what is the path of shortest lengths it will just be the line segment joining those 2 paths, straight line segment.

So, the geodesic in the Euclidean metric they are just straight lines okay alright and straight lines are important for Euclidean geometry right. In the same way these geodesic that we get for the hyperbolic metric okay they will play the same role as straight lines play for Euclidean geometry okay. So, all the axioms all the Euclid's axioms except the parallel axiom that hold for straight lines, I mean parallel axiom also holds for it is also taken for Euclidean geometry.

But this all those axioms will work with the geodesic for the hyperbolic metric except that the parallel axiom you have to throughout the parallel axiom okay, all other axioms that you have for straight lines okay . The same axioms will hold good for the hyperbolic geodesic, so the so all these curves on the unit disc they are the analogues of straight lines on the Euclidean plane.

The analogues of the straight line on the Euclidean plane which are important for Euclidean geometry, the analogues here are these curves the hyperbolic geodesic and you know you can check a lot of statements like you know if you have 3 lines which you know if you take any 2 lines which are if you take 2 lines if they are not 1 in the same .

And of course you know if they are not parallel then they will intersect at you know 1 point, 1 point in the finite plane and of course if you think of infinity as a point then they will also intersect the infinity alright. And the same way here also you can check that if you take 2 geodesic which are not you know one in the same they will hit at one point okay.

And if you take 3 lines you can use 3 lines to form a triangle and every triangle is formed by 3 lines okay which are the lines passing through the lines the sides of the triangle. In the same way you can also define hyperbolic triangle, a hyperbolic triangle will be something like this you know , so you know I can, so if I draw a hyperbolic triangle it will be like this it is given by 3 hyperbolic D geodesic.

So, 1 like this, 1 like this and 1 like this, so this is the hyperbolic triangle okay and you know that in the Euclidean geometry the sum of 3 angles of the triangle is equal to 180 degrees what will happen in hyperbolic geometry is that the is the sum of 3 the 3 angles of a hyperbolic triangle will be less than 180 degrees okay and so you will have all these nice things happening differently from what you know in Euclidean geometry.

So, hyperbolic geometry will give you gives set of properties okay and the basis for all this is the so called hyperbolic length which is defined for an arc but the key to the fact that hyperbolic length does not change under an automorphism of the unit disc is the equality in Pick's lemma okay. So, Pick's lemma we called in Pick's lemma is a beginning point for a whole geometry okay.

And now I will now let me tell you if you take this hyperbolic metric then if you take any automorphism of from the unit disc to the unit disc any automorphism of unit disc to the unit disc will be an isometric with respect to the hyperbolic metric okay, that is what you will get okay. So, the whole beautiful point about Pick's lemma will be that you know any automorphism of unit disc will actually be an isometry of the hyperbolic distance of the hyperbolic metric.

And any map which is not an automorphism of the unit disc will be a contraction with respect to this hyperbolic metric and that is the statement that we need to proceed with the proof of Riemann mapping theorem okay which we will do in the next lecture.