Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-30
The Mean-Value Property, Harmonic Functions and the Maximum Principle

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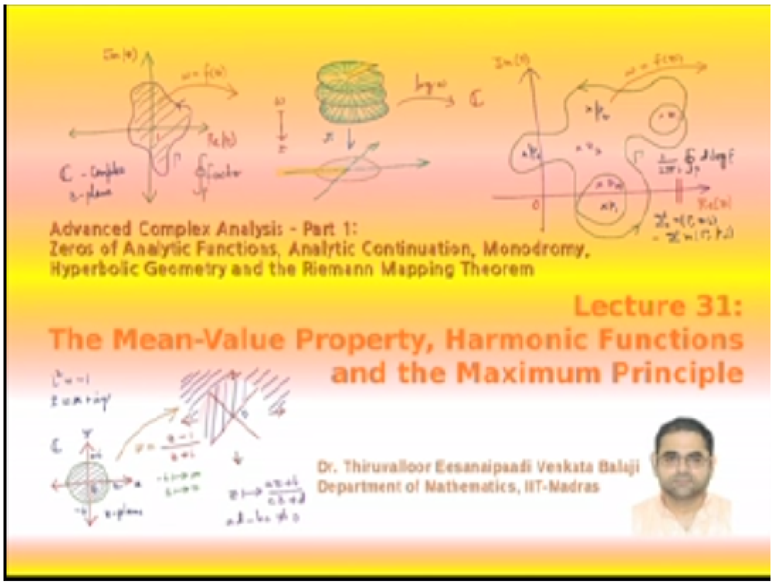

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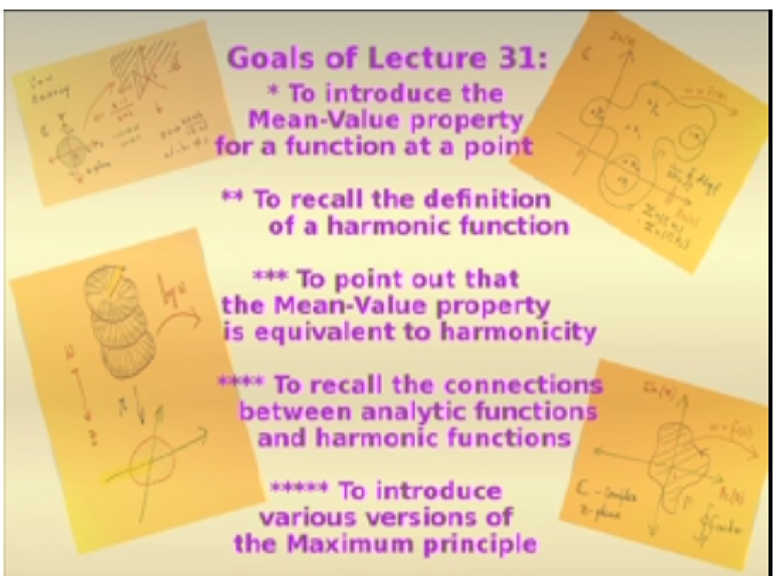

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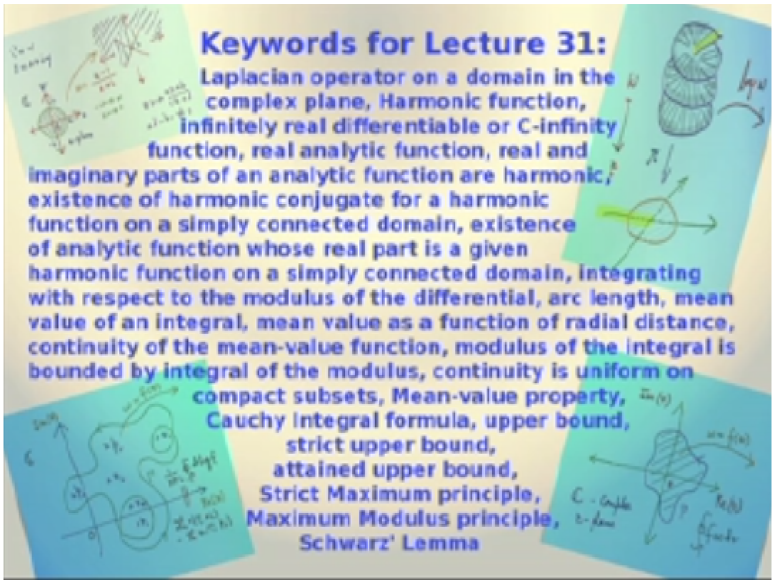

What we are going to do now move ahead to you know try to give a proof of the very important Riemann mapping theorem okay and for this I will need to know I mean we will have to look at other things in as a preparation. So, the first thing is that will be looking at are the so called harmonic functions about which you would have studied in a first course in complex analysis okay.

So, what I am going to do is try to recall harmonic functions the so called mean value property then the maximum principle and then the short which is the fundamental lemma that we need in the context of the that is the simplest lemma that we want in the context over Riemann mapping theorem. So, much of this is something that you would have seen in a first probably you would have seen in a first quotient complex analysis.

But nevertheless it is important, so this will help you to refresh your memory if you have seen it once and if you have not seen it this is an opportunity to learn it.

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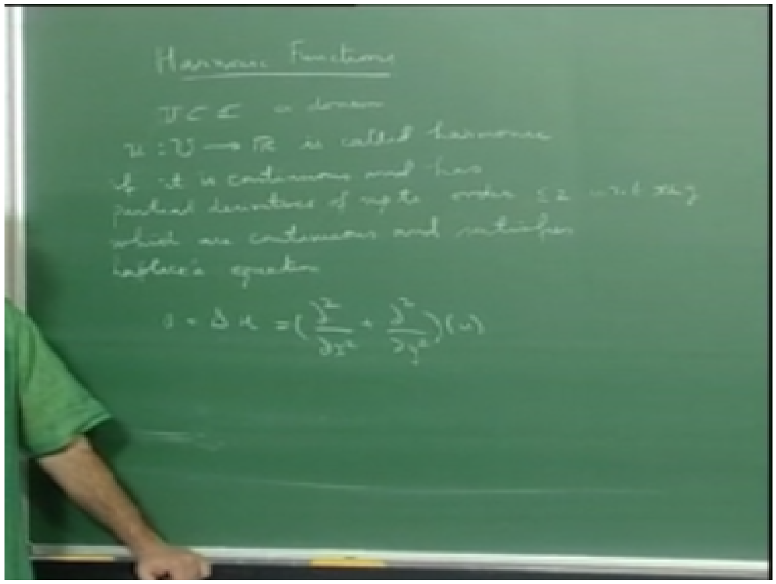

So, we are looking at harmonic functions, so basically you know you take D or rather U in the complex plane a domain and f so let me use the small u form capital U to R , so it is a real valued function okay is called harmonic if it is continuous and has partial derivatives of up to order less than or equal to 2 which are continuous and satisfies Laplesle's equation which is del u=0 where this delta.

There is delta u=0 where delta is the Laplesce operator it is tou square by tou x square+tou square by tou y square okay. So, of course in this definition so what I have done in this definition I have defined a real valued harmonic function on a domain in the complex plane and I am assuming the function is continuous and it has partial derivatives up to order less than or equal to 2 .

Of course you know partial derivatives means I am taking partial derivatives with respect to x and y okay with respect to x and y. So, this means that you know tou u by tou x the first partial derivative with respect to x then tou u by tou y and then for the second partial derivative is you have the pure derivatives tou square u by tou x square tou square u by tou i square that you have also the mixture partial derivative tou square u by tou x tou y.

And tou square u by tou i tou x and we assume that all this partial derivatives exist and they are continuous okay and I mean the point is that somehow you know the at least in the definition it be insist only up to the existence of derivatives of orders up to 2 okay. But the fact is that you know it is rather amazing the fact is that you put this condition.

And then that partial derivative is of all orders will exist okay. so, the requirement of the partial derivatives of order up to 2 existing is just so that this equation can be written down okay. And of course to write this equation I do not need the mixture partial derivative tou square u by tou x tou y or tou square u tou i tou x okay but the point is normally continuity of the function and of the of all these partial derivatives is assumed .

But the the big theorem that you get from complex analysis is that you take such a harmonic function then it is infinitely differentiable okay that is partial derivatives of all orders exist okay and they are all continuous it is a very deep theorem and why it is so deep is because the you are getting infinite differentiability you are getting existence of partial derivatives of any order okay.

And you know let me tell you a few words about this I mean you would have seen an first quotient complex analysis that probably we will revisit that again that you know if you you know that if you take an analytic function okay then the real end imaginary parts of an analytic function or harmonic functions okay and we call the imaginary part harmonic conjugate of the real part okay.

And conversely if you give me a harmonic function alright if the domain is simply connected okay then it will always have a harmonic conjugate that mean so, that means that you give me harmonic function it will be the real part of an analytic function at least on a small disc okay. And the moment it is a real part of an analytic function you know analytic functions are infinitely differentiable.

Because analyticity the beautiful property about analyticity is that you assume differentiability once with respect to the complex variable z and you get infinite differentiability and the moment you know that it follows that the real end imaginary parts of an analytic function are also infinitely differentiable okay. Therefore the fact that you take a harmonic function which has derivatives only up to order less than or equal to 2.

And satisfies Laplace equation okay actually it is a very it is weaker when compare to the result that the function u will actually have derivatives of all orders okay. And it is a and all the derivatives of all possible orders all mixed possible derivatives will exist and they were all be continuous, the **the** reason is because u is locally the I mean u locally is the real part of an analytic function.

And an analytic function is infinitely differentiable that is the reason okay, so so what you must remember is that this condition that we have put that the u satisfies Laplace's equation and all these partial derivatives exist up to order 2 and they are continues as rather weak condition okay . now you see now for such harmonic functions they have a very important property that is called the mean value property okay. So, I will explain what this mean value property is , so maybe I can save some space and rub this off.

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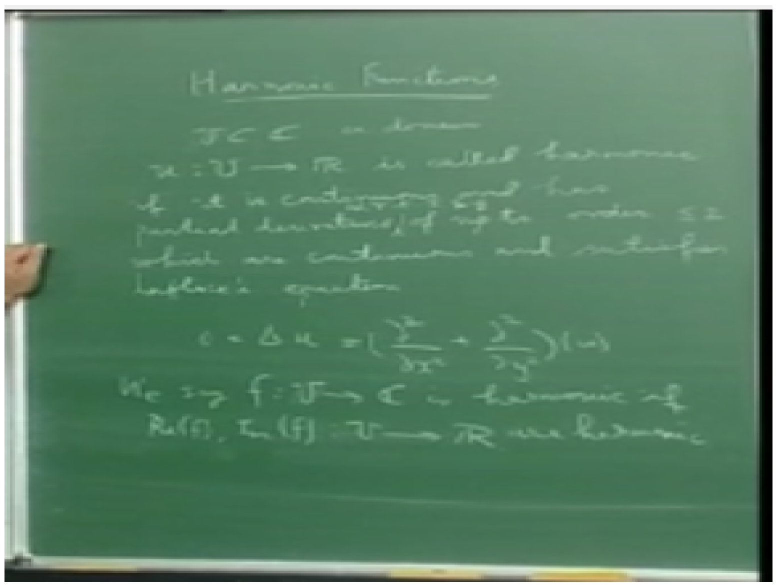

And write partial derivatives here with respect to x and y okay , so let me define this mean value property.

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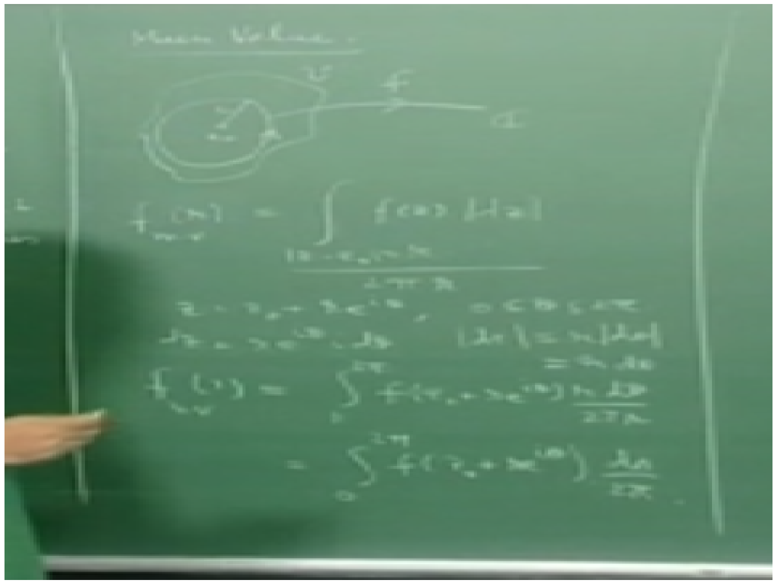

So, well I can of course you know and now I can extend this before I do that I can extend this definition to a complex valued function being harmonic. So, you know if f is now a function from u to c when will say f is harmonic if it is real and imaginary parts are harmonic. So, **so** let me write that down also that is just a extension of this definition we say f from U to C is harmonic if real part of f imaginary part of f from U to R are harmonic.

So, a complex valued function is harmonic if and only if it is real and imaginary parts are harmonic alright. Now what is this business about mean value okay, see suppose you have some domain U and you have a function f defined on U taking values in the complex plane and you take this point take a point z0 inside U right. Then what you do is you take a circle centred at z0 radius R right.

Then well of course I am taking this circle inside U, so that even on the circle at every point on the circle f is defined okay. Now what I do is you know I define f mean value okay this is the mean value of f at over the circle so over this circle centred at z0 radius R I define this mean value, this is to be and this is the function of R okay it will change if I change the radius in the circle.

So, I am looking at circles centred at the point z0 okay of various small r okay and for given one such circle with radius small r I am defining this mean value okay of f with respect to that circle and what is it, it just you know the mean value is defined like this you simply calculate f of z you take f of z that is the value of f at a point z on the circle and then you integrate with respect to mod dz.

Because you know integrating with respect to mod dz will actually give you the arc length okay, integrating mod dz over an arc will give you the length of the arc alright. So, and what is the path of integration is the circle, it is mod z-z0=R okay I do this. And then this is you know this is to be thought of as summing all the values of f as move across the circle okay and this mod dz should be thought of as the arc length right.

So, this is the sum of all the values of f as you take the as you move the point around the circle and then if you want the average value I have to divide by the arc length of the circle which is 2 pie r okay, this is the mean value. So, the mean value is this sum of you know how a mean value is defined it is an average, so what I am doing is I am taking I am summing up all the values of the function on the boundary circle, that is what the numerator gives, that is what the integral gives.

And then I am dividing by the length of the circle the circumference which is 2 pie r okay and this is called the mean value of f for that circle alright. Now you know if you put z=z0 you know you can parameterise the circle as z=z0+r e power i theta where theta varies from 0 to 2 pie okay, you can parameterise this circle and if you do that you know what will this integral changes to see dz will be r into e power i theta into i d heta okay, mind you r is fixed, theta is varying alright.

And if I differentiate e to the i theta with respect to theta I will get e power theta times i right and if I calculate mod dz I will end up with r d theta okay. In fact I will get r mod theta if you want and I can write as r d theta because if I take theta to be increasing then the d theta is a change in theta is also positive okay. So, if you put it in this if you substitute if you for this if you change this integral in z into this integral into an integral at theta what you will get is a the mean value of f over r is well, it is integral from 0 to 2 pie.

So, theta will vary from 0 to 2 pie f of z0+r e power i theta and mod dz is going to give me r d theta and of course I will have 2 pie r, so it also has this expression it can also this integral from 0 to 2 pie f of z0+r e to the I theta into the d theta by 2 pie. So, this is another expression for the mean value okay of f over a circle. Now what is so this is this how you define the this is just a mean value.

And the only thing you must notice is that the mean value is taken on the circular arc the mean of this taken on the circular arc and the fact that you are doing it on the circular arc is reflected by integrating with respect to mod dz okay. Because integral with respect to mod dz over an arc gives you the length of the arc, so this mod dz comes for that reason because you are adding up values of the function on the arc right.

And of course you divide by the length of the arc which is 2 pie r this case, the arc is a circle, the whole full circle. Now what are the properties of this mean value of r okay. The first property is that this so you know I have up a new function cooked up a new function for sufficiently small r I have to take sufficiently small r starting from well starting from r=0 alright.

And I am going to look at sufficiently small r all r below a certain value, so that all these circles are inside my domain alright. And I am getting a function based on that r, I am getting values I am getting mean values of f based on this r okay.

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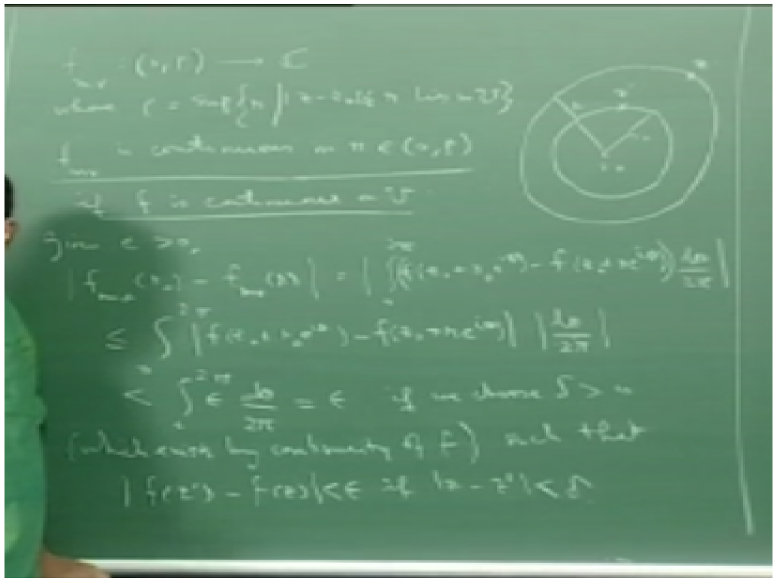

Now what is the property of so you know so f is f mean value is defined from well it is defined on 0 rho if you want well maybe I can even put so the way I have done it, it is 0, rho to C where you know rho is if you want the maximum of all r such that the circle mod z-z0 less than r lies in U I am actually taking a largest possible circle alright .

In fact let me put equal to r I want the circle, take the you take the maximum over all r okay such that mod z-z0=r lies in U and sometimes probably this maximum may not exist okay . so, I think it safer to put supremum here if you want you can put supremum , so if I am if I am want write it like this, write this supremum over all r such that mod z-z0=r is in U .

In fact I want the whole disc inside U, so you know it is very important that I do not many any holes in between okay, I do not want any holes in between. So, I have this mean value function alright. Now the claim is that the mean value function is continuous okay it is a continuous function okay and so f sub mv is continuous that is a observation, why is it continuous.

Because you see it is continuous in what it is continuous in the variable r which lies in 0 rho okay well why is that so because you know it is pretty easy it is actually it is because the continuity of f. The continuity of f sub mv the mean value of f as a function of r is a result of the continuity of f as function of z. Of course here I have not mention that, so you assume .

So, here let me add that if f is continuous on U, so if I take a continuous complex valued function then the mean value function that it defines is also continuous why is that so it is because you see if given epsilon greater than 0 okay. If you estimate the difference between f mean value of let us say some r0 and f mean value of r okay, then this turns out to be I mean it is going to be modulus of if I use this I get integral 0 to 2 pie f of z0 + r0 e power i theta-f of z0+r e power i theta the whole into d theta by 2 pie, this is what I will get by the definition of the mean value.

And but you know I can use this you know this fact modulus of the integral is less than or equal to integral of the modulus. So, this is less than or equal to integral from 0 to 2 pie of mod f the integrant which is z0+r0 e power i theta-f of z0+r e power i theta into mod d theta by 2 pie which again continuous to be d theta by 2 pie right. And well this can be you know this can be made less than integral from 0 to 2 pie epsilon theta by 2 pie which is equal to epsilon okay.

If we choose delta greater than 0 which exist by continuity of f such that mod of f of z let me write z prime-f of z can be made less than epsilon if mod z-z prime is less than delta, you see f is continuous. Therefore you know if you give me any z and z prime then I can make the values of f at z and z prime as close as I want, if I choose z and z prime close enough and of course here I am taking z prime to be any point of this type z0+r0 e theta namely a point on the circle with radius r0.

And I am taking z to be a point on the circle r with radius r okay and I can do this just because of continuity and what does this calculation tell you, it tells you that you know z is lying on circle of radius r, z prime is lying on the circle of radius r0 alright and the fact that this distance can be made less than delta means that you are you know bringing r0 and r close okay.

So, the moral of the story is that if you bring r0 and r close okay, then f r0 and fmv ro and fmv r come close, so that will tell you that if r tends t r0 okay. Then fmv r tends to fmv r0 which means that fmv is a continuous function of r okay. So, this tells you that f sub mv the mean value function it is a continuous function of r alright. So, **so** you know so the diagram is like this I mean I am having this z0, I am having this r0 and I am having this circle.

And this where I am taking my z prime okay which is this argument and then I have this bigger circle which is bigger or smaller it does not matter both ways. So, this is r and I am having a point z there okay and if I bring r close to r0 I am actually bringing z and z prime close okay and if z and z prime come close then f of z and f of z prime come close.

And therefore the difference between fmv ro and fmv ro fmv r becomes small enough okay. So, this is the proof of the fact that the mean value function define by complex valued continuous function is a continuous function of r alright. Then the further thing is actually this mean value function actually is even defined at the origin and it takes the value f of z0 okay.

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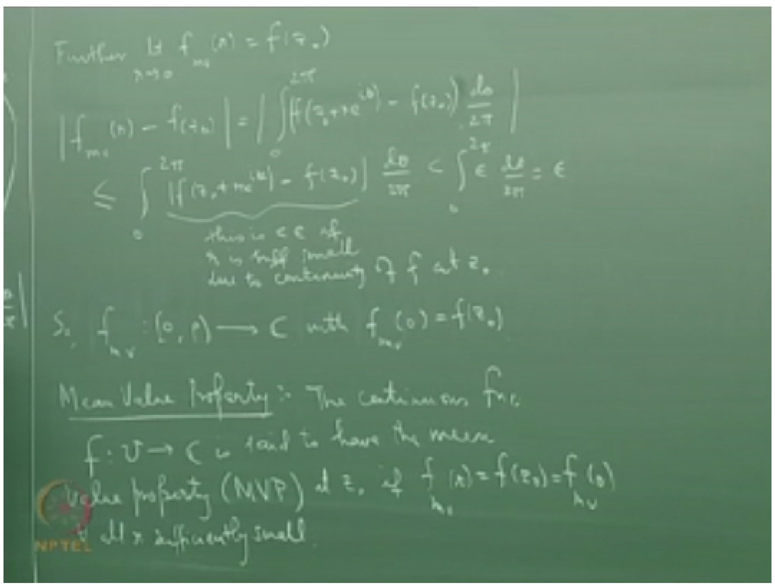

So, further limit r tends to 0 fmv of r is actually f of z0, the mean value is the mean value function tends to f z0 as r goes to 0 or you must understand that r goes to 0 the circle is becoming a smaller and smaller and smaller circle centred at z0 and you know it is natural to expect you know as I make this circle smaller and smaller and smaller the function values are also going to come close to fz0.

Therefore you should expect the average also to be fz0, I mean if all the function values are close enough to **f** fz0 then the averages also become close enough to fz0 okay and then therefore if you take the limit you should get only fz0 okay. So, this is intuitively correct but then you can rigorously prove it by the same method what you can do is you can calculate the value fmv r-f of z we calculate this what you will get is the same kind of , you will get modulus .

The same kind of estimation as here integral 0 to 2 pie I will get f of z+z0+r e power i theta –fz0 d theta by 2 pie. So, again modulus of the integral is less than or equal to integral of the modulus, so I will get this is less than or equal to integral come 0 to 2 pie mod of f of z0+r e power I theta-f of z0 well into d theta by 2 pie and this can be made lesser than integral 0 to 2 pie epsilon d theta by 2 pie which is equal to epsilon, I can make this less than epsilon.

Because I can make this less than epsilon and that is because of continuity at z0 of f if I make r sufficiently small okay . This is less than epsilon if r is sufficiently small due to continuity of f at z0 after all f is continuous also at the centre alright. So, the same so what this tells you is that as r tends to 0 fmv r tends to f fz0, so actually, so you know you have this the mean value function actually extends to 0.

So, 0, rho to C with f of mean value at 0 is fz0 okay, so what you have done is we have defined a mean value function which at 0 the mean value is fz0 which is just the value of f at z0. And for any r it you get the mean of the values of f along the circle centred at z0 radius r alright. Now this is all about the mean value function, now when does a function have a mean value property is the function set to have mean value property.

If the mean values of the function all they are all equal to the value at the centre for sufficiently small discs okay. So, here is a definition so this is a definition of the mean value property, the continuous function f from U to C is set to have the mean value property I will abbreviate it as MVP it **it** has the mean value property at z0 if f mean value of r is actually equal to fz0 which is just f mean value at 0 for all r sufficiently small okay.

So, that is for all r belonging to 0 epsilon okay, for some epsilon greater than 0 okay. In other words the function, a complex continuous complex valued function has a mean value property at a point, if you take sufficiently small circle surrounding that point and you take the mean value of the function you get exactly the value of the function at the centre okay, this is a mean value property.

So, you see this property is some it is a completely it is a kind of integral condition right because after all the mean value is defined by an integral, mean value is defined by this integral here or here okay. And the integral of a continuous function is always exist alright. So, it is very easy define and now comes the big theorem really big theorem.

So, **so** the big theorem is function is harmonic if and only if it has a mean value property a continuous function as harmonic if and only if it does a mean value property at every point it is a terrific theorem. Because you see 1 part of the theorem says that if it is harmonic does a mean value property that is more or less easy to prove okay and you would have seen a proof of that in the in a first quotient complex analysis by for example taking the function to be the real part of an analytic function okay are you .

And if it is a complex valued function you can even take I mean you can do it using analytic functions, you can use analytic functions for example and show that analytic functions have the mean value property okay. In fact the analytic functions having the mean value property is exactly Cauchy's integral formula in a way alright and on the other hand the striking is the other **t**he implication with the other direction.

That you start with a continuous function which has only mean value property and what you get is that it is harmonic and why it is so powerful is because a condition we have a put is an integral condition I mean you have this mean value you have the mean value function associated to that function which is defined by an integral okay, I mean it is a it is defined on just a you know interval to the right of 0 and of course you can include 0 also.

The fact that this is for sufficiently small value is equal to the function value at z0 okay, it is seems to be rather simple condition but that condition if you put at every z0 what it results in is that the function is harmonic and what is harmonicity that is a great deal as I told saying that the function is harmonic means you are saying it infinitely differentiable okay at least it is I have told you that even though in the definition of harmonic function we only require that it is derivatives of to order too exist and or continuous.

But it is actually infinitely differentiable and it is also satisfies laplace's equation, so it is rather amazing that you know you put this simple condition on a continuous function okay, it results in the function becoming harmonic it makes the function C infinity, it makes both and real imaginary parts of the function infinitely differentiable with respect to both variables and that is amazing.

And all the derivatives all mixed partial derivatives of all orders exist they are all continuous and on top of the function also satisfies the Laplace's equation okay. So, you see this condition it is a really a remarkable property.

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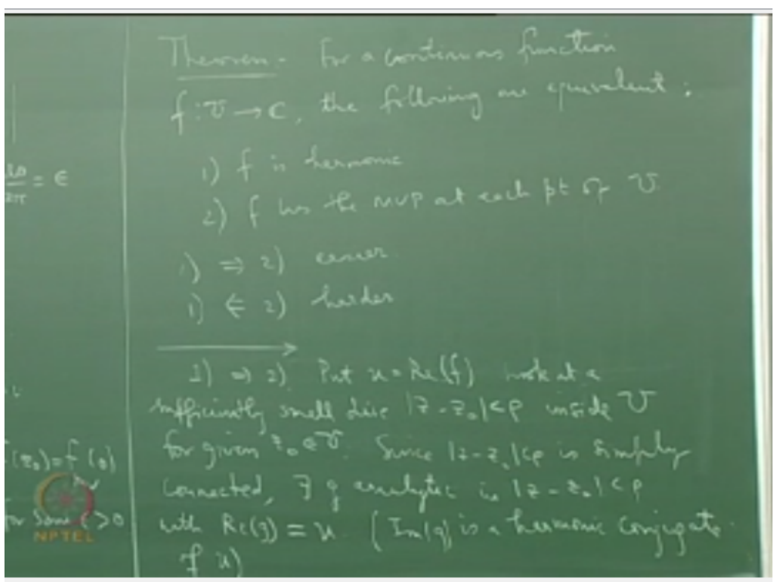

So, here is the theorem is for a function for a continuous function f from U to C the following are equivalent, so what are they number 1 is f is harmonic, number 2 f has the mean value property at each point of U. So, it is an amazing equivalence okay, see 1 implies 2 is easier okay it is easier provided use some Cauchy theory which you would have covered in a first quotient complex analysis okay.

But 2 implies 1 is serious order that if you have you can obviously expected to be harder because the condition 2 seems to be very simple condition you have just saying that some integral is you calculate some integral for small values of r and all the value should coincide to the value the function at the centre okay. The mean values of the function around small enough circles should give you the value at the centre of the circle.

That is a condition 2, it is a relatively simple condition, but then from that trying to conclude that f is harmonic ok which means effectively you are getting infinite differentiability of f which is very very serious alright. So, 2 implies 1 is harder and 1 proof of 2 implies 1 involves the so called (()) (38:27) integral formula using which you can solve the problem for the disc okay.

And then you get 2 implies 1 okay, I will try to see that we can cover that we have enough time to do that in the series lectures. But I can for the moment the easier path 1implies 2 is something that we can easily check okay, so well for **for** 1 implies 2 of course I am going to use some complex analysis to do it alright. So, what I am going to do is you know put U=real part of f okay mind you f is I am assuming f is harmonic alright.

And I take U to be real part of f okay and what I am going to do, I am just going to look at a sufficiently small disc mod z-z0 less than rho inside U for given z0 in U okay. Now mind you U is a real part of f and f is harmonic and you know f is mind you f is only a complex valued harmonic function I am not saying f is analytic, I am just saying f is a complex valued harmonic function.

Therefore the definition is that both the real part and the imaginary part of f are real valued harmonic functions, that is all I have. So, I am taking U to be real part of f okay it is harmonic is given to me, now I am going to use this very powerful theorem that if you have a harmonic function and if it is define on a simply connected domain okay, then it as a harmonic conjugate okay namely I can find another function.

Such that if you put that as a imaginary part of a new function then that with this as a real part then you will get an analytic function okay. So, I am going to use that which is this is the fact from complex analysis okay. So, I am going to use that, so yeah so since mod z-z0 less than rho is simply connected there exist g analytic that is holomorphic in mod z-z0 less than rho with real part of g= U okay.

And imaginary part of g will be harmonic conjugate of U, so imaginary part of g is in a harmonic conjugate of U and you would have seen an first quotient complex analysis that you can get different harmonic conjugates but they will only differ by a constant right. So, of course you know you can prove 1 implies 2 also directly appealing to some version of (()) (42:41) theorem right that is also another way of proving it.

But I am trying to circumvent I am trying to avoid all that and I am trying to give a an elegant proof which anyway uses some powerful results. But my main idea is to you know give you an indication of one line of argument that 4will convince you that at least statement 1 implies statement 2 alright. So, and you see now you know now I am in the following situation.

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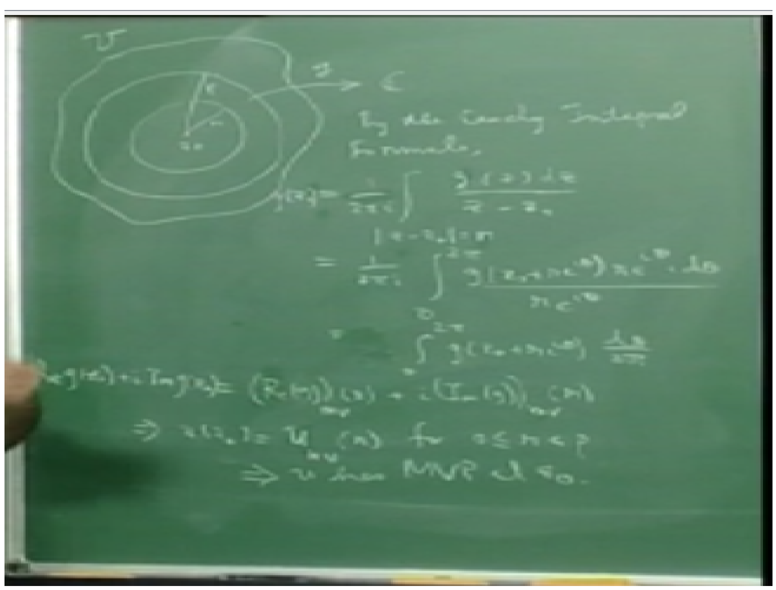

So, you know I have this z0 I have if I take an r such that r is less than rho, so this is inside u and I have f I have this g defined here with real part of g=u okay/. Now you know you apply the Cauchy integral formula okay by the Cauchy integral formula you know that if I calculate the integral over mod z-z0=r of course whenever I am calculating such integrals I am taking the positive orientation every the anti-clock wise orientation.

Suppose I calculate f of z by z-z0 okay, you know Cauchy integral formula tells sorry not f g you know that I will get I think maybe I have to put 1 by 2 pie i you know that I will get g of I will simply get g of z0 okay. This is the Cauchy integral formula alright okay and you know if you write out the integral in terms of theta on the right side, then this the same as 1 by 2 pie i integral from theta=0 to 2 pie g of z is r, z0+r e power i theta.

And just parameterizing this circle radius small r centred z0 as z=z0+r e power i theta where r small r is fixed and theta is varying from 0 to 2 pie. So, then I will get this of course I have forgotten a dz here, you should put dz there is variable of integration and well you know if I write that what that dz is I have already written it out here, it is r e to the i theta i d theta divided by z-z0 is r e to the i theta.

And what I will get is, I get 1 by 2 pie integral 0 to 2 pie g of z0+r e to the i theta so you know I let me write it in this form d theta by 2 pie this is what I okay. And what is this is, this is you if you this g is real part of g+imaginary part of g+i times imaginary part of g. So, you know, so if I write it like that this is integral if I write as g as real part of g+i times imaginary part of g what I will get is I will get real part of g mean value at r+i times imaginary part of g sub mean value at r, this is what I will get.

And that is equal to on the left g z0 which is real part of g z0+imaginary part i times imaginary part of g z0 this is what I will get okay. But then what is real part of g my real part of g is U, so what I will get is if I compare real parts I will get U at z0 is the mean value of U at r for 0 less than or equal to r less than rho and this is the same as saying that you know u has the mean value property okay.

So, I mean what I have done is the same proof actually is the usual proof that you would have seen a first quotient complex analysis that tells you the real and imaginary parts of a an analytic function have the mean value property okay. So, what I am doing is you give me a harmonic function I am using this rather powerful theorem that if you take a sufficiently small disc which is simply connected.

Then it has a I mean if I take a real valued harmonic function then this is real part of an analytic function okay and then I am using the fact that the real and imaginary parts of an analytic function have the mean value property okay. And I am saying I am just trying to deduce it using Cauchy integral formula, so this proves this is 1 way of showing 1 implies 2 okay.

For 2 implies 1 we need lot of machinery, so I would not get into that but my aim is to tell you that you know harmonic functions have the mean value property alright which in some sense you should have seen in the first quotient complex analysis. Now come to the so called maximum principle, I want to say it is maximum principle.

So, you would have again loosely learn this in the first quotient complex analysis as if you take an analytic function on a domain with boundary bounded okay a bounded domain . so, the domain is bounded, so it is boundaries also bounded then the analytic function if you take the modulus of the analytic function that will attain it is maximum only on the boundary and not in the interior.

And if it attains a maximum the interior then it has to be constant. So, a non constant analytic function will attain it is maximum only on the boundary okay, now this is actually a property of harmonic functions . So, the maximum modulus being attained at the boundary is a property of the harmonic functions okay and that is why analytic functions have that property.

Because an analytic function is harmonic, the analytic function has both real and imaginary parts which are harmonic okay. And the mean value property for I mean the maximum modulus principle for harmonic functions gives you the maximum modulus principle for analytic functions. In fact it also gives you maximum modulus principle for complex valued harmonic functions okay.

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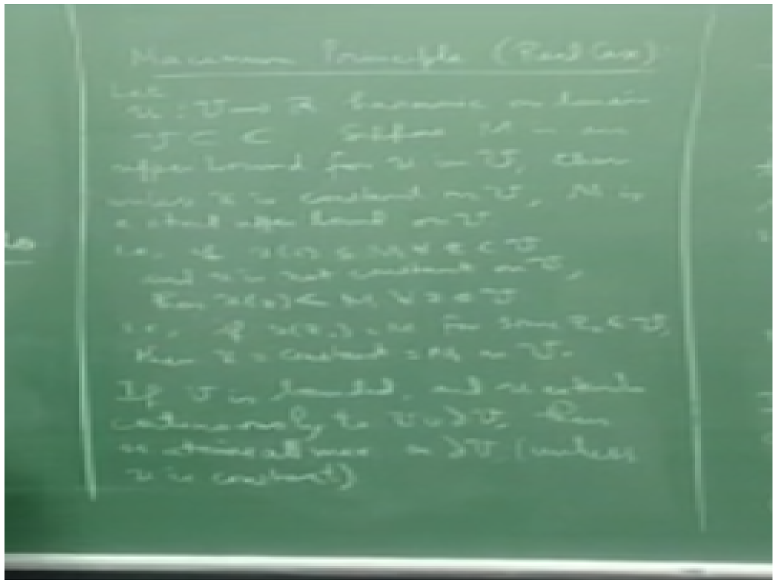

So, let me write this maximum principle so let me write a few cases, so this is the maximum principle for the real case and what is the maximum principle for the real case you have u from capital U to R harmonic on domain U in the complex plane. So, U is a real valued harmonic function alright and so this is let suppose M is an upper bound for u for small u in capital U.

Then unless u is constant on U, M is a strict upper bound I should say strict upper bound on D. So, this is the real this is the maximum principle for real valued harmonic functions, so in the situation is that I have this domain u in the complex plane and I have a harmonic function on that right and suppose all the values of u are bounded by bounded above by a real M okay.

So, that is if so you know if I write it in symbols if u of z is less than or equal to M for all z belonging to U okay and u is not constant on U. Then u of z is strictly less than M for all z U, so this is the so every upper bound is a strict upper bound that is the maximum principle. In other words another way of saying it that if u attains this upper bound that is with the upper bound is not strict and if that upper bound is attained in at some point then it has to be constant.

So, another way of saying it is if u of z0=M for some z0 in U, then u=constant=M on capital U alright this is strict version of the maximum principle alright. This is the maximum principle for the real case and then you have the same statement as a maximum principle for the complex case and then you have a maximum principle for also for analytic functions okay.

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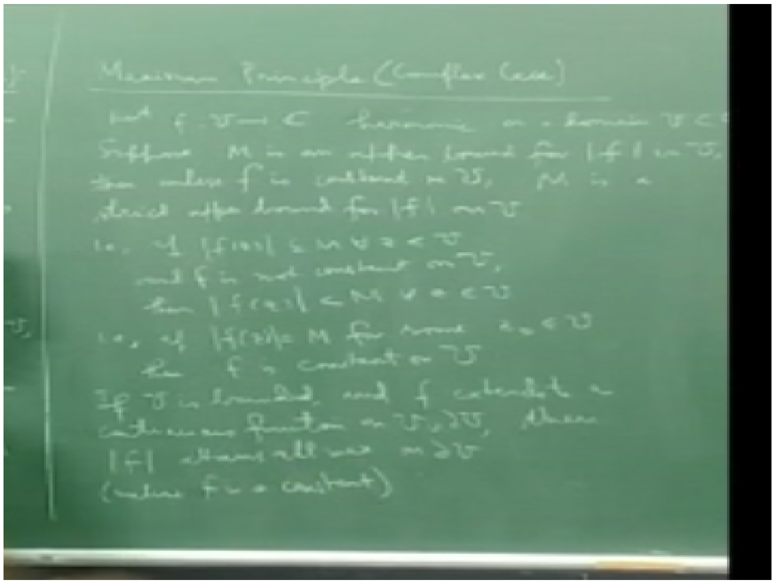

So, let me write out the other case also maximum principle complex case, so here I take f from U to R you see complex harmonic function, so it is a complex valued harmonic function on a domain U with C and so **so** the same statement suppose M is an upper bound for mod f okay. Now because it is complex valued by a bound for f we actually mean a bound for mod f okay in U.

Then unless f is constant on U M is a strict upper bound for mod f on U, so this is a statement okay. The only thing is when you take a complex valued function we have to and when you talk about bounds you know you have to take mod f. Because you know I cannot write complex number 1 complex number lesser than another complex numbers because complex numbers are not order okay.

And the ordering of the real numbers does not extend to complex numbers right. So, I cannot write a statement such as M is an upper bound, so small u here u is real valued I can write that okay. But I cannot write M is an upper bound for f does not make sense because f is complex value, I should only write M is an upper bound for mod f okay. In general whenever you say f is bounded we always mean there is a bound for the modulus of f okay.

And then the rest of a statement is a same thing, so that is if so let me write this if f of mod z is lesser than or equal for z in U and f is not constant on U. Then mod fz is strictly less than M for all z in U, so it is a strict upper bound and of course the other way of writing it is also like this if f of mod fz=M for some z0 in U then f is constant on U okay and therefore you know this applies also for mind you here f is harmonic complex value.

But need not be analytic okay but then even if f is analytic this applies because after all if f is analytic then both the real and imaginary parts of f are harmonic therefore f is also harmonic an analytic function is always harmonic. Because it satisfies Laplace equation both the real and imaginary parts are harmonic but harmonic complex valued function need not be analytic okay.

So, this is also applies to analytic functions right and the usual version that we often use is the contra positive of this which is that if your domain is bounded okay. Then f attains it is in the eal case the maximum is attained on the boundary, in the complex case the modulus of the, the modulus maximum modulus attained on the boundary, so let me write that also.

If U is bounded and u extends continuously to U union tou u, tou u is a boundary of U, then u attains a maximum on the boundary, so this is the version that you are all familiar with which is actually which is also the equivalent to that. so, let me write the same thing here. If U is bounded and f extends to a continuous function on U union tou u mind you in these cases u union tou U is compact because it is closed.

Because I have added the boundary tou u to u and it is bounded, so it is closed and bounded, so it is compact okay. Then mod f attains a maximum on tou u and only on tou u okay, you cannot get a maximum in the interior okay, I am not so I should say in fact in such thing a maximum it say all maximum only on the boundary, here also if it is say all maximum only on the boundary you cannot get a maximum in the interior unless it is constant okay.

So, tou u of course unless u is constant, so here also unless f is a constant okay, so the either the function is the constant in which case it has a same value throughout if it is real valued and it has a same modulus throughout including the boundary if it is complex valued or if it is not constant then the maximum is only on the boundary for the real valued function and for the modulus of the complex valued function.

So, this is a maximum principle for harmonic functions okay I will continue the next lecture and I will give you the proof of this I think it is just will use the mean value property very easily. And then so I am making is a fact that harmonic functions have the mean value property which are more or less sketch the proof of right and then my aim of doing all this is to get to the so called Schwarz' lemma okay. And that is required in a preliminary discussion of the Riemann mapping theorem okay so I will continue in a next lecture.