Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-29
Proof of the Algebraic Nature of Analytic Branches of the Functional Inverse of an Analytic Function at a Critical Point

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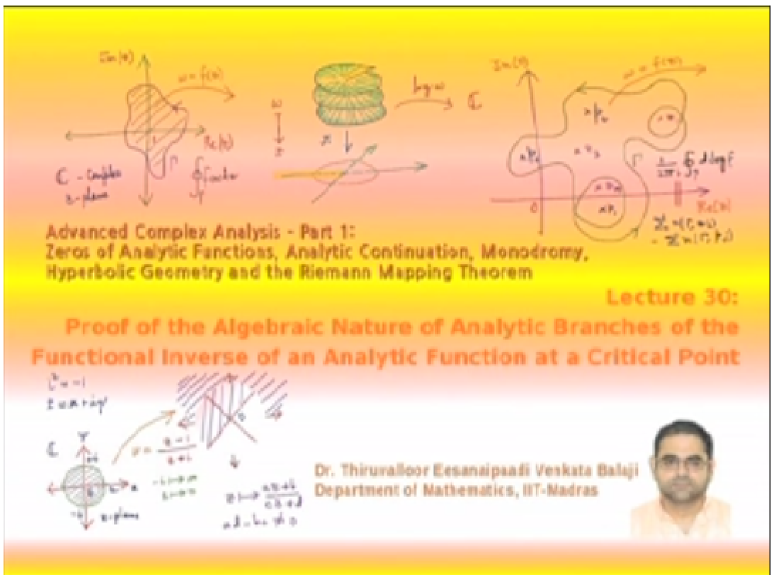

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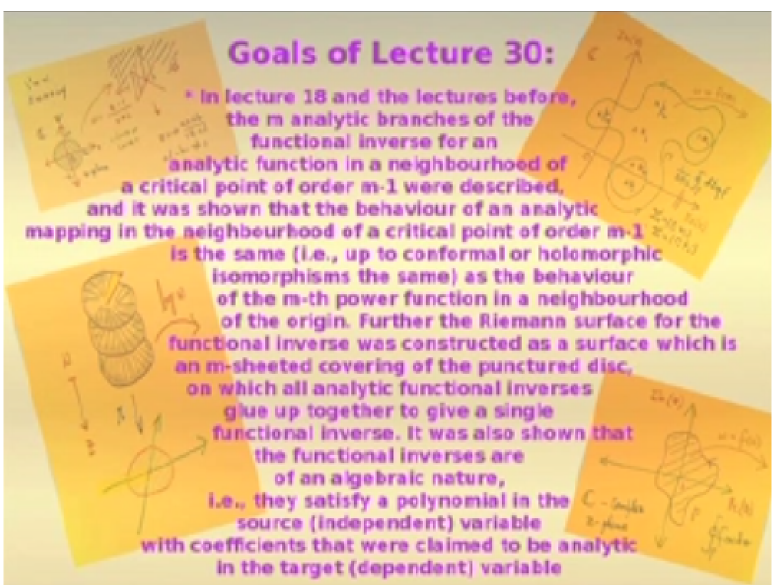

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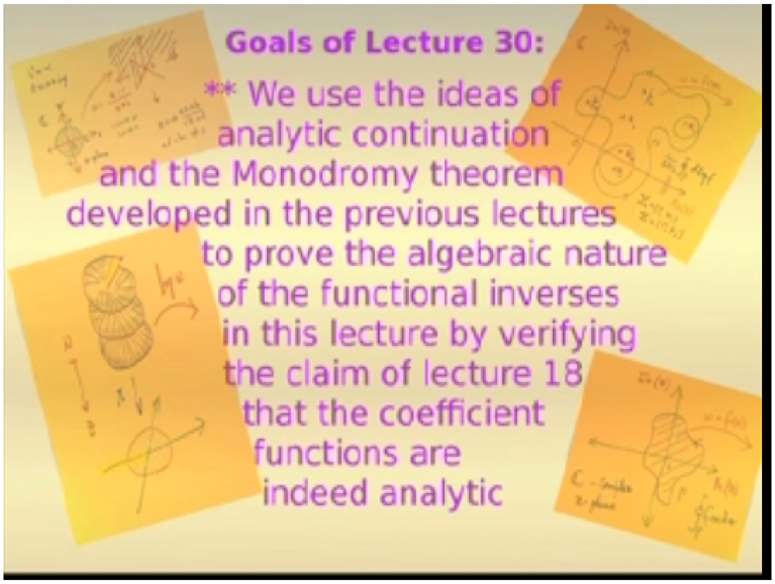

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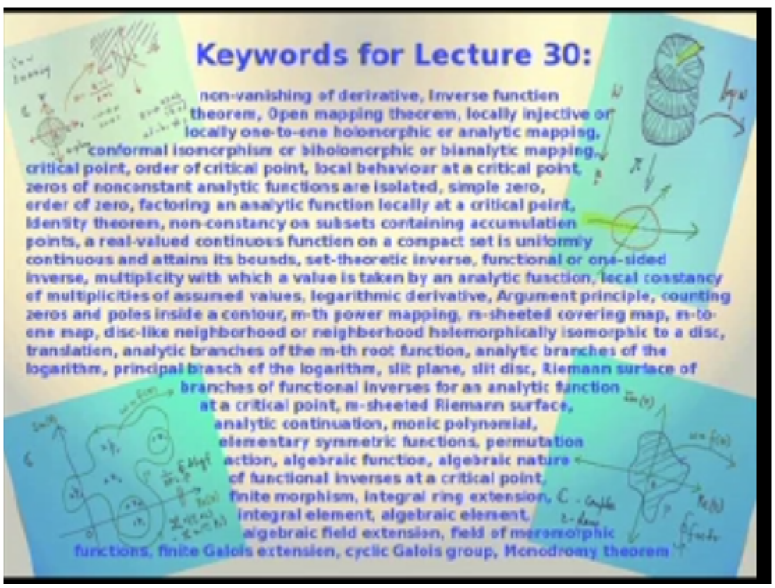

Okay, so **so** we have seen in the last lecture various version of the monodromy theorem, now I just want to discuss the application of all this to study the behaviour of an analytic function at a critical point which we had done which we had begun a few lectures before, I started discussing analytic continuation on the monodromy theorem, I want to conclude that now. So, you know our situation was like this we had so let me draw the diagrams for that and remind you of what we have already seen.

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So, you know we had some we had a point z0 in the on the complex plane and well and you are looking of course this point is in a domain u where an analytic function w=f of z is defined. And well the analytic function f of z takes z0 to the value w0, so well this is the complex plane this is the z plane and the target plane is of course is the omega plane or the w plane, this is just the complex plane which is the w plane.

And the point z0 goes to w0 this is f of z0 okay and what we have assumed is that we are assuming that z of course you know let me recall everything depends on the vanishing or otherwise of the derivative f at z0 okay. If f dash the derivative of f at z0 is not equal to 0, then the inverse theorem will tell you that f is locally 1 to 1 okay. So, it means that sufficiently small disc surrounding z0 will be map to something that looks conformably like a sufficiently small disc at w0 okay.

So, what you can expect is a sufficiently small disc will look like something that is conformably equivalent to a disc okay. For example you can get something that looks like a distorted disc okay for example, that is the situation if the derivative of f at z0 does not vanish and that is the consequence of the inverse function theorem right and this happens when z0 is not a critical point.

And what happens if z0 is critical point that is the other case when the derivative of f at z0 vanishes okay and how do we look at that case we look at we try to look at z0 as 0 of f-w0 okay. so, that is the way we think we so you know right from the beginning of this course we have been looking at zeros of analytic functions we have been translating everything to zeros of analytic functions.

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So, what you do is you see z0 is a 0 of f of z-w0 because f of z0 is a w=w0 and you know and f of z-w0 it is derivative is a same as f dash of z because after all w0 is a constant alright an analytic function-a constant is also an analytic function. Because the constant functions are analytic for the derivatives is of course at the same and suppose inverse function if you know so if f of f dash of z0 is not equal to 0.

Then inverse function theorem implies f is an isomorphism on small disc centred at z0, this is the inverse I mean this is the inverse theorem with applied together with the so called open mapping theorem also, see inverse function theorem will tell you that f is at in a neighbourhood of z0 which is not a critical point that is a derivative does not vanish, f can be inverted okay, open mapping theorem will tell you the image of that neighbourhood is also an neighbourhood.

And open mapping that the fact that it is 1 to 1 open mapping will tell you that the inverse is also continuous. The inverse function theorem will tell you the inverses is actually holomorphic, so it is a holomorphic isomorphism. So, this is the nice picture that you get if the derivative does not vanish at z0 which is a situation when z0 is not a critical point okay. but we are want to look at the situation when z0 is a critical point.

So, if z0 is a critical point of order m-1 okay, so this is the reason I took m, we take m-1 there is because then z0 will become a 0 of order m of f of z –w0, z0 is a 0 of order m of f of z-w0 okay, mind you in this case if the derivative is non 0. Then z0 becomes a simple 0 of fof z-w0, if f dash of z0 is not 0 then z0 becomes a simple 0 of f of z-w0 okay. But if f dash of z0=0 that means z0 is a critical point and if it is 0 if it is a critical point of order m-1 then z0 is 0 order m of f-w0 okay, this is how the definitions are arranged, we know that already.

And well you know so you know what you do is you do a you make a you know we factored this throw a sequence of transformations. So, what we did was we first we now we chose first of all we chose a small enough disc surrounding z0 of radius rho where z0 is the only 0 of f of z-w0 and that we can do because you know zeros of an analytic function are isolated okay.

So, choose rho greater than 0 such that f of z-w0 has is not 0 is non 0in this deleted neighbourhood which is 0 less than mod z-z0 strictly less than or equal to rho okay, I can do this because the zeros after all z0 is a 0 of fz-w0 which is an analytic function and the zeros of an analytic function are isolated. Of course you know I am assuming the analytic function is non constant okay.

And the reason that analytic function is not constant is I mean that is because you know throughout I am working with non constant analytic functions, I am not working with constant analytic functions and for analytic functions that I defined on a domain even they are being constant on a locally will also mean that they have to be globally constant that is because of the identity theorem.

So, I cannot even get constancy on a even a small neighbourhood because identity theorem actually says you cannot get even constancy on a subset which has an accumulation point okay. So, anyway the functions that I am considering as certainly not constant functions, so I mean if the function is a constant then it is derivative is identically 0, so I will not then that is the only case when the zeros are not isolated because every point is becomes a 0 alright.

So, if a function is constant it has a value 0 at a point then it has to have the same value 0 everywhere. So, the zeros will be the whole domain alright and then the zeros are not isolated, so whenever you say that the zeros of an analytic function are isolated of course you are worried about non constant analytic functions okay and we are considering only non constant analytic functions, so **so** the 0 z0 is isolated.

So, I have a deleted closed neighbourhood of the of z0 disc surrounding z0 deleted disc closed disc where there is no other 0 alright. And in particular you know if I look at, so you know fz-w0 does not vanish even on the boundary circle okay. so, if it is modulus on the circle it will have a minimum and I am calling that as delta okay and delta greater than 0 such that delta is lesser than mod of f of z-w0 for mod z-z0=rho okay.

So, this is also this is also it is true because after all mod z-z0=rho is a circle, it is a compact set and mod fz-w0 is a continuous function, real valued non negative real valued continuous function and you know continuous function on a compact connected set will the image will be a an interval and because it is taking non negative values the left end point of that interval could very well be delta okay and give me a positive quantity.

So, you could take delta to be either the minimum value of mod fz-w0 on the circle o anything will anything positive but lesser than that anything will do okay. and now what you can do is now you can now look at this delta neighbourhood of w0 okay. so, mind you the order of the choice is first I choose z0 I mean I have start with this z0, I have choose this rho, then using rho I choose delta and that gives me the delta neighbourhood here. So, the delta neighbourhood here is well I am looking at all w with mod w-w0 less than delta okay.

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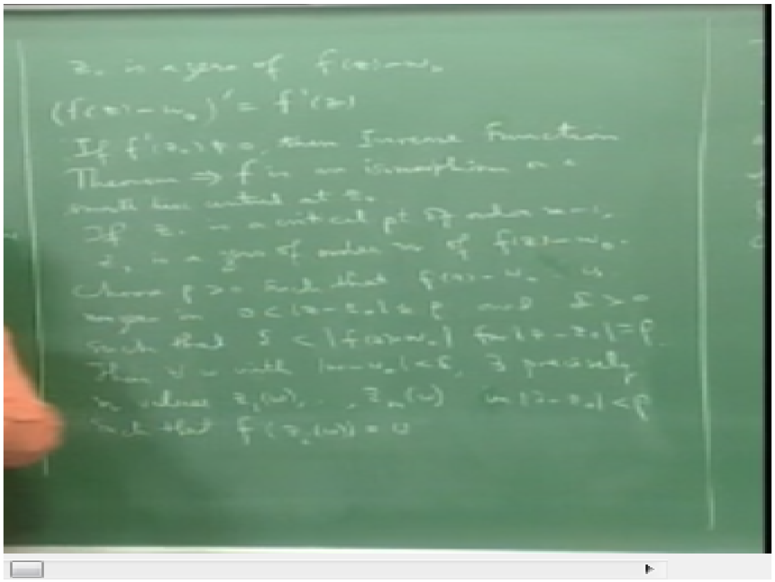

Then so, you have seen then for every w with mod w-w0 less than delta there exist precisely m values z1 of w and so on up to z m of w which are in mod z-z0 lesser than rho. Such that f apply to z i of w will always end give you w okay, so these are the zi are the so called branches of the inverse of f, see when I say inverse of f it is not a set theoretic inverse okay but it is a functional inverse.

Set theoretic inverse will not make sense unless the function is set theoretically injective you cannot talk about the inverse of function unless the function is injective. So, the problem is that it is not an injective function alright, in fact the point z0 itself it is taking with multiplicity m because z0 is a 0 of order m of f of z-w0. Therefore the function f of z-w0 has z0 as a 0 not as a simple 0 it is a 0 of order m and m is greater than 1 okay.

That mean that is the same as saying that f takes the value w0 at z0 m times okay alright, so it is not 1 to 1 okay, it is a many to 1 function. In fact it is a m is to 1 function okay and therefore so when you write the inverse functional inverse for every point you will get m values. So, if you start with a point w here, then you know if I try to go back I will end up with m points which are the various zi of m, z1 of m, z2 of m I am sorry z1 of w, z2 of w and so on up to zm of w, I will get m of w.

Of course the m of them some of them could coincide but the point is if I count them with multiplicities they have to be distinct. And in fact the truth is so long as w is different from w0 they all will be distinct, the truth is they will be distinct if w is different from w0 okay. and how do you see that because you see that because u is branches of logarithm, so what you do is well you know you have , so if you recall what we did in that in our earlier discussion.

We you know f is a we started with a fact that z0 is a 0 of order m of fz-w0 and then you factored out that the term z-z0 power m from the Taylor expansion of fz-w0 centred at z0 okay.

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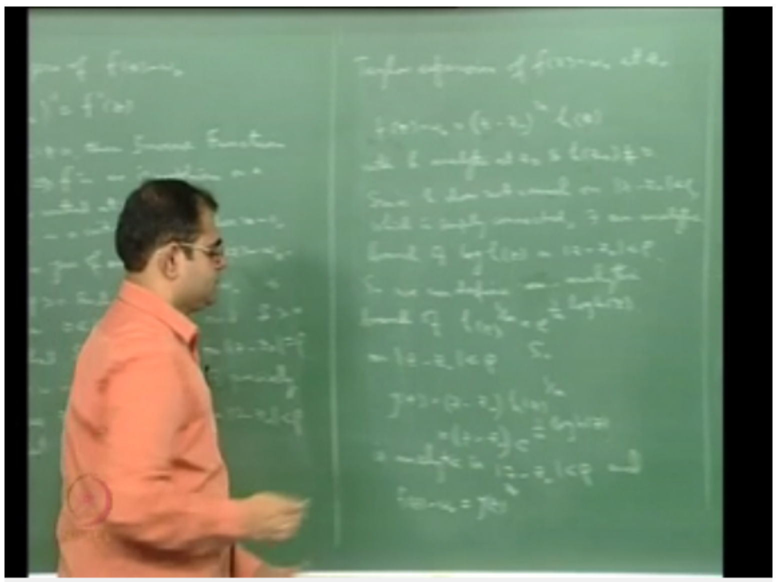

So, what you write if you take the Taylor expansion there is a power series expansion of fz-w0 at z0 what you will get is fz-w0 is z-z0 to the power of m into some h of z it will get something like this with you know h analytic at z0 and h of z0 is not 0. So, this just happens I mean this is just what happens when I mean this is just reflection of the fact that z0 is a 0 of order m of f of z-w0 okay.

So, you can factor out the z-z0 to the power of m out you have seen in this already that is because of vanishing of the corresponding Taylor the first so many Taylor quotients right . And then now what you do is h is not 0 so since h is not 0 you can find a branch of the logarithm I mean you can find a branch of the mth root of h in a neighbourhood of z0 okay.

So, we can get branches of so in fact I should say so you know yeah so before I write down let me tell you what we want to do see we want to take mth root on this side. You want to take an mth root on this side so, that you know here the z-z0 the power of m when you take mth root you just z-z0. And then you get h to the power of 1 by m but to write out h to the power of 1 by m.

You need to know that mth root of h exist okay and mth root of h exist that will happen in a small neighbourhood about z0. Because h does not vanish see h does not vanish at z0 and h is analytic by continuity it will not vanish in a small disc surrounding z0 okay. So, I have a small disc surrounding z0 where h does not vanish in fact h cannot vanish in this whole disc alright.

Because if h vanished on that disc then fz will also vanish on that disc but I mean fz-w0 will also vanish on that disc. But I know the only place where it vanishes on the disc is that the centre okay. So, h really cannot vanish anywhere alright so, h is non-zero on that disc. And the disc is simply connected and I told you that whenever you have a non-zero analytic function an analytic function that never vanishes on a simply connected domain.

That is always a logarithm there is a analytic branch of the logarithm of that analytic function defined on that domain okay. So, I have a branch of log h and once I have a branch of log h it is very easy to defined h power 1 by m. Because it is just e power 1 by m log h h power 1 by m is e power 1 by m log h. So, just to defined h power 1bym all I need is an analytic branch of the logarithm of h.

And that analytic branch of the logarithm of h does it exist in this neighbourhood because this is simply connected and h does not vanish okay. So, let me write that since h does not vanish on mod z-z0 less than row which is simply connected there exist an analytic branch of lo hz on mod z-z0 less than row. There is an analytic branch so, we can define h of the analytic branch an analytic branch of hz to the 1 by m to be exponential of 1 by m log hz on mod z-z0 strictly less than row.

So, I can make sense of h to the 1 by m and once I can make sense of h to the 1 by m I can multiply this h to the 1 by m with z-z0. I will get something whose mth power is fz-w0. So, now you put so, you know g of z which is defined to be this thing z-z0 times h z to the 1 by m by which I mean z-z0 times exponential of 1 by m into log hz okay is such that is analytic in mod z-z0 strictly less than row.

And fz-w0 is just gz to the power of m okay so, you know therefore you know the the whole mapping that goes from that z2 to fz can me know factorized into a serious of transformations. So, what you do is first you apply g so, I will try neta is equal to g of z okay so, what I do is I get so, I am I go to the c plane which is a net a plane okay. Now you see what will happen is that if I plug in z equal to z0.

Then this will go to 0 g of z0 is 0 because you see g of z0 is what g of z0 is I plug in z0 here. And there is a z-z0 factors so, it will become 0. So, g of z0 is 0 mind you therefore what will happen is and mind you note also that g dash is non-zero okay note also that g dash is non-zero that is very important. Because why do you I need it, I need it to say that g is actually one to one in a neighbourhood of in neighbourhood c g dash of z is g dash of z0 is not equal to zero see what I want to say is that at least this argument tells you that g dash is non-zero okay that is all I want okay.

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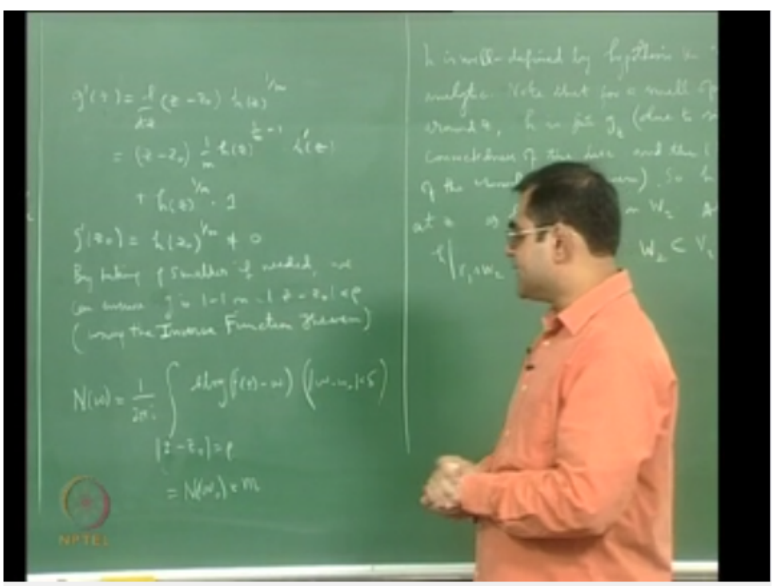

And the whole point is a g dash is non-zero that will tell you that g is one to one on a the inverse function theorem will tell you that g is going to be one to one on a small enough neighbourhood of z0 okay. So, if necessary even further shrink row I can do that always alright so, that is good enough for our discussion so, well let me say the following thing so, by taking rho smaller if needed perhaps if need to do that.

We can ensure g is one on a neighbourhood on mod z-z0 strictly less than row okay. This you can do because you know I am using the inverse function theorem with inverse function theorem says that whenever you have a analytic function which is whose derivative does not vanish at a point

Then there is a sufficiently small disc surrounding that point where the analytic function is one to one okay so, if necessary I take row smaller I can make g one to one okay now since and so, you know g becomes one to one analytic function on this disc okay. And it will take z0 to 0 so, the image of an g is one to one mind you so, it is a holomorphic isomorphism. Because I told you one to one analytic function is a isomorphism on it is image.

The inverse function is also analytic it is a holomorphic isomorphism so, if I take the image of this disc I will get some disc like neighbourhood of the origin okay z0 will go to the origin alright. So, the image of this disc will be another disc like neighbourhood of the origin which you can think of the started disc okay. So, this is the effect of a plane g then now what you do is now from here to here.

You apply you take the map zeta equal to neta to the power of m okay. Then you and you know what this map is this is the power map this power map will you this will take you onto it will map this onto again a dislike neighbourhood alright and you know it is a m:1 map you know that okay. So, **so** the combination of these of two will be z going to g power mz okay.

But what is g power mz z power mz is just fz-w0 so, when you go from here to here you already got z going to fz-w0. So, if you want further to z going to fz I have to add w0 so, that is this final map. So this map is zeta going to zeta +w0 this is the final map it is a translation. So, this is of course this is 0, 0 goes to 0 now the 0 is going to w0 okay. So, finally this is how the map fact us this is how the map w go to fz fact us alright.

So, I should tell you that this map is just going to be z going to g power mz g z to the power of M which is fz-w0 and then if you follow it by this which is translation by w0 you will get fz okay that is how this map fact us alright. Now the fact is that I can write out how I from the starting from this w how I got this values okay how did we get this M values that is because of this you know it is because of that function N of w which we define.

And use to prove the open mapping theorem the inverse function theorem and so on you know so, you know you define this n of w what is this N of w, N of w is 1 by 2piei integral over mod mod z- z0 equal to row of the log f of z-w you know this is integral this is for mod w-w0 lesser than delta. This integral will give you the number of times the function actually it will give you the number of zeroes-number of poles okay.

But there is no pole because everything is analytic so, actually it give you number of zeroes of fz-w and number of zeroes of fz-w is actually equal to the number of values z at which f takes the value w. So, your count the number of times f takes the value w okay so, that is this N of w alright and by of course you know this is a logarithmic derivative. It is f dash of z divided by f of z-w that is what the integrand is logarithmic derivative of fz-w alright.

And you know that this Nw we have already seen this Nw is an analytic function and it is an analytic function which is defined on this disc okay. And it is a and but it is a integer value because of the residue theorem or the argument principle whichever a way to want see it. So, an integer valued analytic function is has to be a constant if it is defined on a domain okay.

Therefore N of w is a constant and therefore N of w is actually equal to N of w0 which is equal to m so, you will get that this is equal to m okay. This is equal to N of w0 which is actually equal to m so, what this tells you is that you take any w with in this disc mod w-w0 less than delta. Then there are precisely N values of z z1 of w z2 of w etc... to zm of w where f takes the value w okay. So, you get precisely m values z1 of w throw zm of w at whichf of that each of those values is w.

You get these distinct values and in that the fact is that all these are distinct there you only this N of w is just the number of points z counted with multiplicity okay. But it is just N of w just tells you how many times fz-w has zeroes okay. But those it is only counts with multiplicity but the analysis actually shows is what it shows is that all these for w differ from w0 all these zset different okay. Why are they different that is because you can write down okay.

So, how will you get the inverse function when you write down the inverse function what will you do is well the of course here I can write g inverse. Because g is one to one okay inverting this is very easy this is just g inverse. Because this is a one to one function translation is also one to one so, the inverse function is translation by –w0 okay the translation by w0 is undon it is reversed by translating by –w0.

So, this inverse function is also easy this is a one to one function so, holomorphic map it is a moebius transformation. So, you can easily reverse this so, only thing that gives you to gives all these all branches of the function of this which are the mth root functions okay. So, here you get all these so may branches of zeta to the 1 by m you will get m branches now what you do is that you combined this with each of these m branches.

And then you follow by g inverse you will get all these m branches which are given by w going to z1 w going to z2 w etc... okay that is how you get all these m branches. And all these m branches are different for w different from w0 why because these m are different. The m branches of the m the m branches of the mth root function are all different because there are m different mth roots for any non-zero complex number.

That is the reason why for w0 equal to w0 the zis you get the zi of w you get there all different okay. So, you get exactly m of those values for w0 not equal to w0 of course when w is equal to w0 you get z0 okay. But you get z0 with multiplicity m if w is not equal to w0 you get various zis and there are distinct zis that is how it behaves. So, what happens is that you get branches like this.

You get all these branches z j of m z j of w okay and **wa** where are these branches defined well the branches are all defined on you know the point is that somewhere in the middle hand using the mth root function. And the mth root function you know pretty well depends on the logarithm and the logarithm is analytic only on a slit plane. So, you know when I defined this zeta to the 1 by m I am using a logarithm here okay see what is zeta to the 1 by m.

It is actually e to the 1 by m log zeta and when do this I get m branches what are those m branches well I will get e to the 1 by m log I will take principle logarithm zeta. And then I will get +2npiei these are the various branches and all I have to do is now I have to read this n mod m. So, I will just how to put N equal to 1 etc. up to m either I start I do from one to m or I start from 0 to N or r rn is equal to 0 to m-1.

You will get exactly precisely m of them okay normally this is 1 by m is no there you will get many but because this is 1 by m is that this N has to be red mod m okay. So, these are you get so may branches and because of these m branches when you invert you get these m branches. And where are these analytic see each of these branches analytic only on this slit disc I have to throughout this**.**

The portion of the negative real axis including the origin and when I translate it I will get I will have to throughout this region which is actually this line segment from omega not –delta,omega not this line segment. I have to throw it out so, the moral of the story is that.

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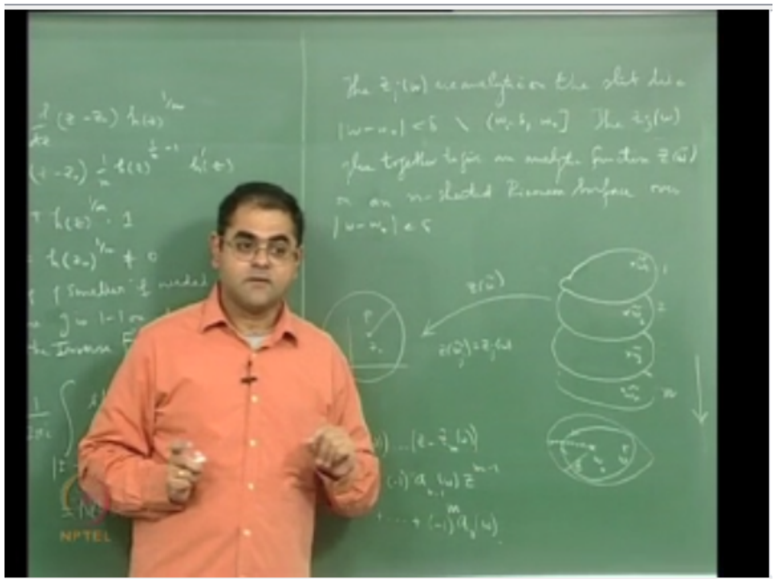

The of m the zj of w are analytic on the slit disc mod w-w0 less than delta- from this throw out the you throw out this, this portion of the diameter to the left of the centre which is w0-delta,w0 to throw this out. This is the slit disc they are analytic on that okay and why you have to do that is because these fellows in between these logarithms. You want the logarithms stably analytic.

So, you have to throw this out and that effects that effectively means you have to throw out here okay. So, these are analytic but the point is where will they all glue to give as single valuehood function that will be on a on the Riemann surface which sits above this punctured disc. So, what happens is that the situation is the zj of w glue together to give an analytic function z of w on so, let me use let may write z of w tilda on a |Riemann surface on an m sheeted Riemann surface over this punctured disc.

So, you know situation is like this I have this punctured disc this is w0 this is delta okay on this I have this Riemann surface. You see yeah it is again slitly difficult to draw but then okay I have this m sheeted Riemann surface 1,2 etc. up to m you have this m sheeted Riemann surface okay. And on **on** sheet so, I have this function z of w tilde where you know w tilde is a point on this which goes down to a point w here okay each w has precisely m inverse images.

This is a cover this is a covering map okay this is a Riemann surface how do I get this Riemann surface is by taking m copies of this slit disc and then you know joining them together so, that all the I am actually doing actually what I am doing is I am constructing the Riemann surface of this logarithm. I mean I am actually constructing the Riemann surface of zeta to the 1 by m.

Ad simply translating it here I am not doing anything else okay so, this is that Riemann surface on this I have this function z of w tilde okay. Now that function will go back to the think that I started with namely it will go to that disc centred at z0 radius row I think here radius row and what will happen is that you know if I start with w I will get so many points w1 tilda w2 tilda w3 tilda and so on.

The last sheet I will get wm tilde I will get these points and if I take z of wi tilda that is wj tilde I will simply get zj of w. I will get all those m those m functions they will all give me if I go to different pre images of this point where is this projection map which is a covering map ever point has m pre images. Because after all I have taken m copies of the slit disc and you know I have cut and paste the negative real axis carefully okay.

So, every point will have m pre images and when I apply z to each pre image I will get the corresponding zj which will be the z **z** restricted to each sheet is the zj okay z restrict the first sheet the z one z restrict to second sheet the z2. This way I get all the zs but the beautiful thing is when I go up this all the zjs they become they all glue together to give a single analytic function and that they give first of all give rise to single function.

And the point is that analytic function that is a beautiful thing if zj if I started from here it is not analytic. Because I will have to throw out this slit with because along that slit it is it cannot be defined. So, that it becomes the analytic but if I go to the Riemann surface I am able to cross this because the reason is because as I go from one sheet to the next sheet I am doing analytic continuation.

I am going from one branch the next branch by analytic continuation therefore I am actually getting one function on the Riemann surface above so, the year is where I am using the fact that you know every zj if I even though every zj is not analytic on this slit I can analytically continue it to the next zj across that slit and that is the reason why on above I get a continuously I get one single analytic function by because there is no obstruction to analytic continuation along that slit.

That is why all of them glue to give you an analytic function now the point is that these the story is you have seen so much already. The story is that all these zjs they are all algebraic so, you know so, this is what I wanted to say so you know in fact I told you that if you write z-z1w you take the product up to z-z mw okay. If I take this product then this product will be just if I write it out I will get the elementary symmetric functions in this zis.

I will get a polynomial monic polynomial in z of degree m with coefficients elementary symmetric functions in the zis which is what I expain so, I will simply get z power m so, I want to m by saying the following thing these a sub ks they are all the symmetric functions in the zis okay. The point I want to make is that these a sub case you seethe zi the each individual zi I snot analytic at that slit okay each individual zi is not analytic at that slit.

So, if you take any elementary symmetric function of these zis which is one of these a case you do not expect it to be analytic on that slit. But the fact is it is and what is the reason the reason is you see if you take a point here okay. And go around once across you across this slit if you go around once what will happen is the zis will get permuted okay. Because if you go around once effectively you have seen that if you go around once the origin the logarithm from one branch of the logarithm.

You moved to the another branch of the logarithm alright so, that will you that if you start with any point here. And you go around once you have going to permute this zis but if you permute this zis the a case will not change because they are symmetric functions so, what is the moral of the story the moral of the story is each of these a case you go around once analytic continuation leaves back to the same ak okay.

Therefore on the punctured disc each of these a case will defined a single valued analytic function. So, all the a case will become analytic further the only problem is that the point w0 and at w0 also the a case will be analytic why because when w is equal to w0 okay. Then **ee** each zi of w0 will become z0 so, left side will become z-z0 power m and therefore uh what happens is that each ak is an analytic function which is has a limit at the point w0.

Therefore it is also by Riemann removable singularity theorem it is also analytic there so, the moral of the story is all the acase are analytic on this whole disc. So, what you have done is you have shown that the all the zis they are algebraic all the branches which live only on the slit disc they are anyway solutions of an algebraic equation it is a polynomial in z whose coefficients are analytic on the whole disc.

The quotients are the a case so, though the zis are analytic only on the slit disc they are symmetric functions are analytic on the whole disc that is the beautiful thing okay that is the whole point that is here is where I am using the fact of analytic continuation alright. So, it gives a very beautiful picture of the behaviour of analytic at a critical point okay so, I will stop with that.