Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-03
Morera's Theorem and Normal Limits of Analytic Functions

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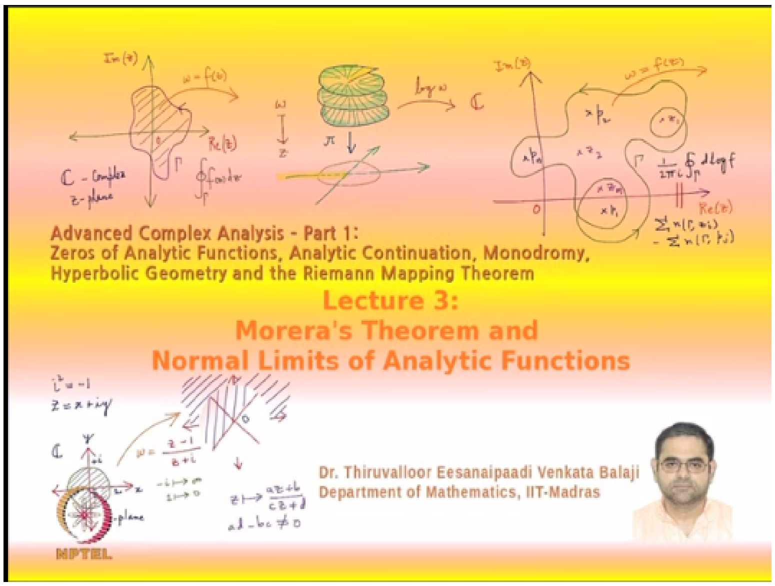

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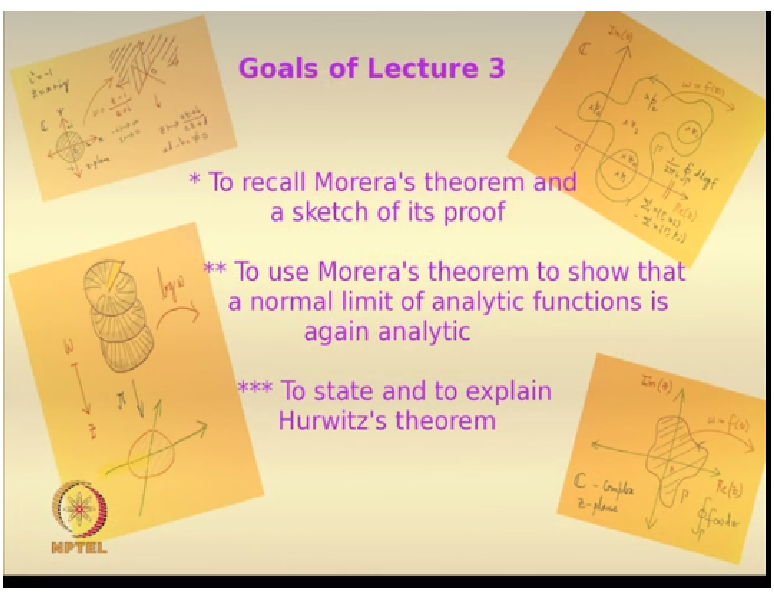

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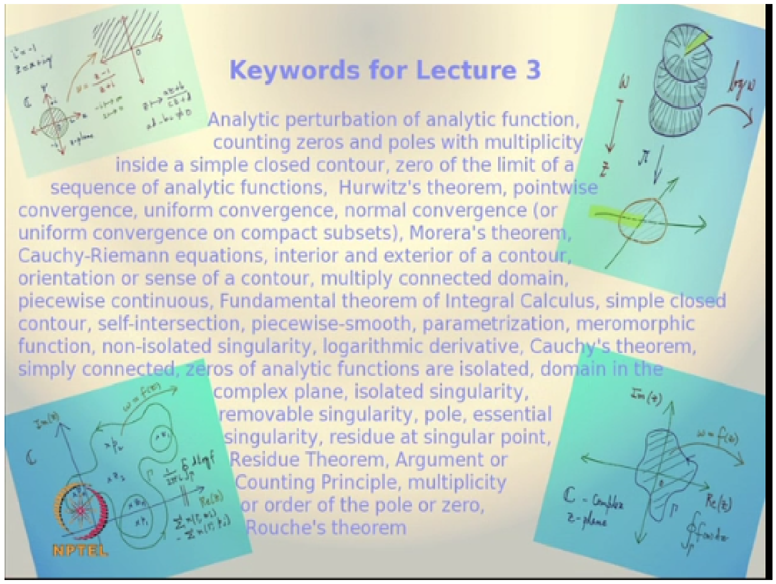

Okay, so let us continue with a discussion so you see what you been doing is looking at a zeros of analytic functions okay. So, what we saw in the last lectures was basically to be so called Rouche's theorem okay and that actually in principle tells you that if you, you know perturb an analytic function by a in a small way then the number of zeros that it enclose that is enclosed in a region is not going to change okay.

Now of course this came out of the basically order the argument principle and the argument principle in term came out of a the residue theorem okay. So, we are now we are going to again continue with this study of zeros of analytic functions and of course I should remind you that the argument principle actually tells you gives you method of counting the number of zeros and poles with multiplicity inside a **a a** closed contour, in the region that is enclosed by a closed contour.

Now what I am going to do this today's lecture is try to actually look at the 0 of a functions an analytic function that is obtain as as a limit of a sequence of analytic functions okay. And what I am going to say is that essentially this 0 of the limiting analytic function is gotten by you know it is gotten by zeros of the analytic functions that converge and taking limits of such zeros okay.

So, you have a sequence of functions that converge to an analytic function if the analytic function has a 0 then that o can be gotten as a limit of zeros of the functions that originally we started with which converse to the given analytic function. So, at essentially this is what is called as Hurwitz's theorem okay. But let me start of the discussion like this.

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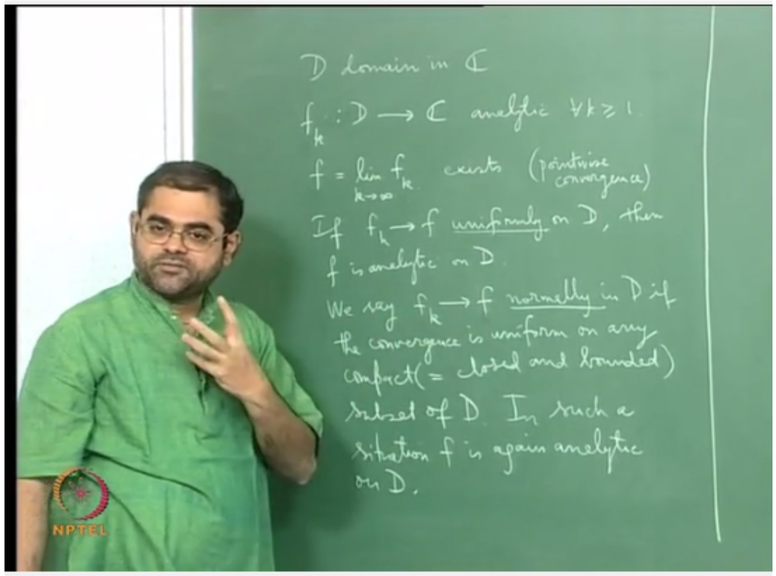

So, you see suppose you have, so you have the following situation , so D is a domain okay which means it is an open connected set okay. And of course the domain in C such as that the complex numbers and you have suppose you have a sequence fk of functions analytic on analytic for every k written in the equator 1, so here is the sequence of analytic functions defined on the domain D.

And you are know from a first course in analysis what I mean when I say that fk converges to a function f point wise. So, f is limit k tends to infinity of fk exists okay, so the sequence of functions converges to a function this is actually point wise convergence what does that mean, it means you take any point z in D small z in D take the value fk of z for various k you will get a sequence of complex numbers.

And you take the limit of this sequence, the limit of the sequence exist and whatever the limit is that is what you are calling as f of z and you do this for every z in D okay. That means fk of z converges to f of z for each point z in D, this is point wise convergence and of course you know that point wise convergence by itself it is not good enough. Because you know to begin with you know if the fk's are continuous okay forget analytic.

Suppose the case I just continuous and f is the limit then f need not be continuous, so what really helps is the notion of uniform convergence you should remember. So, you know if fk converges to f uniformly then and if fk each fk is continuous then f is continuous okay. So, of course so you know if fk each fk is continuous and if I want f to be continuous I need uniform convergence.

But you see what happens is uniform convergence will not happen on a whole domain okay usually uniform convergence happens only on compact subsets of the domain. In general this is the best condition to assume and this is what will happen, so let me write that down and what I am trying to say is that if I assume that fk converges to f uniformly on the whole domain.

Then of course I will get that not only is f continuous, if fk is continuous in fact f will become analytic, if fk each fk is analytic okay. But the fact is you cannot in general expect uniform convergence on a whole open set usually uniform convergence is to be expected on bounded closed and bounded subsets which are otherwise called compact sets, because you know in the Euclidean space any subset is closed and bounded if and only it is compact from basic topology okay.

So, you can replace the condition that fk converges to f uniformly on the whole domain by a slightly weaker condition which is fk converges to f uniformly on compact subsets of the domain okay and this technical condition is refer to in some of the literature as normal convergence okay. So, if fk converges normally to f okay then the fact is that since each of k is analytic, f becomes analytic okay.

So, it gives you the nice situation that a sequence of analytic functions does converges to an analytic function okay. So, that is the first piece of information that we need, so let me write this down if fk converges to f uniformly on D then f is analytic on D okay. This is a and of course you know what do I say what do I mean when I say fk converges to f uniformly.

So, you know there is a difference point wise convergence in uniform convergence let me remind you what does point wise convergence means mean, it means that if you take a point small z in capital D and you take the you evaluate this sequence of functions of that point you will get a sequence of complex numbers, you will get the sequence fk of z and that fk of z will converge to f of z okay.

And fk of z converges to f of z means given an f epsilon you can find index n such that f fk of z-f of z can be made in modulus lesser than epsilon whenever k is greater than that index. But if but all this is for a fixed point z in D okay, but we change the point the index that you will need if you change the point z then the index that you will need to make the distance between fk of z and f of z lesser than epsilon will also change.

In general that index will depend on z also but if does not depend on z that is you are able to get a an index such that fk of z, the distance of fk of z can be made lesser than epsilon from f of z for all z irrespective of what z it is you chose the domain, that is when you say that the convergence is uniform okay. So, the uniformness is and if that there is no dependence on which point of D you choose okay, so uniform convergence is of course stronger than point wise convergence.

And what I am making the statement that fk converges to f uniformly that is what when I make the statement that is what it means okay. So, you must have come across this in first course in real analysis or complex analysis but anyway let me remind you. Now but of course is a very strong condition to expect uniform convergence on a whole domain is in general too much what you normally will get is you will get uniform convergence on closed and bounded or compact subsets of domain.

And that condition which seems which is certainly weaker than this is called normal convergence and the fact is even if you weaken this condition to normal convergence still the limit function continuous to be analytic okay. So, let me write that down we say fk converges to f normally in D if the convergence is uniform on any compact which is equal to closed and bounded subset of D okay.

So, this is called normal convergence, so normal convergence is actually uniform convergence on compact subsets okay. And of course if you have uniform convergence on the whole domain then of course you have normal convergence because uniform convergence holes on the whole domain then it also holes on any subset of the domain okay and therefore this is a stronger condition than this, this is a weaker condition.

But you see this is what will happen in (()) (12:03) in practice and but the fact is even if you weaken this condition you still get that the limit function is analytic. So, let me write that in this in such a situation f is again analytic on D okay, so having looked at this of course what is our aim, our aim is actually to show that in this situation that is if you have sequence of analytic functions converging f normally in a domain.

Then you give me a 0 of the limit function okay give me 0 of the limit function then that 0 is gotten by zeros of the functions in the sequence by convergence okay, that is a 0 of f is an accumulation point of zeros of fk that is essentially what Hurwitz's theorems okay. So, but before that let me take a small diversion to explain why is that if fk converges to f uniformly or even normally why is it that the limit function is again analytic okay.

So, I mean I am doing this purposely many of you must have come across this in a first quotient complex analysis but I just doing this a like you recall some basic things and it is helps you refresh your memory okay. So, you see you know, so how does have prove this that is what I want to say.

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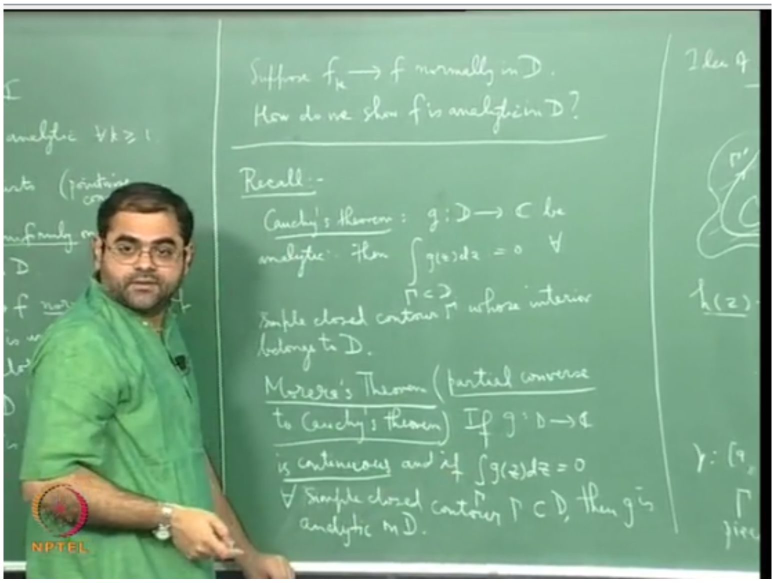

Suppose fk converges to f normally in D suppose is the case okay, now how do I show that and given of course that each fk is analytic how do I show f is analytic okay. So, just think about it you know if you want to show function is analytic there are so many ways of showing, one is of course you show that it is differentiable at every point, it is differentiable 1 at every point which means you have to calculate the derivative at every point.

And show that the derivative exist okay, the other way is of course to write out the Cauchy-Riemann equations and check that the a valid and also to check the first partial derivatives, we satisfy the Cauchy-Riemann equations are continuous okay but of course there is a deeper theorem which says that you do not have to check the continuity of the first partial derivatives it is enough you to check the Cauchy-Riemann equations hold.

And of course it is very important that you assume f is will have to use the fact f is continuous okay. So, first of all because your normal convergence it means that if I take a point and if I take a closed disc surrounding that point, the closed disc surrounding that point in in the domain D then of course that is a compact subset of D because it is closed and bounded and fk will converge will f there uniformly.

And since each fk is continuous analytic of course means it is continuous, so the limit function f will also become continuous okay. So, the continuity of the limit function will come automatically just because of uniform convergence okay. And now I could choose such a disc at every point of the domain and therefore I get continue at all points. So, what I want to first understand is the movement I make such an assumption to begin with f is automatically continuous on the whole domain okay.

Then of course the our question is how do you show that, that f is analytic, so the of course as I told you 1 is to show is the differential at each point and the other one is to show it differential at each point, the other one is to show that the first I mean the f satisfies the Cauchy-Riemann equations. And the third way is of course to show that f is locally represented by a convergence power series okay.

You see and but these things are not so easy to do, in general they are not so easy to do what really you can use in principle to show that the function is analytic is to check the conditions of Morera's theorem which is a you know kind of converse to Cauchy's theorem. So, you see if you remember so let me say the following thing how do we shows it let me write this how do we show f is analytic.

So, this is our question okay, so I will draw a line here, so see if you recall first is Cauchy's theorem okay suppose f so let me use g so let me use D I need not to use the same D but in way let it be so let g from D to c be analytic okay then integral of g of zdz over gamma a simple closed curve is 0 for every simple closed contour, gamma whose interior belongs to D**.**

And of course whose interior and of course I should say gamma itself a sub of D the contour itself should be the domain and the interior of the contour should also mean the domain. So, that means there is a standard orientation on the contour, the orientation is such that the interior of the contour lies to your left as you traverse the contour.

For example if you traverse if the contour is a circle, if you traverse it in the anti-clockwise sense then the interior will circle will lie towards your left if you walk on the circle that is called the orientation. And whenever you do path integral you have to orient the path, path has to be oriented, the direction has to be given and if it is a closed if it is integral over a closed path or a lube then the orientation is always given it said to be positive.

If the region inside the contour is lies to the left as you walk along the contour in the direction prescribed okay. So, and of course when I say simple closed contour by contour I mean a curve which is piece wise smooth. So, it is a continuous image of an interval the closed interval with starting point equal to the ending point. And the interval can be divided into closed sub intervals such that the parameterizations are given by smooth functions.

They are given by functions which are not only differentiable but the derivative is continuous okay, so contour is a piece wise smooth curve okay. So, this is Cauchy's theorem right, so what it essentially tells you is that you integrate analytic function over a closed curve like a loop you are going to get 0 okay. Now what is Morera's theorem it is a like but partial converse to Cauchy's theorem.

So, here is Morera's theorem is and what does it say, it says if g from D to c is continuous and if integral over gamma g of z dz is 0 for every gamma as above okay. So, I should say something here for every not gamma as above slightly weaker for every gamma in D for every closed simply closed contour gamma in D then g is analytic okay, this is Morera's theorem, Morera's theorem is like a converse to Cauchy's theorem.

But it is a partial converse because you see in Morera's theorem you have to assume already that the function g is continuous okay. And then so that is you need continuity of g that is 1 extra thing that you need but what you leave out is you are just saying that over any loop sitting inside D the integral of the function is 0 which is the condition of Cauchy's theorem but it is weaker because here the loop had to be such that the interior of the loop had also to be in D whereas here I am not putting the condition there.

I am just saying that the interior of the loop need not be in D it is not required okay, that means this will work for a region with a hole and to **to** give you an idea why this is true**.**

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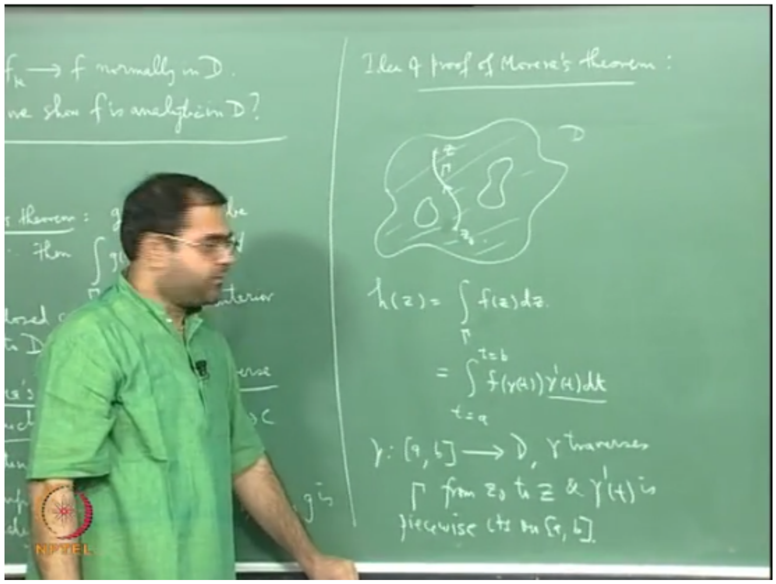

So, you see suppose you have idea of proof Morera's theorem see suppose i have a suppose my domain D is like this it might have some holes, suppose this is my domain okay the shaded region it might have holes okay. So, suppose but the domain is of course an open connected set, if it has no holes then it is called simply connected and that is the condition that any closed group can be continuously shrunk to a point without going out of the domain.

For example if there is a hole then loop surrounding that point cannot be shrunk continuously to a point without going outside the domain. So, this is not simply connected this I mean domains like this are called multiply connected they are not simply connected. And see what look at the condition see the condition is see suppose I fix suppose I how do I use that condition the integral over any closed loop is 0, see you take a point z0, you fix a point z0 okay.

And take any other point z okay what you do is you just join z0 to z by a path gamma okay and then what you do is you look at the function h of z given by integral over gamma of f of zdz look at this function okay. Now first what I want it to understand is that f is continuous therefore f is continuous on the whole domain okay therefore f is continuous also on the arc.

And you need continuity on the arc to be able to compute the arc integral or the path integral because the continuity is basically done using the notion of a Riemann integral you just form Riemann sums over the arc, you parameterize the arc okay which means that you think of this as an image of an interval of the real line and then you are actually integrating over that integral interval.

The composition of f with gamma, the gamma is a parameterization of the arc okay, so the point is that and of course that will involve that will need the fact that the that the parameterization of this arc is piece wise smooth okay. So, what I want to tell you is that this is well defined okay, this is well defined because f is continuous okay, so in you know if you want I can in fact write it as t=a to t=b, f of gamma of t into gamma dash of t dt, where is the map from the closed interval a, b into D.

Such that gamma traverses capital gamma from z0 to z and gamma and gamma dash t is piece wise continuous on a, b. so, this is how you write the so let me see I have written it correctly I think probably I should not put gamma yeah z is so yeah so you formally write it as z on the curve on the on this path capital gamma you write a point omega, as omega=gamma of of t. Then D omega will become gamma dash of D dt and that is how you get this gamma dash of Ddt okay. So, I think there is a little bit of the probably I should avoid z here.

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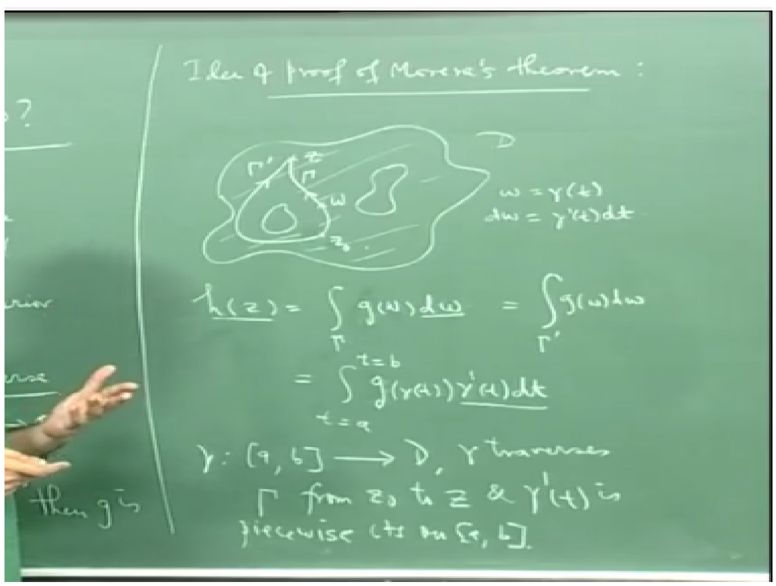

Because the same z also is used here, so let me use omega because that is better because you know what will happen is omega if you take a point omega on the arc**.** Then omega is just gamma of t where t varies small t varies from a to b and then you will see that D omega will be gamma dash of D dt and that is how I replace formally this D omega by gamma dash of D dt.

We do this formally but then you can actually make a trigger as by the definition of the Riemann interval okay this is actually limit of Riemann surface okay. But mind you what you are use in the process is mind you to do define a Riemann integral now you see this is a integral on the real line on the on this closed interval on the real line and you know to integrate a function you know it should be at least piece wise continuous f is already gamma is continuous and f is continuous.

So, the composition this is just the composition f circle gamma so that is continuous, so it is a continuous function of t and gamma dash of t is piece wise continuous because that is what I told you what a contour means, the contour a path is always assume to be piece wise continuous. That means the parameterization small gamma of the contour capital gamma must have a derivative which is piece wise continuous.

So, you see unless I do not unless I have these continuity of these 2 things I cannot define as a integral okay, that is where the technicality comes in okay you have to notice that okay. So, the point is in any case I can define this but the more important thing is I defined it based on gamma and I am but I am writing it only a h of z where z is end point of gamma, so the question is what is a dependence on gamma, am I been careless about the dependence on gamma.

But the answer is that exactly where I am using this condition that the integral of okay I think I have messed up something so it should be g here sorry, it should have been g because it is this g I am trying to show is analytic okay. So, I am thinking of this g okay which is of course or somehow I change notation from f to g okay, so it is g that I am worried about.

So, my situation is like this I have the domain D I have the function g defined on D with values in complex numbers and it satisfies the conclusion of Cauchy's theorem that integral over every loop you see okay**.** And the extra condition is g is already assume to be continuous so I look at this integral okay and the fact is that this does not depend on gamma.

Because you know if I took some other path gamma prime okay it will also this will also be equal to integral over gamma prime of g of g omega d omega this is also be true that is because integral over gamma followed by the inverse of the path gamma prime which is –gamma prime will be 0 that is here is where I am using the condition that the integral over a closed path is 0.

So, this therefore hz is well defined it really does not depend on what path I am choosing and it also does not worry if there are holes in the region mind you okay. Now you see now it is a matter of now it is very easy it is something like the fundamental theorem of integral calculus, see what you are actually saying is you are saying that h is the integral of g, so you can immediately say that the derivative of the h must be g okay.

So, that is a fundamental theorem of calculus kind of statement and what this will tell you is that you will get from this.

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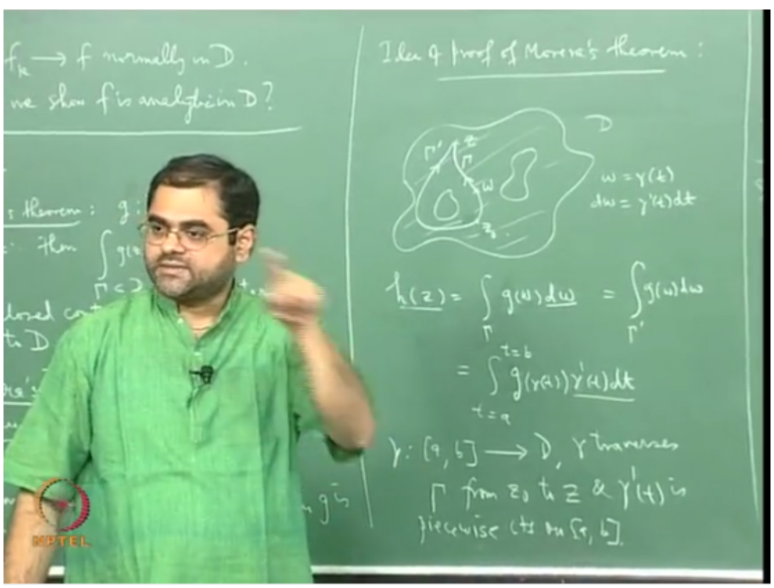

That g dash of sorry h dash of z is actually g of z you will get this, this is just a version of the fundamental theorem for integral calculus. So, all you are saying is in other words what you are saying is that you know if g is continuous and g satisfies this condition that integral over every loop is 0. Then g has a an anti derivative okay the in other words there is a function , so I should say then g is a derivative of function okay**.**

But you see but what is this tell you see tells you h is differentiable it tells you h is differentiable on D that means h is analytic on D okay. But if but you know one thing if a functions analytic then all derivatives exist and the derivatives are also analytic. Therefore since so this equation in one go tells you that not only is h analytic because it is differentiable once everywhere but because **is** it is analytic it is derivatives is also analytic.

And then tell you that g is analytic and that is how you prove Morera's theorem okay, so this implies h is analytic and that implies h dash is analytic, h dash is g is analytic okay. So, that is the condition sketch of the proof that is how Morera.s theorem is to prove okay. Now it is this Morera condition that is very useful to check a function is analytic for example in this situation.

So, you know so let us go back to that situation, suppose fk converges to f normally in D suppose this is true. So, you know so you take a so here is your D maybe it may have some holes one does not bother and I take a point z0 in D, so here is I take a point z0 in D what I have to show is that I have show I want to show that if each fk is analytic I want to show that f is analytic that is what I want which fk is analytic.

Then so is f this is what I want to show how very simple to show us function is analytic on a domain so it is analytic at every point okay. So, what I will and mind you showing analytic at every point means it is saying that it is analytic at every point it is same as same analytic in a neighborhood of every point okay, so I have to concentrate I should take a an arbitrary point.

And I am just concentrate on a neighborhood of the arbitrary point which I will take it to be a disc. So, you know what I will do that I will choose a small disc here which is given by let ne call this is as D sub z0 and what is this choose these sub z0 is open disc centered at z0 small enough radius set of all z in D set of all z such that mod z-z0 is let us say lesser than some rho sub set of t choose for z0 in D.

Such as small disc you can you will because you know D is a basically domain therefore D is an open set, therefore given any point in D there is a small disc surrounding that point which is in the which is in your domain. So, choose such a disc but what I will do is I will also assume that the boundary if you want the way of drawing even the boundary of the domain is inside the but probably I do not need it okay.

Now you see now what you do is you look at you try to check f is analytic inside this Dz0 okay if I want to check is check that f is analytic inside this D z0 that I will have to show that the mind you f is already continuous why it is continuous because fk converges to f normally means fk converges to f in every compact subsets, so it will converge to f in the closure of this disc which will this disc along with the boundary circle okay.

So, if you want let me also, so let me write that assume that D z0 closure is contained in D which means is just saying that mod the set of all points such that mod z-z0 equal to rho is also in D okay that is the boundary of the disc that boundary is circle is also in D okay.

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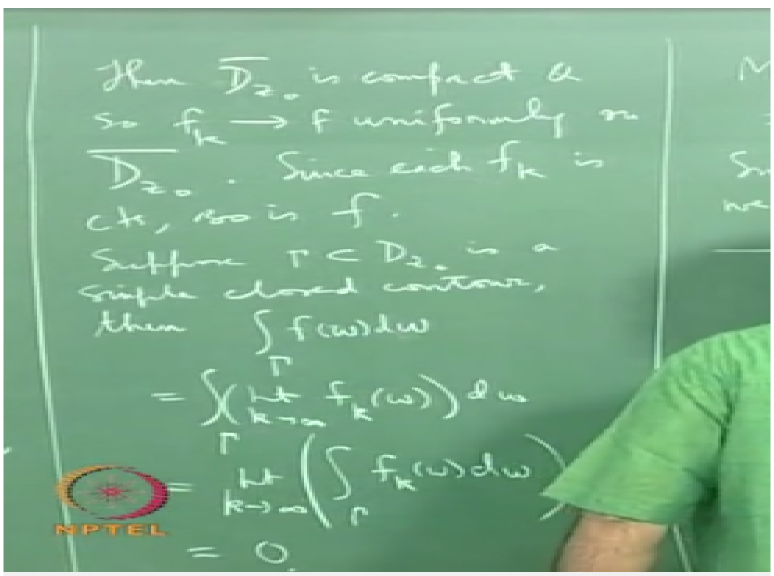

Then you see then f then D z0 bar is compact and so fk converges to f uniformly on Dzo there is uniform conversions. Because that is what normal convergence, it means uniform convergence on compact subsets okay. And by let me remind you a compact set is something that is closed and bounded when I take a closure of this disc it is of course close and bounded okay, so it is compact.

Now so **so** what happens is since each fk is continuous, so is f because uniform limit of continuous functions is continuous okay, there is no problem bounded. And so I have a continuous so if I restrict my attention to this disc I have the function f it is continuous I want to show it is analytic I can set the condition of Morera's theorem all I have to check is that give me a loop inside that disc.

If I show that for any closed loop inside the disc, a simple closed contour inside the disc, the integral of f vanishes then Morera's theorem will tell me that f is analytic inside the disc okay and in this way I can cover the whole region by small discs and cover all the points that will tell me if f is analytic everywhere on D, that is what I want. So, you know suppose gamma inside Dz0 is a simple closed contour, suppose it is simple closed contour.

Then integral over gamma of f of f of z dz or let me keep you f omega d omega is by definition integral over gamma limit k tends to infinity of fk of w omega d omega okay and now I am again using other important fact you see when a sequence of functions converges uniformly to a function, this is uniform convergence is so powerful that not only does it make the limt continuous.

If the original function is continuous, it also allows you to interchange integral an limit okay, so long as on the region you are integrating in this case is a loop on that the convergence is uniform okay. And of course this loop is contain this simple closed contour, this loop is contained inside this and there of course uniform convergence is going on. So, uniform convergence gives you the authority to interchange limit in integral.

So, what will happen is this can be written as limit k tends infinity integral over gamma fk omega D d omega I can do this. The reason why I can go from here to here is because the uniform convergence mind you but then each fk of omega is analytic by Cauchy's theorem this is 0 therefore this is 0. Therefore by Morera's theorem f becomes analytic in Dz0 but then since z0 is arbitrary f becomes analytic in D and that is the proof okay.

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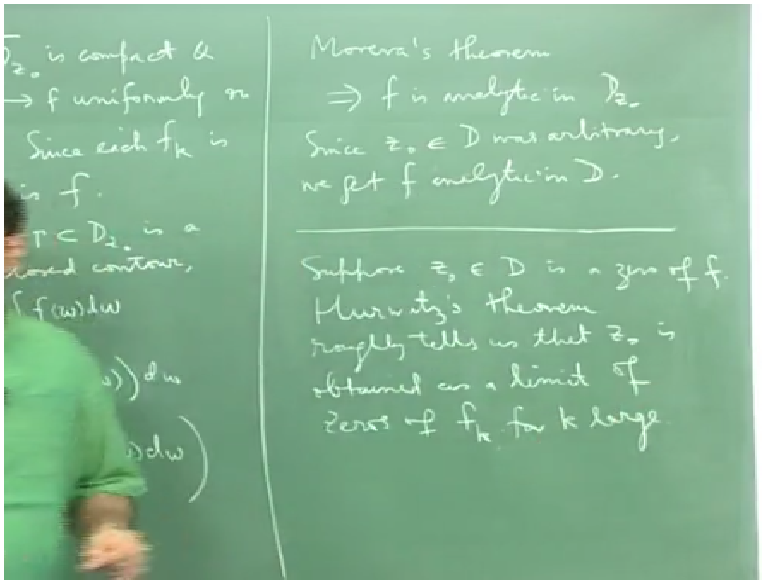

Morera's theorem implies f is analytic in Dz0 since z0 belong into D mose arbitrary we get f is analytic in D okay. So, this is something that you need to know if you take a normal limit of analytic functions on a domain then the limiting function is certainly analytic okay fine. So, alright, so now we come back to this question which is the question that is answered by Hurwitz's theorem.

And that is about what is going to happen if you take a 0 of f okay and Hurwitz's theorem basically says that the 0 of f is going to comes from zeros of fk be on a certain stage okay. So, let me write that down . Suppose z0 it an equal to D is 0 of f, f is uniform f is the normal limit of these analytic functions f case okay, suppose z0 is a 0 of f.

Hurwitz's theorem tell at a roughly tells us that z0 is obtained as a limit of zeros of fk for k large, this is what this is essentially what Hurwitz's theorem says okay. So, 0 of the limit function, the limit analytic function of a normal form family of functions that normally converge to a an analytic function, then the 0 of the limit function comes as a limit of zeros of the original functions beyond a certain stage okay.

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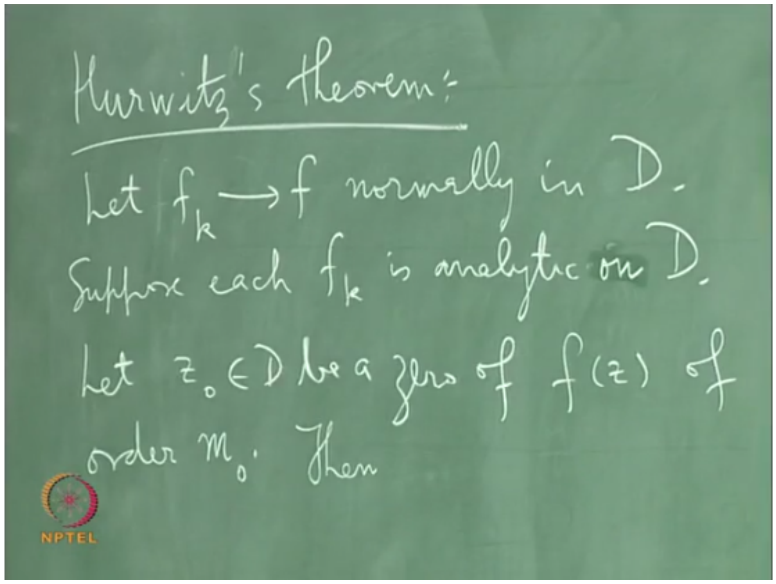

Now, so let me let me write down the let me write down Hurwitz's theorem properly, let me write down the exact statement that we will try to see how to prove it. So, here is Hurwitz's theorem, so let fk normally in D, suppose each fk is analytic in D then by what I have told f is also analytic on D okay, I should say on D if you want. Let z0 in D be a 0 of f of z of order m 0 okay, then their exit rho greater than 0 with mod z-z0 yeah, so I should say.

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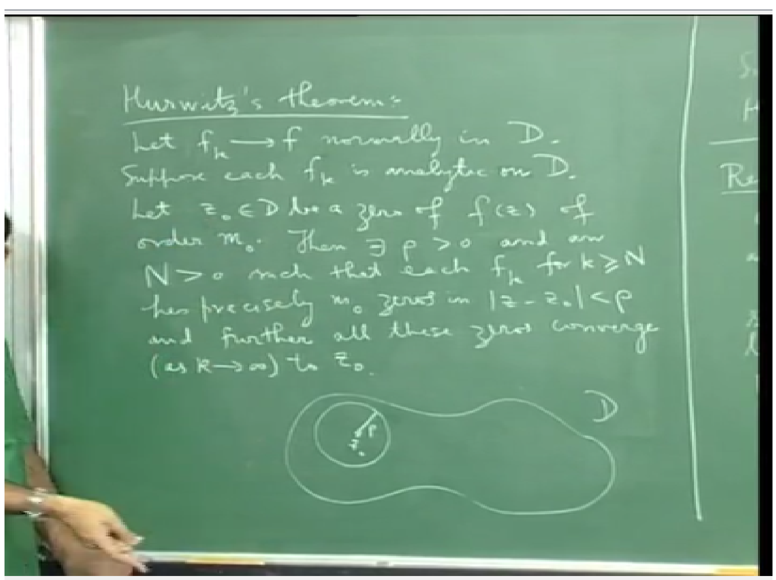

Then there exist rho greater than 0 and an N greater than 0 such that each fk for k greater than or equal to N has precisely m of zeros in mod z-z0 lesser than rho and further all these zeros converge as k tends to infinity to z0 okay. So, you see the statement is something like this let me again explain. So, here is the our domain D because I have been drawing a bounded domain need not even be bounded okay, I am just drawing a bounded domain, mind you domain is an open connected set.

So, it could be unbounded okay, but we really not worried about bounded inverse because the normal convergence, because normal convergence ensures that so long as you restrict your attention to compact subsets the convergence uniform, that is what you always need okay. So, you see the point is if you give me z0 which is 0 of the limit function f then I can find a rho disc of radius rho centered at z0.

Such that beyond a certain stage all the functions in the sequence they also of the same number of zeros as the order of the 0 of the limit function and all these zeros as you decrease rho okay all these zeros so in fact I should say I should say all these zeros converge in fact I must say as rho tends to 0 to z0 okay.

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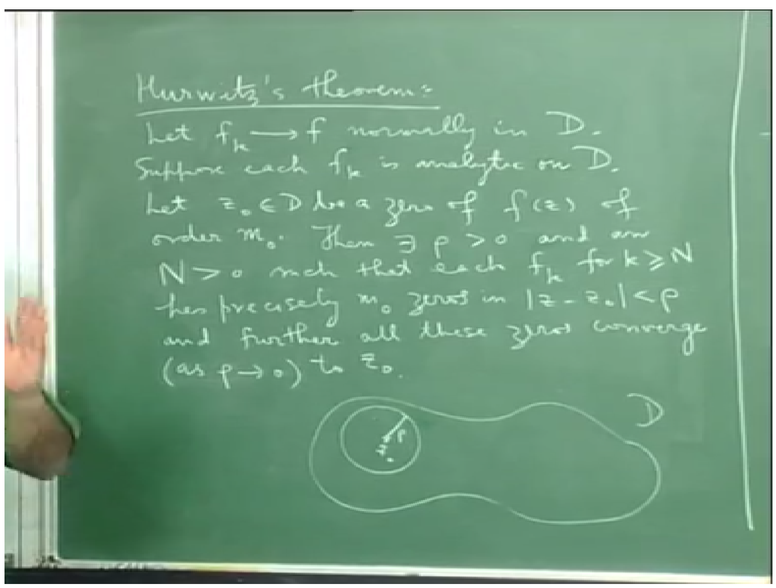

So, you know if I take a particular rho then there is rho first of all such that f sub capital N f sub capital N+1 and so on all the functions beyond the index N, all of functions have exactly the same number of zeros as f dash has the limiting function f has. But when I say number of zeros of course 0 should be counted with multiplicity okay. And the fact is that if you make rho smaller then you will get zeros which are you know closer I am closer to z0.

So, obviously z0 will be an accumulation point of the set of all zeros and in other words as you make as you choose any one of these zeros for each rho and make rho smaller and smaller you will get a sequence of zeros of the corresponding f is and they will all go and converge to z0. So, if you think of it diagrammatically it is like it looks so you know it looks something like this, if you look at fk probably you will have if you zeros.

Then if you look at fk+1 suppose M0 is 5 I have 1, 2, 3, 4, 5 then maybe you know and this is z0 okay, then if you take if you so maybe is should draw bigger diagram here expand this. So, the situation is like this here is z0 and these are the zeros of fk, k sufficiently large and if you count all this zeros with multiplicity the total will add up to M0 which is a multiplicity of this 0 of f at z0.

And the point is as you make this rho smaller all these zeros will converge to z0 which means you know they will just coreless, they will just blend into one another and become one point, it will become a point of order M okay, this is what happens, this is what Hurwitz's theorem says okay, so I stop with that and in the part we will in the next lecture I will explain how to prove this theorem okay.