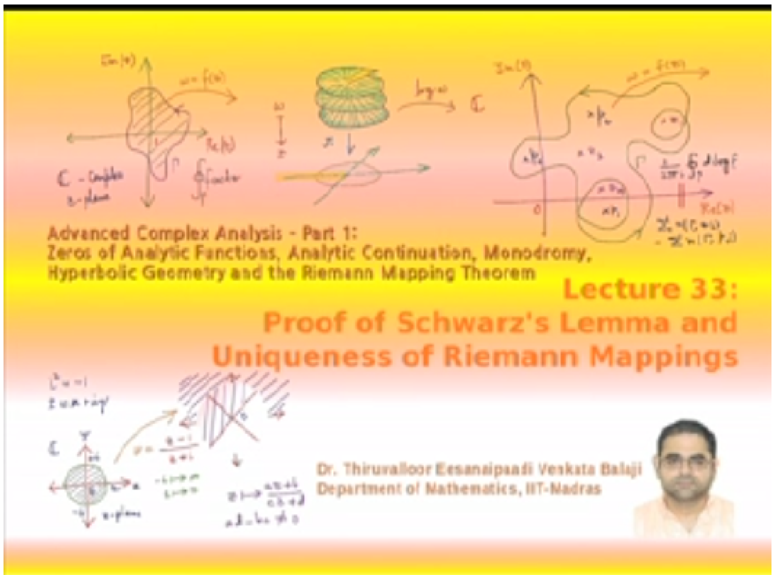
Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

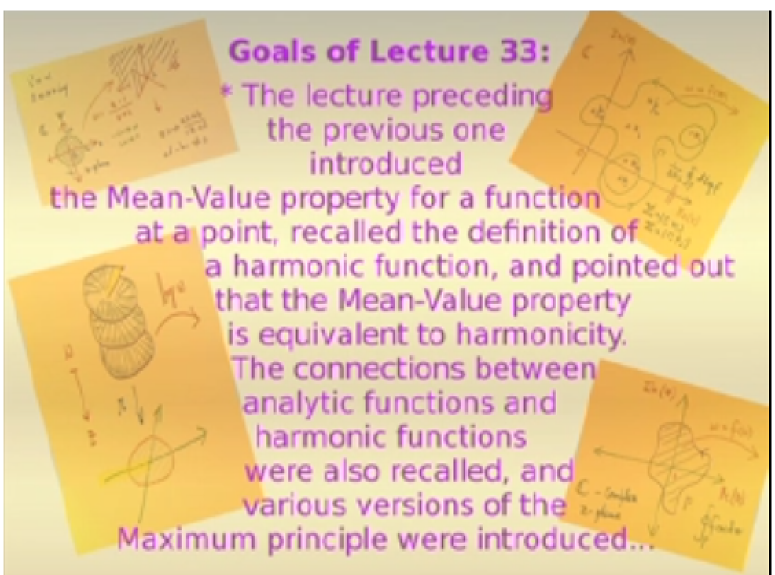
Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology-Madras

Lecture-32
Proof of Schwarz Lemma and Uniqueness of Riemann Mappings

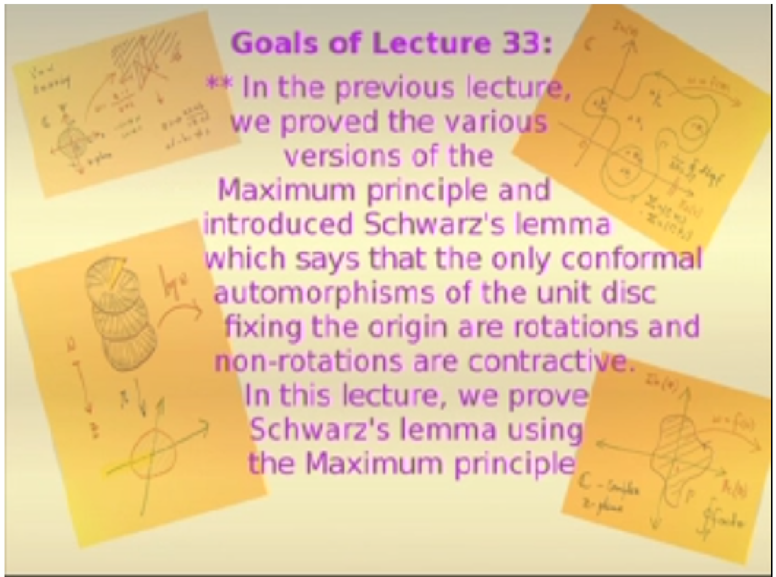
(Refer Slide Time: 00:05)



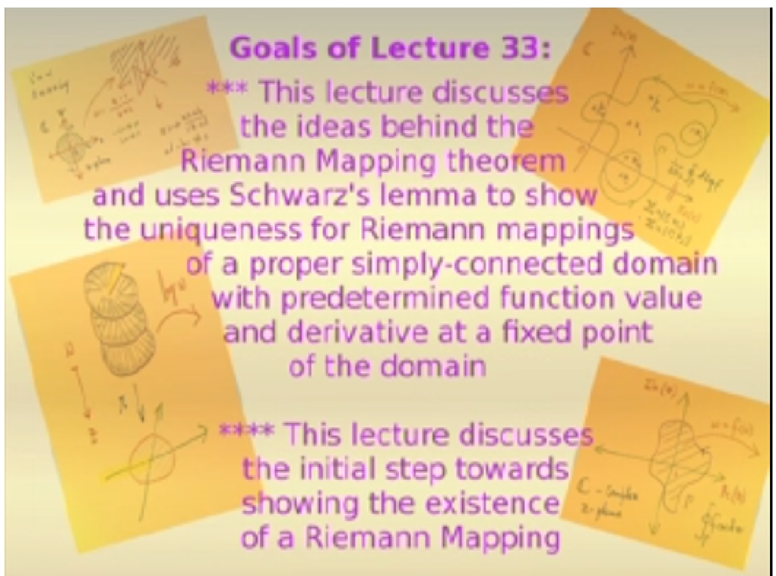
(Refer Slide Time: 00:09)



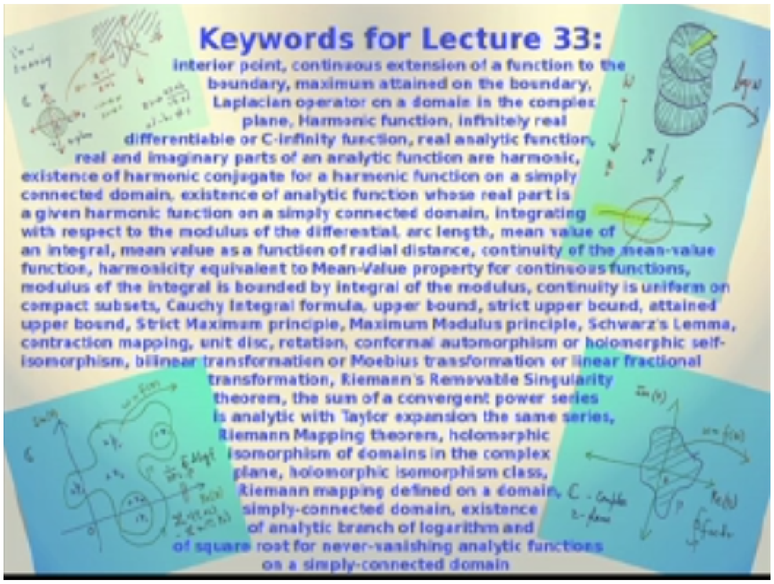
(Refer Slide Time: 00:18)



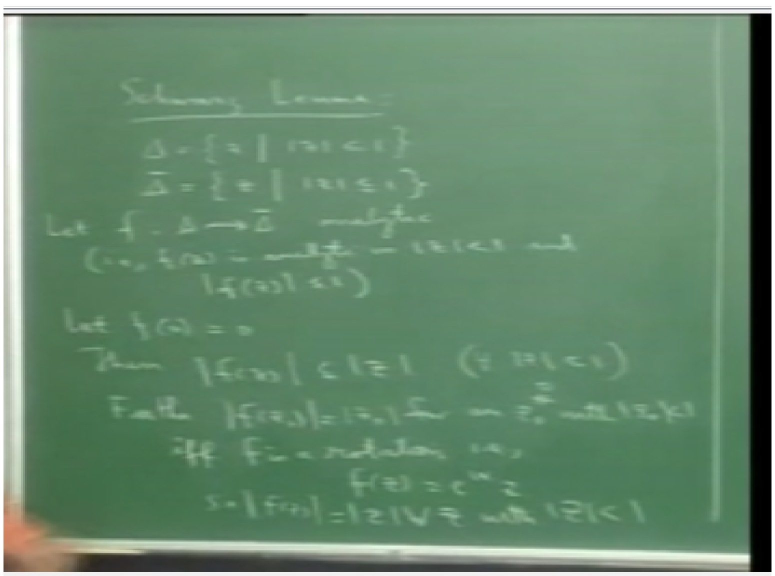
(Refer Slide Time: 00:26)



(Refer Slide Time: 00:34)



(Refer Slide Time: 00:47)



Alright, so what we are discussing is a short Schwarz lemma where we are looking at a function f which is defined on the unit dsic and it is taking values in the closure of the unit disc okay. And we assume that maps 0 to 0 okay then the Schwarz lemma says that the if you take any complex number in z in the unit disc. Then the modulus of it is image cannot exceed the modulus of the image of that complex number under f cannot exceed the modulus of the complex number okay.

So, and the fact is that you get equality even at 1 value if and if it is a rotation okay. And in which case you get equality everywhere okay and of course you know rotation is bilinear transformation and it will map the unit disc isomorphically onto the unit disc and it fixes the origin and corollary to Schwarz lemma is that every automorphism of the unit disc every holomorphic self map of the unit disc onto itself which is an isomorphism namely which has a inverse which is also holomorphic. And which fixes the origin has to be a rotation okay so, now so, let us try to prove these things.

(Refer Slide Time: 02:22)

so, the first thing is so, let me Schwarz lemma so, the idea of the proof is very easy actually you just going to apply the maximum principle and nothing else okay. So, what you do is put g of z is equal to fz by z okay put g of z is equal to fz by z for z0 equal to 0 okay. Because I am dividing by z I should put z0 equal to 0 then but the beautiful thing is that of course g is analytic on the punctured unit disc namely if z is not zero.

Then fz by z is also analytic because numerator is fz which is analytic denominator is z which is analytic and you know the quotient of analytic function is analytic wherever the denominator does not vanish okay. Therefore this g as I have defined is it as I have defined it is analytic on the punctured unit disc. But the fact is it is even analytic at the origin the reason is because f of 0 equal to 0 okay.

So, you see f of z if you write the power series of f of z centred at 0 namely the Taylor expansion of f of z okay. So, what will you get the Taylor expansion of f of z at z equal to 0 which is a you know classically call (()) (03:56) expansion the Taylor expansion at zeroes called the machloren expansion. So, the machloren expansion is what it is just f of z is equal to you know f of 0+z f dash of 0+z square by factorial 2 f double dash of 0 and so on.

This is what it is use the Taylor expansion but what if f of 0, f of 0 is 0 because f is supposed to fix the origin **it** it maps 0 to 0. So, what I will get this I will get f dash of 0+z square by factorial 2 f double dash of 0 where hz is analytic on delta on the unit disc why is hz is analytic because you know hz will have this power series expansion that is gotten by taking this power series expansion.

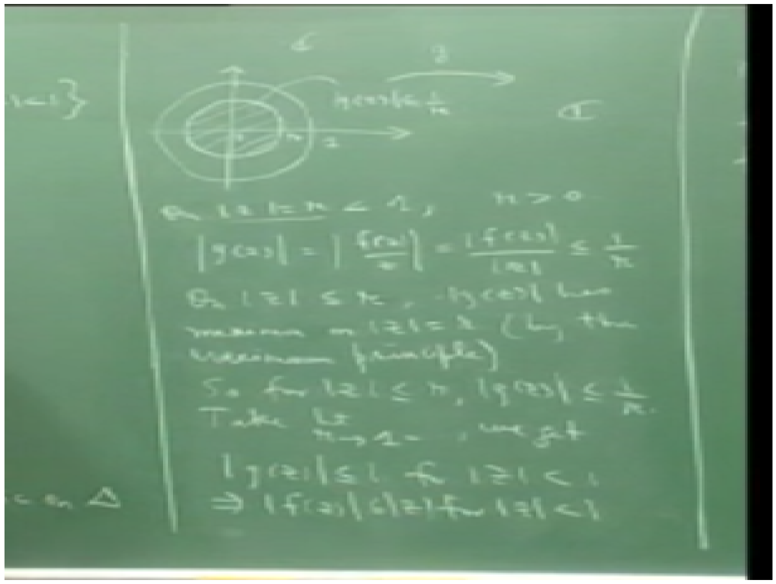
And you divide it by z okay namely h of z will be f dash of 0+ z by factorial 2 f double dash of 0 and so on and that will be a convergent power series okay so, hz is given by convergent power series the origin so, it is analytic at the origin alright. But on the other hand outside the origin h of z is actually f of z by z okay so, what you have saying this function g of z extends to an analytic function hz the origin.

In other words g of z itself is an analytic function in the at the origin okay. So, as h of z is equal to g of z or z0 not equal to 0 this shows that actually 0 is a removable singularity for g of z okay see g of z is like sine z by z which apiary cannot be defined at z equal to 0. Because z in the denominator but actually if you write it as a power series sine z by z is also defined at z equal to 0.

Because at 0 it has a limit it has a finite limit so, remarks the removable singularities theorem says that if you have if a function at a point where it is not defined but suppose it is a function which is analytic and delete neighbourhood of a point okay. Then it can be extended to analytic function at that point if one of the following three conditions satisfy or satisfied is satisfied namely the first condition is that I tends to limit has you tend to that point.

The second condition is that if the function is bounded in a deleted neighbourhood of that point and of course the third thing is if the function has the function can be extended it has a power series expansion at that point okay. And in fact all the three are happening here okay so, g of z is analytic on a whole unit disc. So, you know I will keep writing fz by hz I mean I simply write gz is equal to fz by z and just remember that this is the expression for g when z is not equal to 0. When z is equal to 0 it is actually h okay and h is actually the power series expansion of g at the origin at and the origin is removable singularities.

(Refer Slide Time: 08:13)



So, it is a point to which g can be extended analytic okay so, well fie so, after that remark what we do next is now we are in a position to apply maximum principle okay. So, what will you do is the following thing. So, what I going to do is you going to take this unit disc and so, I have this function g which going from unit disc what I am going to, I am going to take circle centred at the origin radius small r okay.

So, and I am going to look the function g of z which is **is** fz by z alright. And I am going to look at what it is modulus is on the circle okay. So, on mod z=r which is less than 1 okay, so I am looking at all points on a circle centred at the origin radius 1, radius r small r where r is fraction okay what is mood gz, mod gz is mod fz by z because mod z=r and r is positive okay. so, certainly z is not 0 and g has a expression f of z by z when z is not 0.

And if I calculate mod g I am going to get mod fz mod z and that is well that is less than or equal to 1 by r. Because mod fz is always less than or equal to 1 is something that is given to me that is just analytic expression for the geometric fact that f takes values in the closed unit disc okay. And mod z you see mod z=r is already assumed because I am summating the **val** mod gz on the circle where mod z=r, so I get this. If you take mod z less than or equal to r mod z less than or equal to r.

If you take this closed disc centred at 0 radius r including the boundary okay and if you look at the function gz it is analytic there okay. The maximum principle will tell you that it is modulus will be maximum on the boundary, so here is where I am applying the maximum principle. So, on mod z less than or equal to r mod gz ahs maximum on mod z=r okay but on mod z=r mod gz is less than or equal to 1 by r.

And therefore maximum principle will tell you that on this whole disc mod gz is less than or equal to 1 by r, so the upshot on this whole closed disc on here what you are getting is mod gz I mean see the point is you make an estimate of mod gz on the boundary of that closed disc which is mod z=r and that will also be an upper bound for the values inside because this is the maximum principle.

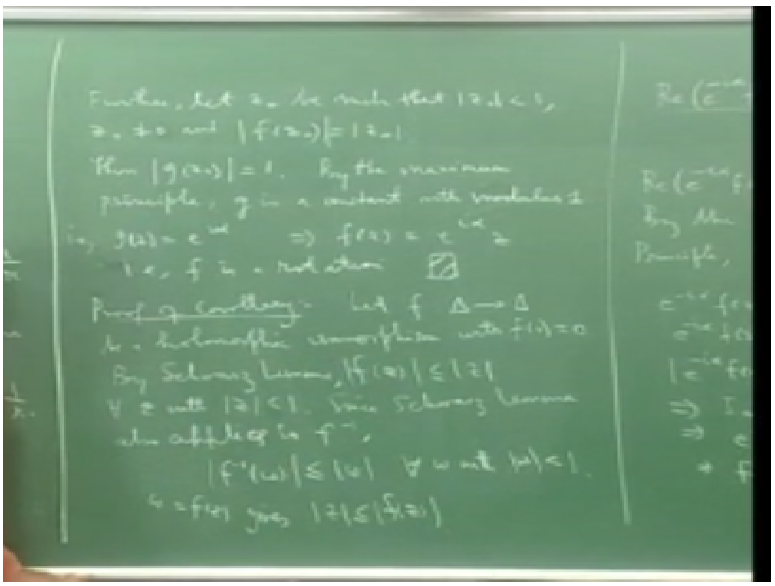
The maximum principle tells you that the mod will attains maximum only on the boundary, so if you know a bound for the function on the boundary that bound will also be a bound for the function values on the interior. And of course the function in this case is the modulus of the analytic function okay. So, by the maximum principle okay you know when in applying the maximum principle I am using the fact that you see g is analytic.

And therefore harmonic because g is analytic means that both it is real and imaginary parts are harmonic and therefore g is also harmonic and I am using the maximum principle for harmonic functions okay. So, it applies to g, so mod g has a maximum of mod z=r but on mod z=r it is bounded by 1 by r therefore 1 by r is a bound for g on the whole closed disc okay. So, for mod z less than or equal to r mod gz is less than or equal to 1 by r, this is what you get.

Now what you do is that you take the limit as r tends to 1- okay, if you take the limit as r tends to 1- we get mod gz is strictly less than is less than or equal to 1 for mod z less than rho okay. So, and mod gz less than or equal to 1 will tell you that mod fz by z is less than or equal to o1 and that translates to mod fz less than or equal to z for mod z less than 1 which is the first assertion in this short lemma okay.

So, **so** this implies so you get this of course you know there is a little bit of trouble at z=0 but at z=0 this holds because mod f mod f0 is 0 and of course here I have to put mod, mod f0 is 0 which is equal to mod 0.

(Refer Slide Time: 14:02)



Further let z0 be such that mod z0 less than 1 I will of course need f is at z0 not equal to 0 and mod of f of z0=mod z0 okay. So, in this statement I should correct myself , so z0 should be non 0 okay because if I do not put that condition then this is always true for z0=0 okay, so z0 should be non 0. So, suppose there is an z0 which is not 0 where mod f z0=mod z0 okay.

So, then what you are going to get is that you are going to get then mod gz0 is going to be equal to 1 okay. Because after all z0 is non 0, so mod g I mean gz0 is just fz0 by z0 and mod gz0 will be mod fz0 by mod z0 and that will be 1 okay but then again you apply the maximum principle to mod g okay, that will tell you that mod g will always be it will tell you that g has to be constant okay.

So, because g can attain it is maximum only on the boundary of the unit disc it provided it extends to boundary of the unit disc. If it does not extend to the boundary of the unit disc it can never attain the maximum, so whereas mod g is bounded by 1. So, the maximum value 1 cannot be attain in the interior if it is attain in the interior then g has to be a constant, this is the maximum principle okay.

So, **so** by the maximum principle g is a constant, so and with g is constant and mod gz0 constant with modulus 1 okay. so I will get g of z==e power i alpha because a constant a complex number with constant complex constant with modulus 1 has to be the form e to the i alpha and that means that because g is f of z by z it will tell you that f of z is just e to the i alpha of z, so it is a rotation okay.

So, that finishes the proof of the short lemma okay, that finishes the proof of the short lemma and I must again point out that I admit earlier the mistake of not saying that z0 is non 0 okay, that is important. Because if I did not say that then I already have mod f0=0=mod 0 right. So, **so** this is the proof of the short lemma I mean what the what you must understand is that the moment f is a rotation then f becomes a bilinear transformation.

Therefore it is 1 to 1 also it becomes the conformal isomorphism okay, so what it what your short lemma actually says is that if you take a analytic function from the unit disc into the unit disc the closure of the unit disc which takes the origin to the origin. Then either it is strict contraction in terms of length okay that is mod fz is strictly less than mod z for all z with mod z less than 1 or it is a rotation okay.

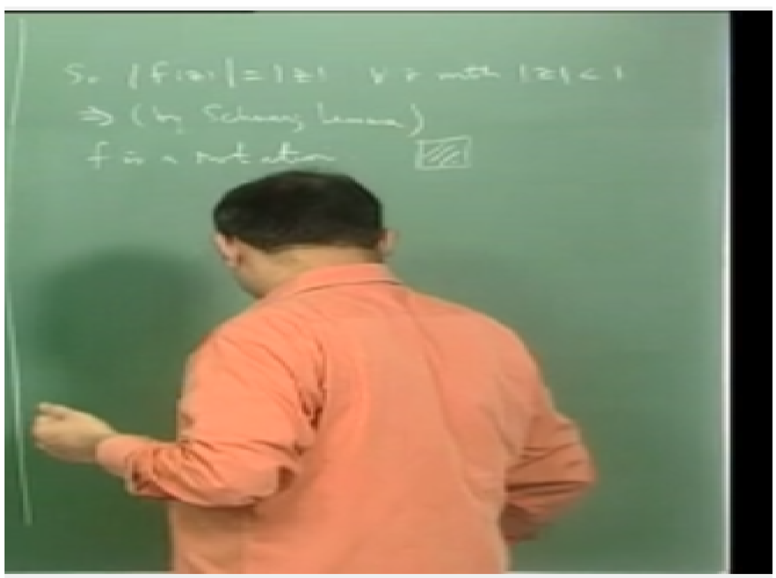
So, let us look at this corollary, let us try to prove this corollary, proof of corollary . So, what I will have to so is, I will have to take a holomorphic automorphism of this with fixes the origin and I have to say that it is a rotation. So, let f from delta to delta the a holomorphic that is analytic isomorphism such with f of 0=0 okay, so you take a holomorphic isomorphism self isomorphism of the unit disc right.

Now of course to f I can apply short lemma because conditions of short lemma is that f should be analytic on the unit disc, it should take values inside the closed unit disc. In this case if taking values in the unit disc itself and it is defined on the unit disc and it takes 0 to 0. So, all the conditions short lemma satisfied, so I can apply short lemma and I will get by short lemma f of z mod fz is less than or equal to mod z for all z with mod z strictly less than 1, I will get this by applying short lemma to f.

But mind you what is given is that f is a holomorphic isomorphism which means f inverse is also like f, f inverse is also a holomorphic map from delta to delta. And f inverse will also take 0 to 0 because f of 0 is 0, f inverse 0 will also be 0, so I can also apply short lemma to f inverse okay. since short lemma also applies to f inverse mod f inverse of w is less than or equal to mod w for all w with mod w less than rho, I can apply short lemma to f inverse which is the inverse of f which is given to me to exist.

Because f is given to be a holomorphic isomorphism which means f has an inverse and that inverse is also holomorphic okay. But then you put w=fz in this will give you that mod z is less than or equal to mod fz okay and then you compare these 2 opposite inequalities mod fz less than or equal to mod z, mod z less than or equal to mod fz and you will get mod fz=mod z okay and that is it that should tell you that that f is a rotation.

(Refer Slide Time: 21:56)

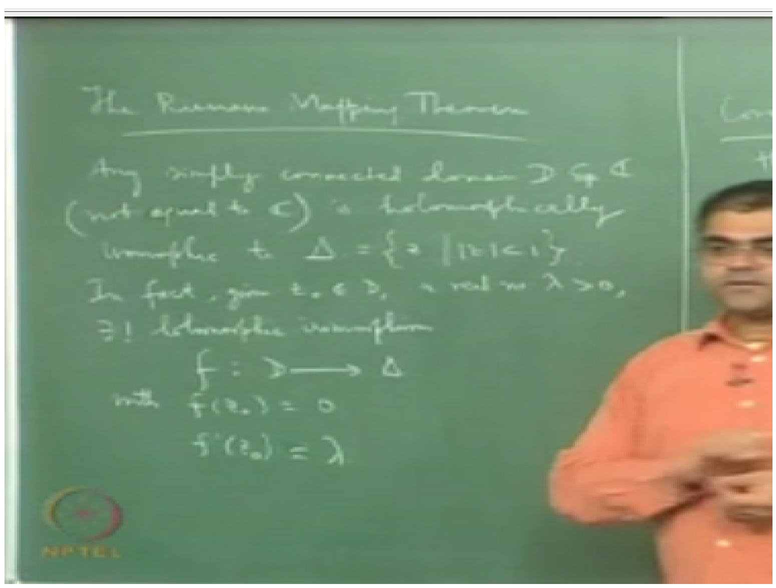


So, mod fz=mod of z for all z with mod z less than 1 but we have already seen the proof of the short lemma that whenever mod fz=mod z holds for a single z0 different from the origin, then f has to be a rotation. So, here it is what you are getting is that it holds for the all the points on the unit disc okay, you have more than what you need okay. So, this will imply again by short lemma that f is a rotation.

So, that proves the fact, that proves the corollary which is that the only holomorphic automorphism of the unit disc that fix the origin they are all rotations okay fine. So, having done this what I am going to embark a is to go into discussion of the Riemann mapping theorem which is something that I want to whose proof which is **is** what I would like to discuss in the coming lectures, it is a very deep theorem.

And it **it** the proof is not easy, it involves several facts and so but to make a preliminary discussion about it, I wanted this fact about automorphism to the unit disc. So let me start with a Riemann mapping theorem which is what our long term goal in the next few lectures is the proof of the Riemann mapping theorem okay.

(Refer Slide Time: 23:51)



So, the Riemann mapping theorem okay, so this is a theorem which says that you take any domain in the complex plane which is simply connected and assume that it is not the whole complex plane. Then that domain can be mapped by a holomorphic isomorphism onto the unit disc, so in other words if you take simply connected domains in the complex plane and you go modulo holomorphic isomorphism you will get only a set contained in 2 elements, 1 will be the isomorphism class of the whole complex plane.

And the other will be the isomorphism class of the unit disc and mind you the unit disc is isomorphic to the upper half plane. In fact it is any disc is any disc is any open disc is holomorphic isomorphic into any open half plane because you can always a Mobius transformation that will map the interior of a disc to any half plane okay. So, geometrically up to holomorphic isomorphism unit disc is same as a half plane, any disc is like the unit disc alright any finite disc, it looks like a half plane that is 1 holomorphic isomorphism class.

The other holomorphic isomorphism class is a isomorphism class of the whole complex plane and these are the only 2 holomorphic isomorphism classes of a simply connected domains in the complex on the complex plane okay and that is the statement of the Riemann mapping theorem. So, **so** let me state that any simply connected domain D not equal to C, so I am writing it also in words, I also put it in symbols D simply connected.

So, the proper domain in the complex plane is holomorphically isomorphic to the unit disc okay. So, this is the celebrated famous Riemann mapping theorem right and so in other words what you are saying is that if you give me a simply connected domain which is different from the complex numbers. Then there is a holomorphic isomorphism from d to delta which is the unit disc okay.

And that holomorphic isomorphism can be made in fact unique in the following way, so let me explain that in fact given z0 point D a real number lamda greater than 0 there exist a unique holomorphic is the same as analytic or conformal isomorphism f from D to delta with f of z0=0, f dash of z0=lamda. So, the Riemann mapping theorem says that any simply connected domain which is not the whole complex plane can be holomorphically mapped onto the unit disc.

And you can make that mapping unique if you fix a require that a point of D, a fixed point of D goes to the origin under this map and also that at that point the derivative of the map at that point is a fixed real number okay. These conditions make the map f unique okay and it is so you know there are 2 parts to this 1 part is to find a map okay, then which is rather the hard part.

The easier part is to say that once you have a map f like this, it is unique and it is the uniqueness part that will use short lemma okay or the corollary of short lemma. So, what I will do is I will first try to apply short lemma which we have just seen to show that you know if a map like that exist it is unique such a map like that is called is given a special name it is called Riemann map, it is called a Riemann map of the domain D okay.

And the Riemann map is unique if you fix the value of a point on the domain and the derivative of a map at that point okay I of course you I have fixed the value of the point z0 to be 0 for convenience and most people are in several text books you would see that lamda is taken to be equal to 1 okay. But in principle you could take lamda to be any positive real value okay.

(Refer Slide Time: 29:51)

Now so let us **let us** first prove uniqueness okay, suppose f1 and f2, f1, f2 are from D to delta or holomorphic isomorphism with these f1 of z is 0 which is also equal to f2 of z0 and derivative of f1 at z0 is the given lamda which is equal to the derivative of f2 at z0. I just want to show that f1 and f2 are one and the same map right. So, what I do is that I compose f1 with one of the fis with the inverse of the other.

And realise that I get a conformal automorphism of the unit disc which by Schwarz lemma is a rotation. Because it will fix the origin okay so, you see so, the situation is like this so, I have D, I have delta, I have f1 then I have f2 then I have delta here okay. So, if I go like this I will get the map which is f2 inverse followed by f1 okay. And this f2 inverse followed by f1 what is the property of this map.

This is holomorphic it is a holomorphic map it is a holomorphic isomorphism because it is a composition of two holomorphic isomorphism f1 is a holomorphic isomorphism and f2 is a holomorphic isomorphism the therefore f2 inverse is also the holomorphic isomorphism. The inverse of a holomorphic the inverse of an isomorphism is always an isomorphism okay. So, this is an composition of isomorphism.

Therefore this is also an isomorphism and so, this is an isomorphism is a holomorphic isomorphism. And where does 0 go to 0 goes to 0. Because you see z0 under this map goes to 0 and under this map also goes to 0. So, if I composite I will get 0 goes to 0 alright so, it is a holomorphic isomorphism from delta to delta which fixes the origin and as we have seen just now seen as a corollary of Schwarz lemma .

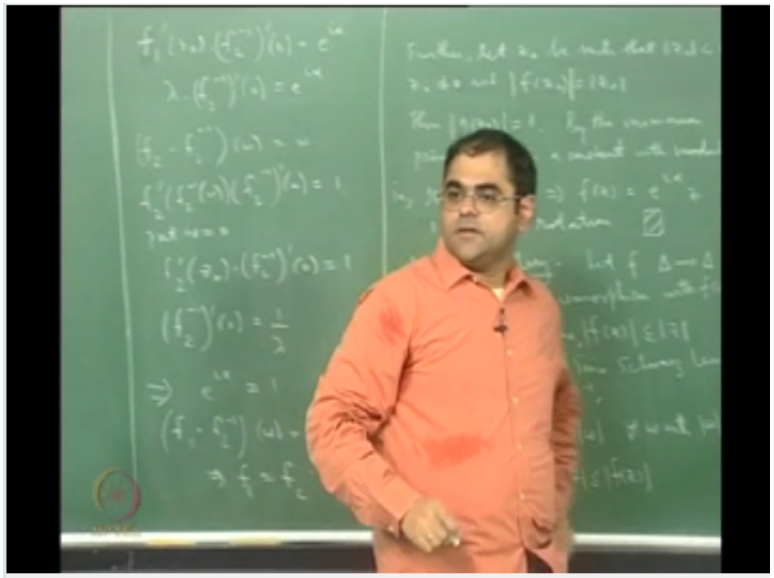
This has to be a rotation okay so, by corollary to Schwarz lemma f1 circle f2 inverse or w is equal to e power ialphaw it has to be a rotation of course whenever I write e power ialpha of course I assuming alpha is real okay. Because if I write e power ialpha with alpha complex when it is no longer a rotation a whenever I have write e to the ialpha I am always assuming alpha is real.

So, **so,** this is because of the corollary to Schwarz lemma okay and you see you know if I calculate f1 circle f2 inverse so, if I take the derivative of this okay. If I take the derivative of this what I will get is that I will get f1 circle f2 inverse derivative at w is equal to derivative of e power ialpha at w which is e power I alpha okay. The derivative with respect to w of e to the ialpha w is just e to the ialpha okay.

And on this side I will get derivative of this expression but then for this you apply the chain role okay so, what you will get is you will get f1 dash of f2 inverse of w into f2 inverse dash of w is equal to the ialpha okay. This is just applying the chain rule of differentiation alright and mind you in particular you know I can put any value for w here.

So, because it is a identity for all w so, I can put w equal to 0 okay and what I will get, I will get f1 dash of f2 inverse of 0 times f2 inverse dash of 0 is e to the ialpha this is what I will get okay. But then you have to remember that f2 inverse of 0 is actually z0 because f2 of z0 is 0 okay.

(Refer Slide Time: 35:22)



And therefore the so, what will you get is you will get f1 dash of z0 right into f2 inverse derivative at 0 is e to the ialpha okay. Now you see f1 dash z0 is given to be lamda so, I will get lamda and the fact is so, let me write that separately okay. Now I claim that f2 inverse derivative at 0 is also is equal to 1 by lamda okay that is just because again chain rule applied to f2 and f2 inverse okay.

So, f2 circle f2 inverse of w is w okay and if I apply the chain rule you will get f2 dash of f2 inverse of w times f2 inverse dash w is equal to 1 okay. You put w equal to 0 and you will get f2 dash of z0 into f2 inverse dash of 0 is equal to 1. And this will you tell you that f2 inverse dash of 0 is 1 by lamda okay mind you lamda is a non-zero it is positive.

And therefore you know well if you look at both what will you get is that you will get e to the ialpha is equal to 1 okay. And 1 c to the ialpha is 1 you will get f2 f1 circle inverse w is equal to w and this will actually tell you that f1 equal to f2 okay. So, that, so by using the Schwarz lemma are a rather the corollary of the Schwarz lemma namely that every automorphism of the unit disc, that fixes the origin is a rotation.

You are able to show that a Riemann map if it exist which is specified at a point of the domain simply connected to domain which is not equal to the complex plane. And it is if it is images specified at one point and it is derivative at that point is specified then the map is unique. The uniqueness comes from Schwarz lemma actually okay now this is a easier path this is the uniqueness of this Riemann surface.

But now you have to prove the existence of the Riemann map okay you have to show that there is a map from the given simply connected domain which is not the whole complex plane to the unit disc okay. So, let me recall the fact that the domain is simply connected is by definition it means that the any two any curve any path in the domain starting at any point can be continuously shrunk to a point to that point.

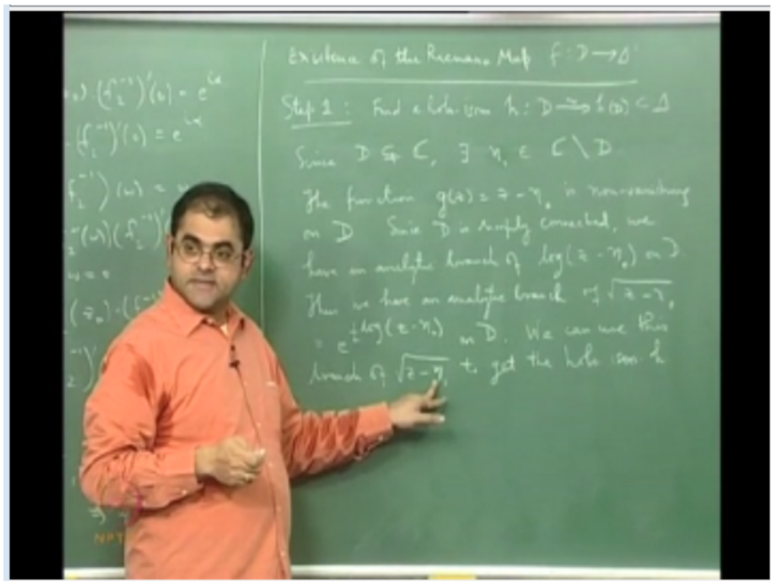
That means that there are no holes in that domain okay if you think of it as in terms of the region that in that is enclosed by any closed curve in the domain. Then you do not want any holes there the domain should not contain any holes okay. Because if there is a hole then you cannot continuously shrink a closed curve to a point okay fine so, it is very very important that you have simple connectedness.

So, the question is how do I produce a holomorphic map from to this domain to the unit disc okay which is an isomorphism. So, first step is you try to get hold of some holomorphic map which maps the domain into a sub domain of the unit disc okay. So, let me explain to you the first step towards the proof of the Riemann mapping theorem namely the first step towards the proof of the existence of the Riemann map.

The first step what you do is you show that the domain D can be conformably that is isomorphically it can be mapped in onto a sub domain of the unit disc first of all you do that. Then you modify that map so, that it can feel out the whole unit disc okay. And I am saying it loosely modify means it is not just modify. You will have to do a lot of things okay first of all given any simply connected domain how do I land at least into the unit disc.

So, the beautiful point here is the existence of a logarithm for an non-vanishing holomorphic function on a simply connected domain that is essentially used. So, **so** let me explain that.

(Refer Slide Time: 41:05)



So, what I am going to do is step1 so, this is a existence of the Riemann map so, I am going on with step1 find holomorphic isomorphism h from D to h of D which is a subset of the unit disc okay. So, the first step is to map first of all D into holomorphically isomorphically into a sub domain of the unit disc. So, what do you what we do is the following thing so, **so** here we exploit all the we exploit the fact that.

The domain is not the whole complex plane and the fact that the domain is simply connected okay so, what we do is since the domain is not the whole complex plane there exist eta not the complex number which is in which is outside the domain okay. I can find such a complex number because it is not the domain is not the whole complex plane. So, that is something outside the domain which is in the complex plane okay.

You take this we have now the next thing is I am going to use of the fact that this domain is simply connected okay. The function z- g of z is equal to z-eta not is non-vanishing on D that is obvious because z because the point eta not is not in D. So, z- neta 0 which z varying in D can never be 0 okay. And since it is a non-vanishing function and it is of course analytic okay.

It is analytic function after all it is a translation it is translation by –neta not okay so, it is an analytic function this analytic function is non-vanishing on this domain D which is simply connected therefore it has a logarithm okay since D is simply connected we have an analytic branch of log z-eta not on D okay. And therefore the movement I have log z-zeta not I will have an analytic branch of the square root of z-neta not on D.

And my claim is that function will do the job of mapping D I can use that function carefully to map D onto a sub domain of the unit disc okay. So, let me write that and the we have an analytic branch of root of z-eta not as exponentially of half log z-eta not on D we can use this branch of root of z-eta not to get the holomorphic isomorphism okay. So, I will stop here we will continue in the next lecture.

So, I am just trying to say that you are using a square root of z-neta not to get to map cleverly to map D first into a sub domain of the unit disc oaky. And the fact is that you are able to write this z-neta not because there is a neta not which is outside you are shift connected domain you are simple it that uses the fact the simply connected domain is not the whole complex plane. And the fact that you have a analytic branches of the square root is because uses. It uses the fact that the domain D is simply connected okay. So, I have stop here.