Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-34
Differential or Infinitesimal Schwarzs Lemma, Picks Lemma, Hyperbolic Arclengths, Metric and Geodesics on the Unit Disc

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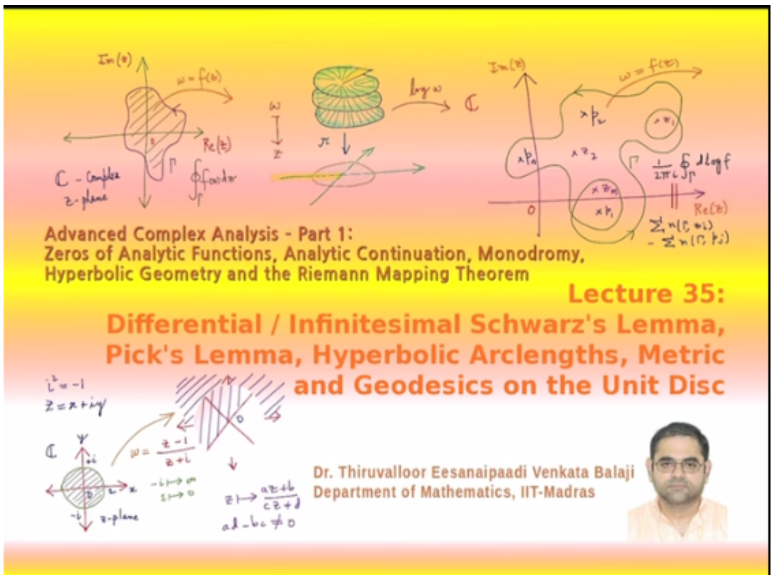

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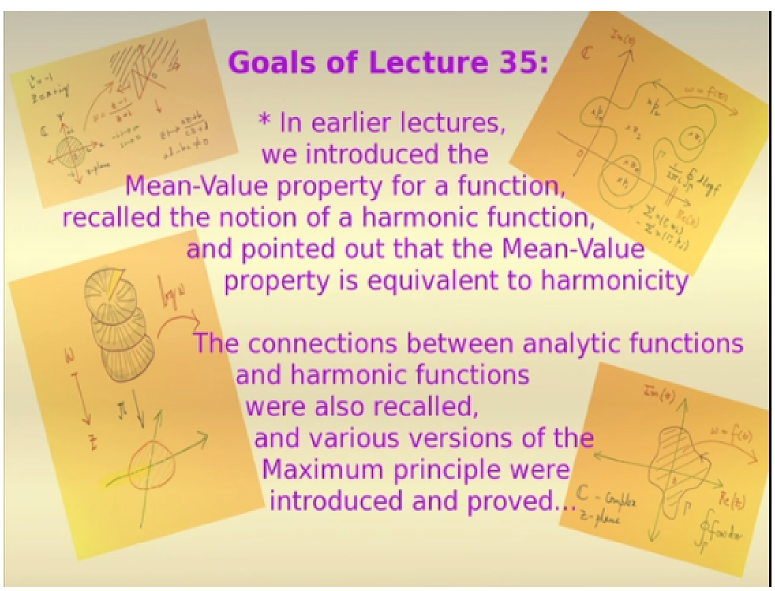

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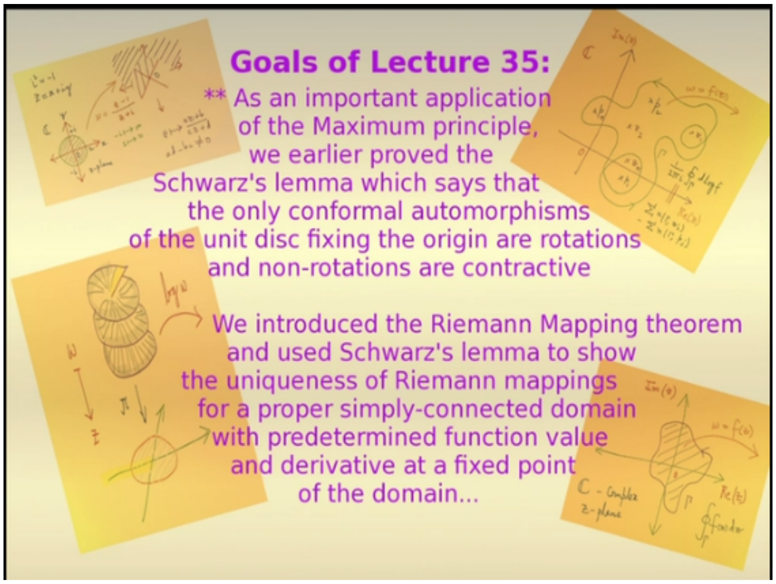

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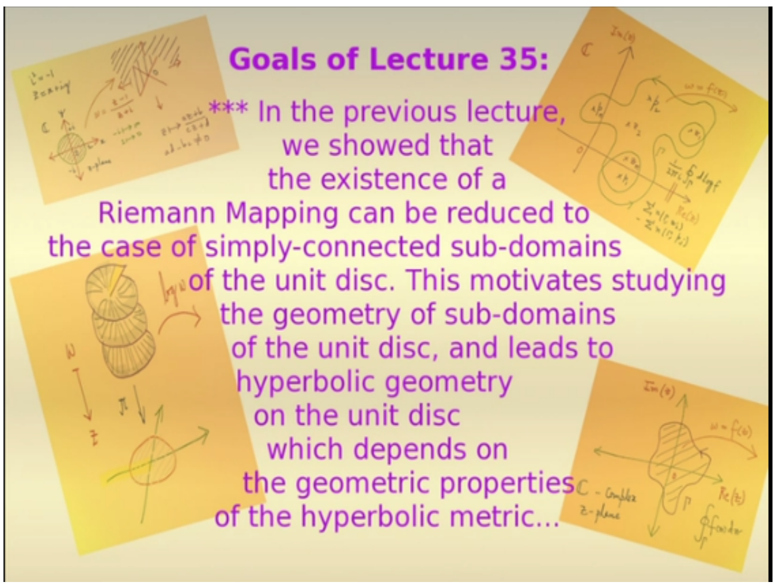

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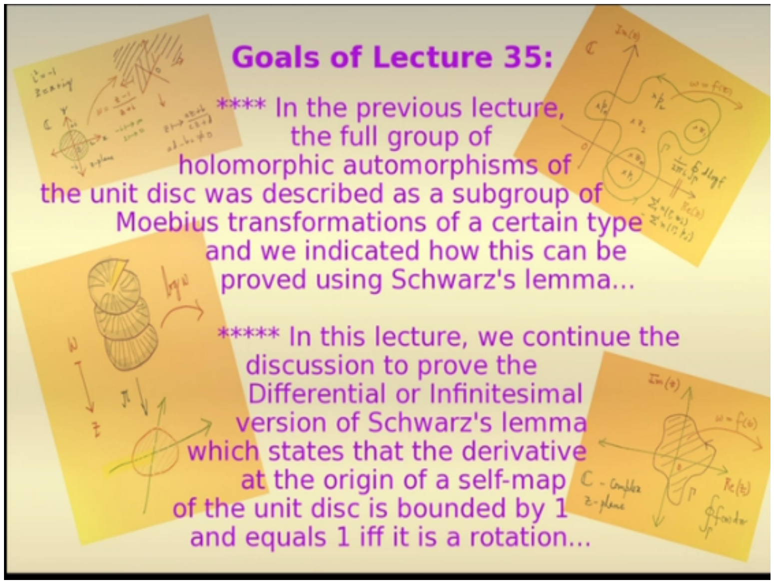

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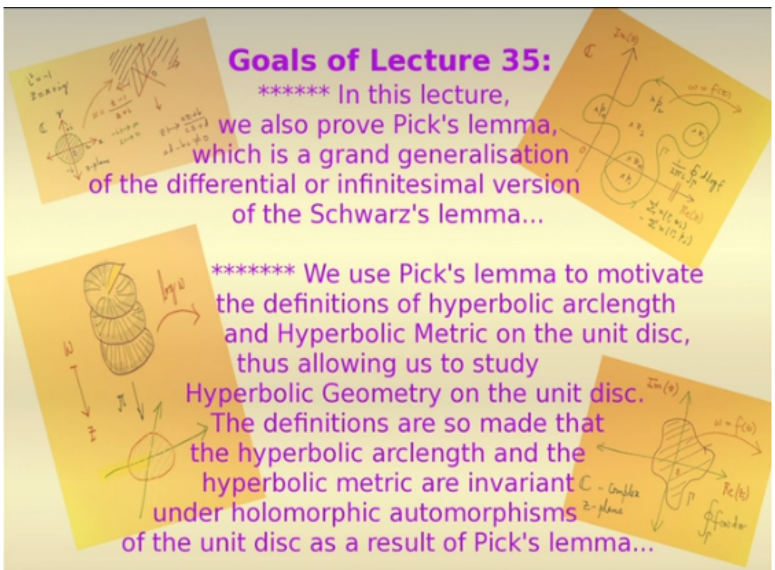

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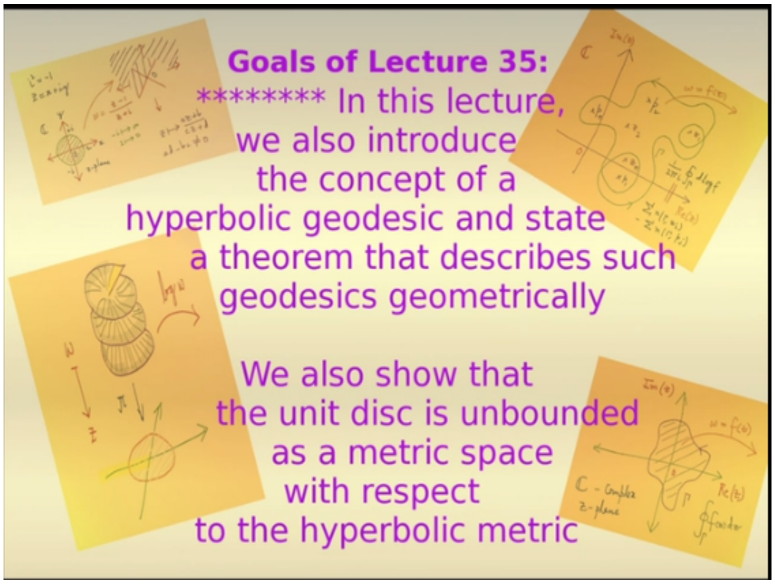

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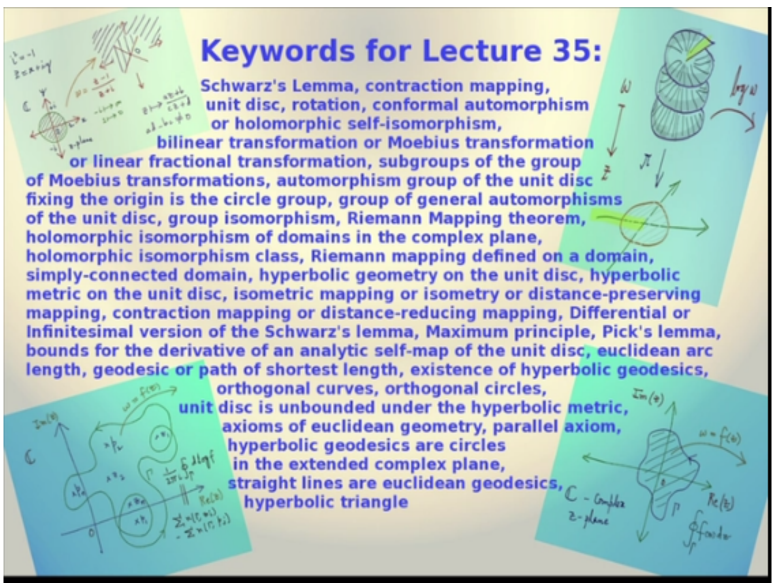

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Okay, so this is lecture number 35 is a continuation of the previous lecture so what we have discussing was the proof of the following statement.

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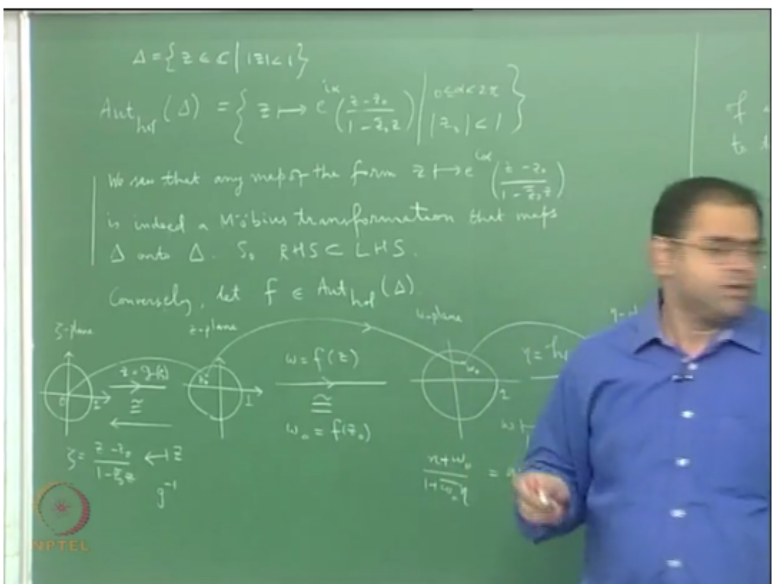

That the automorphism, the holomorphic automorphism of the unit disc, so delta is unit disc and we were trying to show that the set of holomorphic automorphism of unit disc are exactly maps from the form they are moving as transformations. They are given by maps in the form z going to e to the i alpha z-z0 by 1-z0 bar z where alpha is of course an angle and z0 is an element in the unit disc okay.

And what we saw in the towards end of the last lecture was that any we saw that any map of the form is it going to e to the i alpha into z-z0 by 1-z0 bar z is indeed a **a** Moebius transformation. So this Moebius is this o with to dots on top of it, it is called o who got on top of it and it is suppose to be the German symbol for oe put together okay, so this is Moe **oe** half o and half e okay.

So, this is Moebius transformation it is of course also called as you know bilinear transformation or linear fractional transformation okay so it is indeed a Moebius transformation that maps the unit disc on to the unit disc okay. So, the right hand side is of course contained in the left hand side, so my aim into here is here what I will have to show is that conversely any holomorphic automorphism of the unit disc is of that form okay.

So, what I am going to do is , so I conversely let f be a holomorphic automorphism of unit disc, so you what is happening is see you have a f here all I know about f is it is a isomorphism it is holomorphic, it is inverse is also holomorphic and it goes from the unit disc to the unit disc and you know and I have to show that f is of this form okay, so I have to find an alpha and z0 such that f of z is actually e to the **i** alpha into z-z0 by 1-z0 bar z alright.

Now what you must understand is that see if f takes 0 to 0 then we are already done because if f takes 0 to 0 then it will be an automorphism of the unit disc which fixes the origin. And you already seen as a corollary to starts is lema in earlier lectures that it has to be a rotation. So, you so f will be of the form e power i alpha times z okay which is just this map with z0=0 okay, therefore the problem is when 0 does not go to the origin right that is a issue.

So, what we do is you know we just take see we well let us look at what 0 goes to alright. So for that matter you know I can fix any point z0 okay and suppose z0 goes to the point w0, so my map is w=f of z, I take a point z0 it goes to the point w0 alright. Now what I am going to do is I am going to write on this side a holomorphic automorphism of the unit disc which takes 0 to z0 okay, I am going to do that.

And you know what I am going to do I am going to use the first statement I am going to use this statement. So, what I am going to do is it is very very simple you know if I take this map to be z going to z-z0 by 1-z0 bar z suppose I take this map okay. Then I already told you that this map is a Moebius transformation that maps the interior of the unit disc to the unit disc okay and it will take z0 to 0.

So, the point z0 will go to the origin okay and what I am going to do is this is this map this is going to be the inverse of that map okay. So, you know so I am going to call this as g okay and this is my g inverse and I know g inverse is a map that takes unit discs to unit disc and it will map z0 to 0. Because you know in this if you substitute z=z0 you will get 0 and this a Moebius transformation and I am going to take it is inverse.

So, the inverse map will take 0 to z0 okay and then this f will take z0 to w0 okay because w0 is f of z0 right and then you know I am going to then again apply another map here that will move w0 to 0 alright. And what is that map you know what that map is will call that is h, so and this map is going to be, so I am going to just map w, w0 to 0 and you know what that map is, it is just w going to same formula w-w0 by 1-w0 bar w okay.

This map will map w0 to 0 and it will be an automorphism the unit disc okay, so both for this side and for this side I am using this fact that I have already proved in the last lecture right. And now watch if I take so this is h okay, if I take the composition what will I get, see this is also an automorphism unit disc f is also an automorphism unit disc and h is also an automorphism unit disc.

Therefore if you compose you will get a the composition of all the 3 will give you an automorphism unit disc to unit disc. And it will fix the origin because it will take 0 to 0 that is the whole idea of putting this g on this side and h on that side. So, it will be an automorphism unit disc which fixes the origin which by an earlier corollary of this Schwarz's lemma we have seen it is (()) (09:34).

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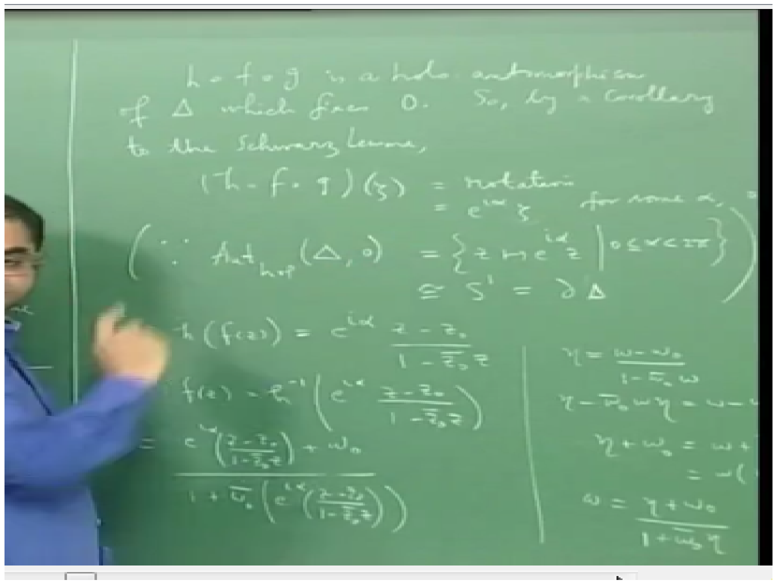

So, what you will get is you will get that if you take first apply g then apply f, then apply h is an automorphism **a** a holomorphic automorphism of the unit disc which fixes the origin okay. But you know automorphism holomorphic automorphism the unit disc which fixes the origin it is a rotation that is the first one of the corollaries of the that is the corollary of the Schwarz's lemma that we have seen okay.

So, by a corollary to the shorts lemma what you will get is that h circle f circle g of w is just going to be a rotation and therefore it is all the firm e power i alpha w okay. Since you know this is something that we have already seen the set of automorphism holomorphic automorphism of the unit disc which preserve the origin. So, this is a subset of this set, these are all holomorphic automorphism bijective holomorphic maps of the unit disc onto itself.

And these are the subset of maps which send 0 to 0 and this is the same as this is identified with this is equal to rotations, the set of all z going to e to the i alpha z where alpha is varying from 0 to up to 2 pie okay. And this is identified with you know S1 which is the boundary of the unit disc we have already seen this and this is what I am using here okay, so h circle f circle g is an element here therefore it is a rotation.

So, it is just the variable multiplied by an e power i alpha for suitable for some alpha okay 0 less than or equal to alpha less than 2 pie. Of course alpha is just an angle and you have to read it mod 2 pie okay. So, I think I will have to worry about my variables here, so here this is the z plane, this is z plane and this is omega plane and let me call this as neta plane, so this is neta .

So, neta is h of w okay and w is fz and here let me use zeta and zeta yeah z is g of zeta, so this is zeta zeta goes to z under z goes to zeta is g inverse and zeta goes to z is g okay. So, here you have the variables, so if I use these variables then I should let me change this variable to neta okay because not neta should be zeta, so, you know you will get so g zeta g of zeta is z okay.

So, I will get h of f of z see h of f of g of zeta is h of f of g zeta but g zeta is z, so I will get f of z, so I will get h of f of z= e power alpha into zeta and what is the formula for zeta, the formula for zeta is z-z0 by 1-z0 bar z okay. So, I will get so f of z will be h inverse of this okay and what is the formula for h inverse it is going to be the map in this direction .

So, it will be h inverse of neta will be what you have to solve this for omega okay, so you have to solve this for omega and what will you get well if I make that calculation somewhere let me make it somewhere here, see I will get neta=w-w0 by 1-w0 bar w, you cross multiply will get neta-w0 bar w neta will w-w0, so I am solving for so, it is neta going to w, I want to write w interms of neta that is h inverse, h inverse of neta is w.

Once I have a formula for h inverse neta I will put I will apply it to this okay, so **so** I will have to calculate w form this or omega fro this. So, I will get omega into so I will get neta+w0=omega+ w+w0 bar w neta which is w into 1+w0 bar neta, so you will get w=neta+w0 by 1+w0 bar neta, this is the formula for w, so this is neta+w0 by 1+w0 bar w, w0 bar neta, so this is the formula for h inverse of neta right and apply it to this**.**

And you will get again an expression of this form okay, you I mean if you must have probably done this such exercises when you have taken a first quotient complex analysis. But it is very easy to do it, if you have not done it, so I will have to apply, so this is mu neta okay, so I substitute it in here I will get e to the i alpha into z-z0 by 1-z0 bar z +w0 by 1+w0 bar into neta which is this e to the i alpha into z-z0 by 1-z0 bar z, this is what I will get okay.

In this whole calculation I have chosen z0 I have chosen sum z0 in the unit disc and I have taken w0 to be image of z0 okay. But you know I can make special choices, for example I can take w0-0 if you want I can reduce the calculation by I can make the calculation simple by simply taking w0=0, w0=0 means I am taking z0, so I am taking the unique point z0 that goes to 0 under f okay.

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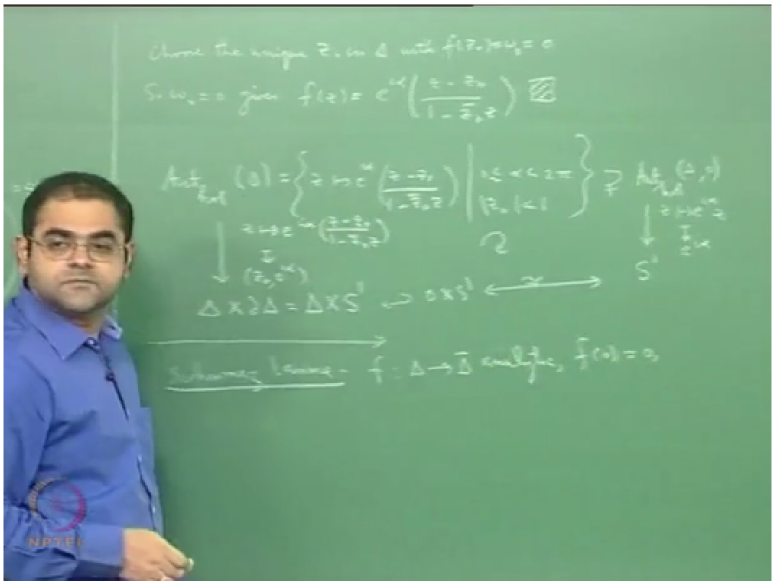

So, you make that assumption choose the unique z0 in delta with f of z0=w0=0, this has to happen because after all f is given to be a holomorphic automorphism of delta. Therefore there is something is map to 0 okay whatever is point is map to 0 you call that as z0 okay. So, if you do this then you are actually putting w0=0 and if you put w0=0 you get f of z in the form that you want okay.

So, w0=0 gives f of z to be simply you put w0=0 you see this is gone, this is gone you will simply get e power i alpha, z-z0 by 1-z0 bar z okay immediately and that is the proof, that is the end of the proof. So, what you have done is you started with an element here f here and showed that it is of this form okay, so these 2 are orient the same okay and so in fact there is so in fact the truth is that this is also a group.

The holomorphic automorphism of the unit disc that is also a group because this is subgroup this is also certain sub group of Moebius transformations, Moebius transformations mind you form a group because you can compose Moebius stand the group law, the group multiplication is just composition okay, so under composition Moebius transformations form a group and this is a subgroup of Moebius transformations okay.

These are the these are certainly I mean these are Moebius transformations of this particular form and they are all they are exactly the automorphism unit disc okay. And it is a subgroup under composition of mappings alright and there is a much more or subgroup among these you have this smaller subgroup which are further those which fix the origin and they have the rotations okay, there are not rotations there is this part is a rotation there is something more here okay.

So, the fact is just like the this subgroup of Moebius transformation that fix the origin or the rotations it can be identified with the boundary okay. That also can be identified in a nice way okay, it can be identified with delta cross the boundary okay. So, you see what will happen is that if you take the holomorphic automorphism for delta that is that I have written down, set of all maps of the form z going to e to i alpha into z-z0 by 1-z0 bar z where alpha is an angle z02 pie.

And z0 is a point with point in the unit disc okay, now from here I can you know I can give you a map into delta cross tou delta which is delta cross s1, s1 is the unit circle okay, set of all complex numbers with modulus 1. So, it is numbers of the form e power i alpha okay and you know what is this map, this map is you know you sent the map you sent the Moebius transformation z going to e to I alpha z-z0 by 1-z0 bar z, you simply send this to a pair where this first member is this z0.

Because you see z0 is an element of the unit disc and the second member is that e power i alpha and because e power i alpha will beyond the boundary of the unit disc which is unit circle okay. And what will happens is that this will be also an isomorphism of groups, this will be an isomorphism of groups alright. So, and what will happen is that this will contain properly the automorphism holomorphism I mean holomorphic automorphism of the unit disc which preserve the origin okay which are only the rotations.

And only rotations means that it will correspond to z0=0, so only rotations okay and what you will get is the same map will give you this isomorphism it will you are going to send z0 going to e to the i alpha z to simply s1 you are going to send this to the element e power i alpha of s1 okay. And this is identified as 0 cross s1 which is sitting inside s1 okay, so the holomorphic automorphism of unit disc are identified with delta cross s1 okay whereas the subgroup of holomorphic automorphism which fix the origin are simply identified with s1.

So, s1 so you have an identification of as a group of both the automorphism of delta and the automorphisms of delta which fix the origin alright, so this is the remark. Now of course this diagram commutes okay and these vertical arrows are both group homomorphism okay, they are just bijective maps they are group homomorphism right. So, now you see we come to so you know see what we have just looked at are the automorphism of the unit disc okay.

The question is what can you say about a general analytic map of the unit disc what are the most general statements you can make. So, I mean what can you say about maps which are not automorphism of the unit disc but which are maps analytic maps of unit disc into itself okay what can you say about such maps. So, I will tell you what the main result is that on the unit disc okay.

There is a metric call the hyperbolic metric okay it is a way of defining the distance between 2 points in the unit disc. Of course you know metric is something that gives you the distance between 2 points in a space okay, so on the unit disc there is a special metric it is called the hyperbolic metric and we will see what that metric is later but the point is what the Schwarz's lemma actually says is that, if you take an analytic function from the unit disc to itself which is not an automorphism.

Then with respect to this hyperbolic metric it is a contraction mapping okay. So, this is the philosophical statement, I mean this is the ideological statement. The ideological statement is I mean this is the statement that we need this is this statement is needed for proceeding with proof of the Riemann mapping theorem okay, you see last in the last lecture what we did was we proved that.

If you give me a simply connected domain which is not the whole complex plane the ni can conformably map it to the sub domain of the unit disc that is what we prove okay. But then somehow you have to alter the map, so that the image fills out the whole unit disc. Because finally the Riemann mapping theorem requires I mean the proof the theorem requires you to find an isomorphism holomorphic isomorphism of the given simply connected domain is not the whole complex plane with the whole unit disc okay.

So, that is the reason we have to study the hyperbolic geometry on the unit disc right and the point is that the Schwarz's lemma actually in a disguised avatar of itself actually tells you that with respect to this hyperbolic metric on the unit disc. If you take an arbitrary analytic function which is not an isomorphism of the unit disc of onto itself, then it has to only contract, so the moral of the story is if either it is an isomorphism of the unit disc on to itself okay.

If it is not an isomorphism unit disc of on to itself it will be a contraction mapping which means you take 2 points okay and you take their images the distance between the images will be smaller in the hyperbolic metric than the distance between the original parts, that is not a contraction mapping, it is a contraction mapping between 2 metric spaces is a map that reduces the distance okay.

The distance between 2 source points **is** is greater than the distance between the image points, that is what a contraction mapping is and what the statement of the result that we need is that if you take an analytic map of the unit disc into itself which is not an isomorphism which is not a member here. Then it is certainly a contraction map for the hyperbolic metric alright, so for that I will have to introduce the hyperbolic metric okay.

And then of course there is also the question that what will happen to these maps what will happen to automorphic I mean holomorphic automorphism of the unit disc under hyperbolic metric. The answer is that they have preserve the metric they will be isometrics okay, a map between 2 metric places is called an isometric if it preserve distances, that is you take 2 points in the source metric space and you take their images under this map.

In the distance between the images the same as the distance between the source points, if the distance is preserved the mapping is called isometric okay. So, what happens is whenever you take a map of the unit disc to itself which is analytic either it is an isomorphism and it preserves the hyperbolic metric or it is a contraction, so it will shrink the image of the domain okay, so this is the geometric fact in hyperbolic geometry which we need okay.

And that is what I am trying to explain , so you know everything comes from Schwarz's lemma you know so the fact is that Schwarz's lemma gives you a description of the holomorphic automorphism of the unit disc which fix the origin and also the general holomorphic automorphism of unit disc not only that it also gives you information about any other analytic map of unit disc on to itself okay.

So, **so** the first thing is let us recall the Schwarz's lemma, so here is the Schwarz's lemma, the shot lema is well f is defined from the unit disc and taking values in the closed unit disc analytic f takes 0 to 0 okay. Then mod f z is less than or equal to mod z for all z in delta, this is the Schwarz's lemma which already tells you that you know any analytic mapping of the unit disc is contraction okay, in the sense that if you measure the distance from the origin then it is a contraction okay.

The mod fz is just the distance from the origin of the image fz of the point z under f and the distance from the origin of the image cannot exceed the distance from the origin of the source point, that is what it says. So, the fact that there is a contraction is already there in Schwarz's lemma okay what we are going to do is we are going to improve upon this okay and of course what is the other statement in Schwarz's lemma.

If you get equality here sorry even a single z0 then it is a rotation okay, equality occurs for some z0 in delta z0 not equal to 0, if and only if f is a rotation okay.

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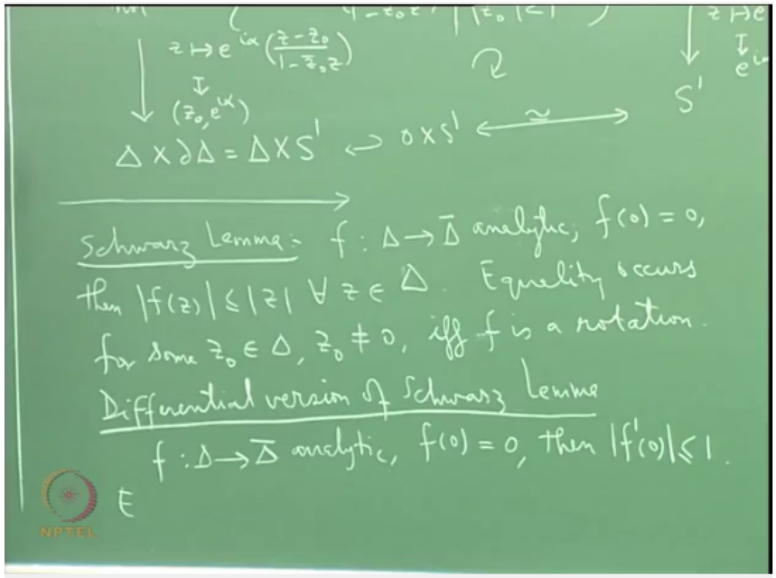

So, this is the this is a Schwarz's lemma from this version of a Schwarz's lemma I am going to get another version of the Schwarz's lemma which is called the differential version of the Schwarz's lemma okay. So, let me write this differential version of the Schwarz's lemma, so what is the differential version of the Schwarz's lemma you see again the differential version is that you have a the result in terms of derivative at the origin okay.

So, again you take f different of the unit disc and taking values of the closed unit disc analytic and f of z=0 f is takes the origin to the origin okay. Then the differential version says you take the derivative at the origin of f that cannot exceed 1 okay. The derivative at the origin of any analytic map of the unit disc into the closed unit disc the derivative cannot exceed 1 okay and again equality occurs if and only f it is a rotation, okay.

So, this is the differential version of short lemma or infinite decimal version of Schwarz's lemma because it involves the original statement in the original Schwarz's lemma involves f okay whereas the differential version we involve it is derivative, it gives you a bound for it is derivative. The derivative of any analytic map from the unit disc in the unit disc at the origin cannot exceed 1.

And it will be 1 only if it is a rotation if this not a rotation then it will be strictly less than 1 that is what it says okay.

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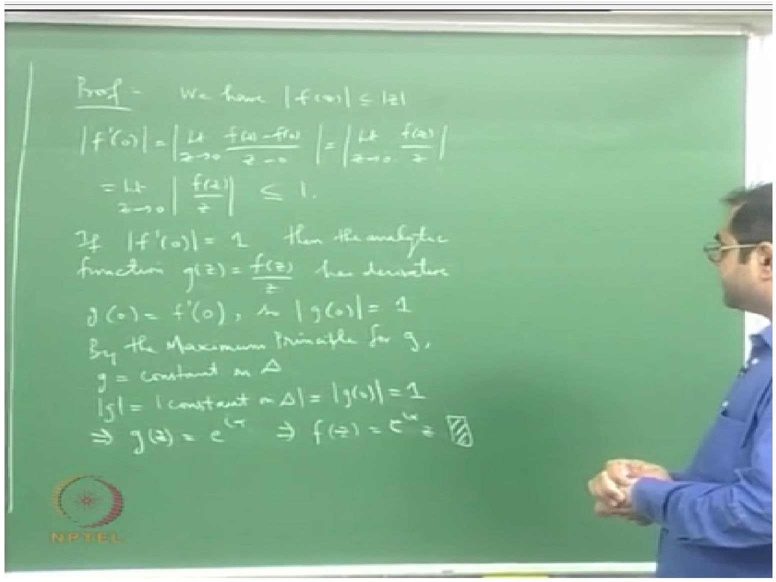

Now so what is the proof, of the differential version you can get the proof of the differential version from the original version okay and what is the proof of that. so, you know you see you we have mod f of z is less than or equal to mod z this is already there by the Schwarz's lemma alright, so what you do is for z0=0 you divide and you take a limit and you will get this.

Because after all limit z tends to 0 f of z by z is just f dash because f of 0=0 okay, so you get it so you know so this implies so modulus of f dash of 0 is modulus of limit z tends to 0 f of z-f of 0 by z-0 which is modulus of limit z tends to 0 f of z by z but you know if the limit of some expression exist okay, then modulus is a continuous function so I can remove the limit outside okay.

So, I can write this is as limit z tends to 0 mod fz by z but mod fz by z of course when I say limit z tends to 0 I am assuming z is not equal to 0 okay. So, if z is not equal to 0 mod fz by z is less than or equal to 1 and this is inequality because of the original Schwarz's lemma okay, therefore So this is a quantity less than or equal to 1 and you are taking the limit, so this is also going to be less than or equal to 1 okay.

So, you get this is the differential form of infinite decimal version of the Schwarz's lemma okay, then we have to say one more thing we must say the derivative is 1 if and only if it is rotation okay. And why is that true if the derivative=1 okay, then the function the analytic function g of z=fz by z okay. So, if you remember the in the last lecture or the lecture before that where we proved Schwarz's lemma what we did was, we define this function g of z to be fz by z.

And apply the maximum principle to it okay because it was analytic it was not defined at z=0 but then at z=0 it had only a removal singularity okay which you can see because the z will cancel off if you write out a Taylor expansion of f at 0 namely if you write a (()) (37:24) expansion of f at 0 the first constant term will be 0 that is because f of 0=0 alright and the z will cancel out and therefore g of z will be represented by a convergent power series at the origin.

Therefore it is also analytic at the origin, therefore this is actually though I am writing fz by z it extends to an analytic function at z=0, it just like sign z by z. If I take limit z tends 0 I will just get g of 0=f dash of 0 okay.

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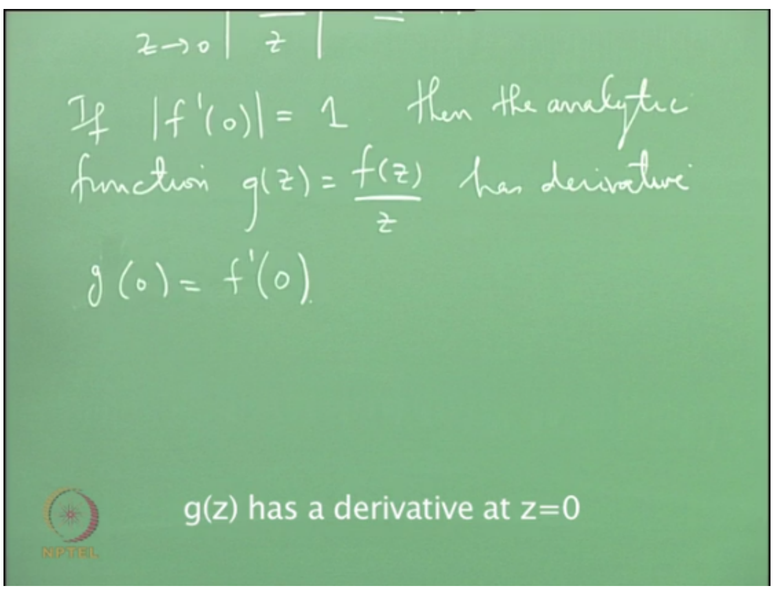

And what this will tell you, so modulus of g dash of modulus of g of 0 will be 1, if mod f dash of 0 is 1 then mod g0 will be 1 okay and you see now go back again to the proof of the Schwarz's lemma. In the Schwarz's lemma what we did was we apply the maximum principle to g which is f by f of z by z and showed that the maximum value of mod g is always less than or equal to 1.

Actually we proved mod g is less than or equal to 1 which translated to mod fz less than or equal to z which is the conclusion of the Schwarz's lemma okay. So, and what is the maximum principle what it say, if the maximum modulus if the maximum value is occurs at an interior point then the function is a constant okay. So, by the maximum principle for g, g is a constant on the unit disc okay.

You will get g is a constant on the unit disc and that constant and mod g will be mod of that constant on delta which will be mod g 0 which will be 1. So, this implies that this constant has modulus 1 okay, so mod g is mod of that constant and mod g is mod of that constant it has to be equal to mod g 0 but mod g 0 is 1.

So, modulus of that constant is 1 okay, it is a complex number it is modulus 1, therefore it is all the form e power i alpha. So, g of z is all the form e power i alpha it is a constant function but what is g of z, g of z is f of z by z. So, this will tell you that f of z is a rotation, so in other words you know you have proved that the differential form a Schwarz's lemma, derivative at the origin of an analytic of the unit disc on itself cannot exceed in modulus 1 okay, it is modulus cannot exceed 1.

And it is equal to 1if and only it is an automorphism and that automorphism has to be rotation, so if it is not a rotation the derivative of the origin is strictly less than 1 that is the of the Schwarz's lemma r okay. So, that finishes the proof of this okay, so this is the differential version of the Schwarz's lemma and now we come to you know this question of trying to generalise this differential version of the Schwarz's lemma.

So, you can ask what will happen if I put instead of 0 is z here okay, so take an analytic functions from the unit disc to the unit disc okay, I know f dash 0 in modulus is less than or equal to 1 but my question is what will happen if you take f dash of z for z an arbitrary point on the unit disc not necessarily the point 0 okay, then also you get a bound and that is grander version of the Schwarz's lemma and that is called pick's lemma okay.

And which is the fundamental lemma that is required for hyperbolic geometry okay, so I will just state pick's lemma somewhere I think .

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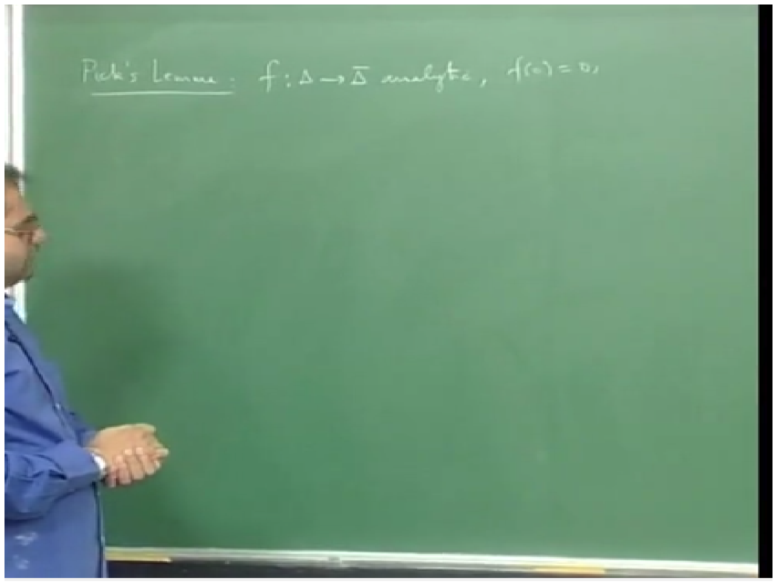

So, here is pick's lemma, so again same assumptions f from delta to delta bar analytic f takes 0 to 0 okay I do not think I need f takes 0 to 0 I do not need that this is very general, f is any analytic map from unit disc taking values in the closed unit disc okay.

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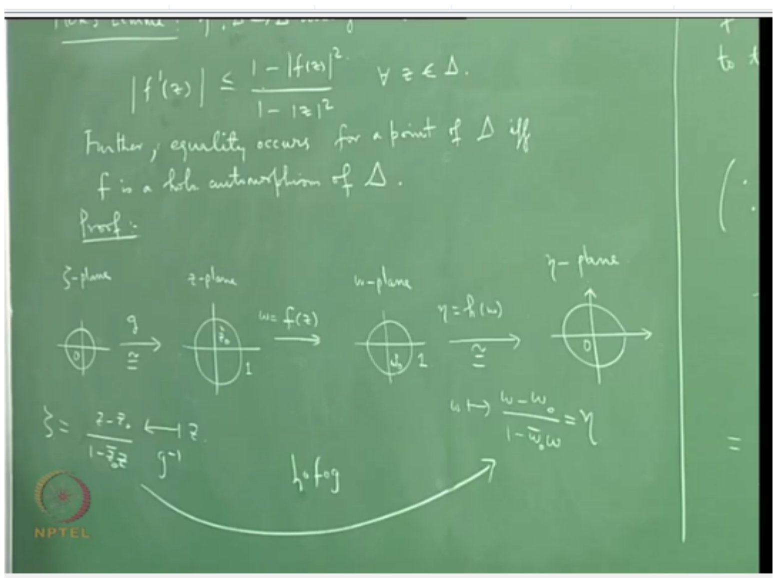

And then you see mod f dash of z okay is less than or equal to 1- mod fz square by 1-mod z square for all z in delta okay, you see if you in this you know if I take in particular an analytic function which takes 0 to 0 okay. Then if I substitute z=0 what I will get is mod f dash of 0 is less than or equal to 1-0 by 1-0 because I am putting z=0 and f of 0 is also 0 I must if I assume 0 goes to 0 then I will get mod f dash of 0 is less than or equal to 1 which is the differential form of the Schwartz's lemma.

So, this is a this pick's lemma is a generalisation of the Schwartz's lemma okay, the only point in Schwartz's lemma is that you are looking at the derivative at the origin the modulus of the derivative at the origin. And you are also assuming that the map takes 0 to 0 now you are relaxing both the things, you are taking any analytic map of the unit disc into itself and you are saying that the derivative modulus of the derivative at any point has a bound and this is the bound okay.

And again just like that case you will get equality if and only if f is a automorphism okay otherwise you will have strict inequality right. So, further equality occurs for a point of a delta if and only if f is an a holomorphic automorphism of delta. So, this is pick's lemma okay, so you see you can see the flavour of our arguments near more and more worried about the mappings of the unit disc into itself, the Schwartz's lemma itself is a statement about the mapping of the unit disc into itself .

So, we are worried about the mappings of the unit disc into itself we are worried about which of these mappings are automorphism we are worried about properties of such mappings and this focusing on the unit disc is actually leading us to studying hyperbolic geometry on the unit disc which is essentially given by defining the so called hyperbolic metric on the unit disc okay.

So, **so** this is Pick's lemma and what you do to prove Pick's lemma the nice I mean it is a pity I erased this diagram here I have to use the same diagram to prove Pick's lemma but I will have to finally apply the infinite decimal or differential version of Schwartz's lemma. So, let me remove I do not want it to take values on the unit circle alright, so let me remove this let me just assume that I am taking the map is from the unit disc to unit disc okay.

Let me not think the let me not take main the assumption that f takes a boundary value alright from the unit disc into the unit disc right. So, you know the diagram is a same, so I have **I have** this unit disc I have the map f and it goes from the unit disc to the unit disc okay and this is w=fz, this is z plane and this is a w plane okay. And well if you take a point z0 it will go to a point w0 and I want this w0 to lie inside the unit disc.

That is the reason why I did not take f to be a mapping from the unit disc to it is closure okay. So, **so** I want w0 to lie inside the unit disc right and well if you take any point in the unit disc it is going to go to a point inside the unit disc. Because I have remove the bar there okay and what I am going to do is as I did before I am going to put here this map which is an automorphism of unit disc which takes z0 to 0 okay.

And you know what is that map, so you know if you remember I call this the zeta plane earlier in this lecture and then I wrote a map like this which is g which is an isomorphism it is an automorphism of the unit disc which takes 0 to z0 and it is inverse is a automorphism of the unit disc it takes z0 to 0 and you know what that map is it is simply z going to z-z0 by 1-z0 bar z, this is map and this is our zeta okay.

And this is g inverse okay and I am taking g to be this map, so that it takes 0 to the z0 it has g inverse takes z0 to 0, g will take 0 to z0 alright and I have f here mind you f is not an automorphism okay f is not an isomorphism it is not even given to be 1 to 1, it is only given to be analytic nothing more is given about f okay. And then here what I do is I take this map from here neta=h of w to the is goes langs in the neta plane.

And I will take an automorphism of the unit disc that maps w0 to 0 okay and you know what that map is what is that map it is w going to w-w0 by 1-w0 bar w which is your neta okay. So, I have this set of maps alright, this is an isomorphism the last one is an isomorphism the centre is just an analytic map, the centre is analytic I do not know it is a 1 to 1 I do not know onto I do not know anything about it.

Now what I do is I compose if I compose what I get is g followed by f followed by h, if I compose I get h circle f circle g, it is a map from the unit disc to the unit disc okay I cannot say anything about this map except that I can only say 0 goes to 0 okay. Because 0 goes to z0, z0 goes to w0, w0 goes to 0, so 0 goes to 0 but that is good enough for me apply the Schwarz's lemma, the infinite decimal version of the Schwarz's lemma which will essentially give me the proof as you going to see.

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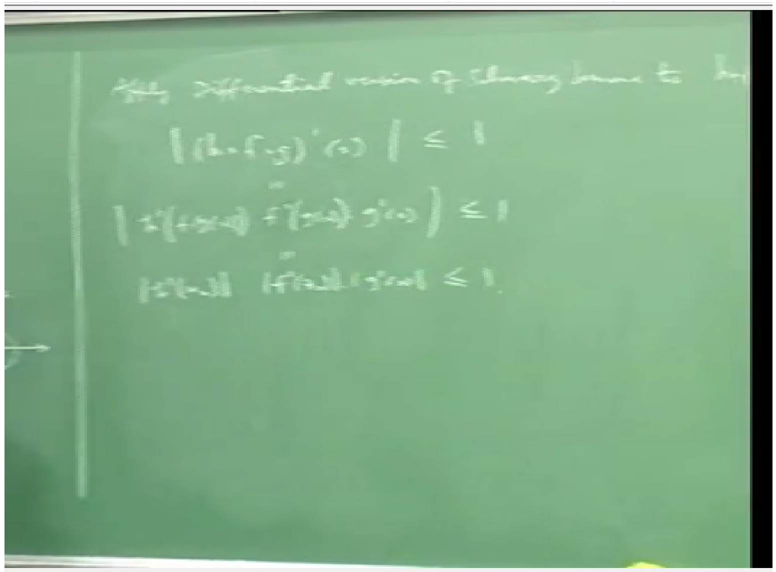

So, apply differential version of Schwarz's lemma to this composite function h circle f circle g what you will get, you will get h circle f circle g derivative at 0 modulus is less than or equal to 1, that is what the infinite decimal version of the Schwarz's lemma says, infinite decimal version of the Schwarz's lemma says whenever you have an analytic function unit disc to unit disc which take the origin to the origin okay which fixes the origin.

Then the derivative at the origin has modulus less than or equal to 1, so h circle f circle g is an analytic function from the unit disc to unit disc which takes 0 to. So, it is derivative modulus of the derivative at the origin cannot exceed 1 okay but what is this, this is if you see this is g dash you see **see** I will have g dash of 0 okay. So, if I differentiate it using chain roll I will get h dash of f of g of 0 into f dash of g of 0 into g dash of 0, this is what I will get this.

Because you know if I differentiate this by chain roll I differentiate the outer most function then the next enough function and then the next enough function this is just the chain roll okay and you know but this is what you see this is mod g dash of 0 mod f dash of g of 0. But what is g of

0it is z_0, so g of 0 is z_0 and what is and I will have mod h dash of f of g of 0 which is w_0, this is less than or equal to 1, I will get this alright.

So, I will tell you what to do is you calculate g write out g alright g will be just zeta going to zeta+z_0 by 1+z_0 bar zeta you just have to change the –to+ that will give inverse one okay, you differentiate it and substitute 0 okay you literally calculate it and you will get mod f dash of z_0 is less than or equal to 1-f of z_0 the whole square by 1-z_0 the whole square, you will directly get it, it direct calculation ordinary calculation nothing to do nothing complicated. And since is true for any z_0 you get that estimate alright, it is just straight calculation ordinary differentiation right.

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