Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-28

Proof of the First (Homotopy) Version of the Monodromy Theorem

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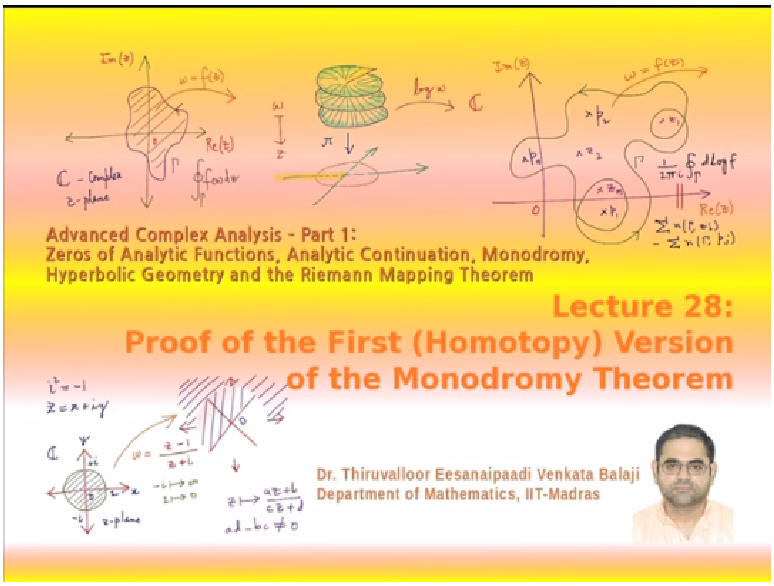

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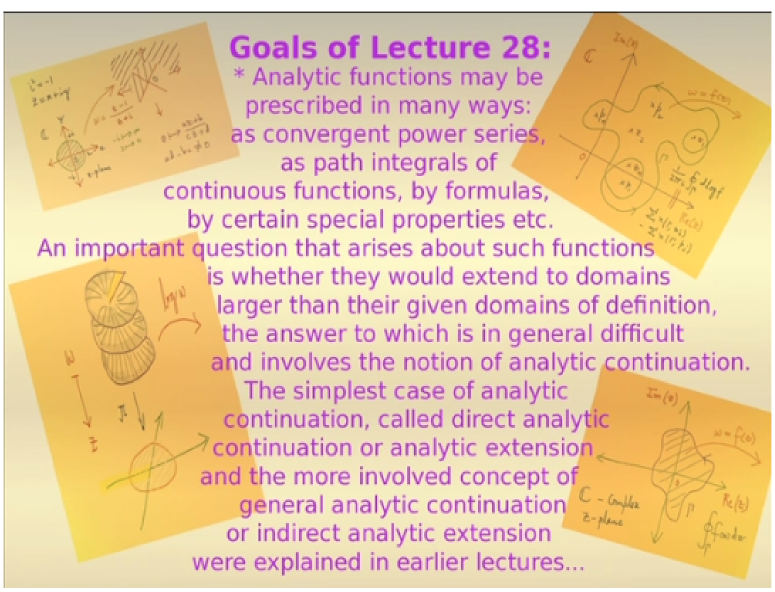

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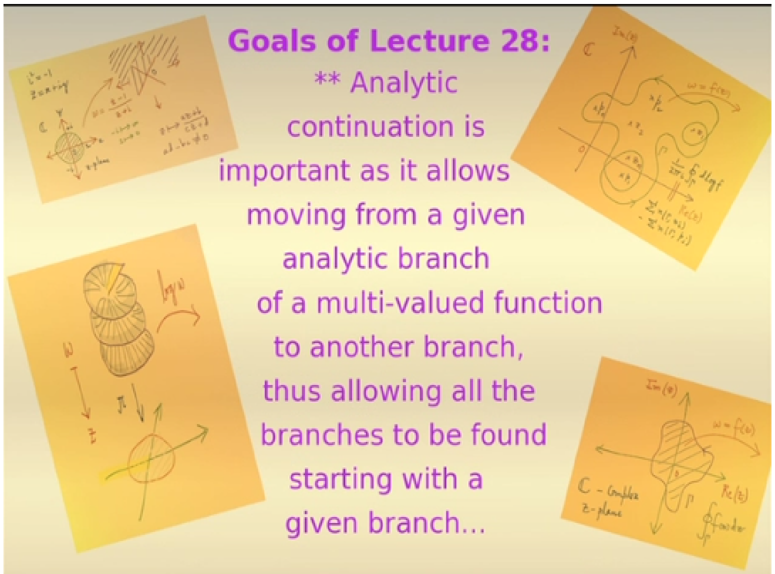

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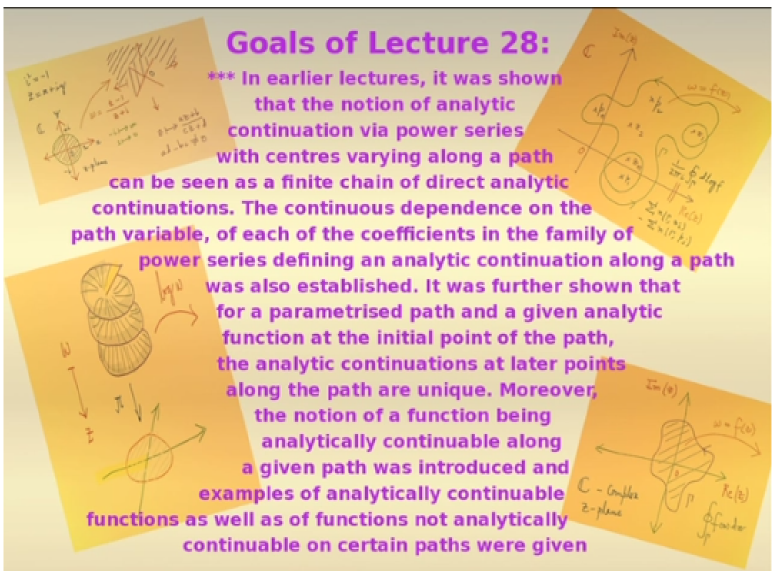

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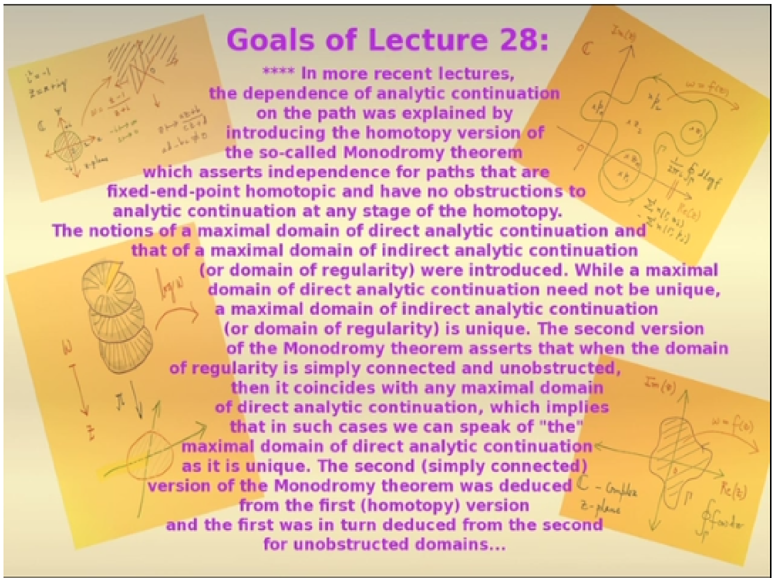

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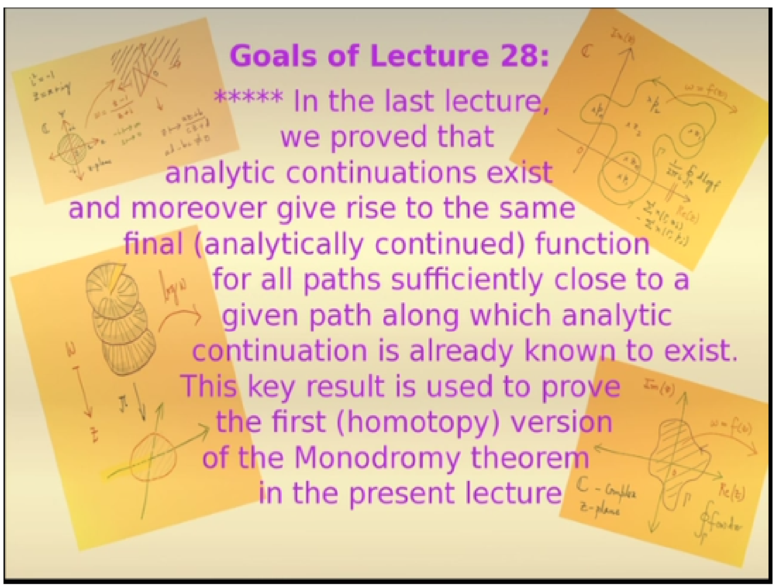

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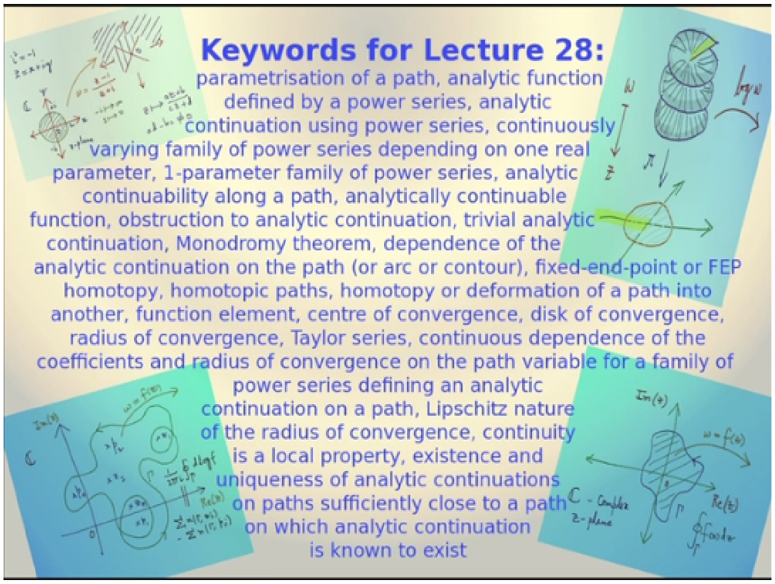

Okay so we continue with the proof of the Monodromy theorem.

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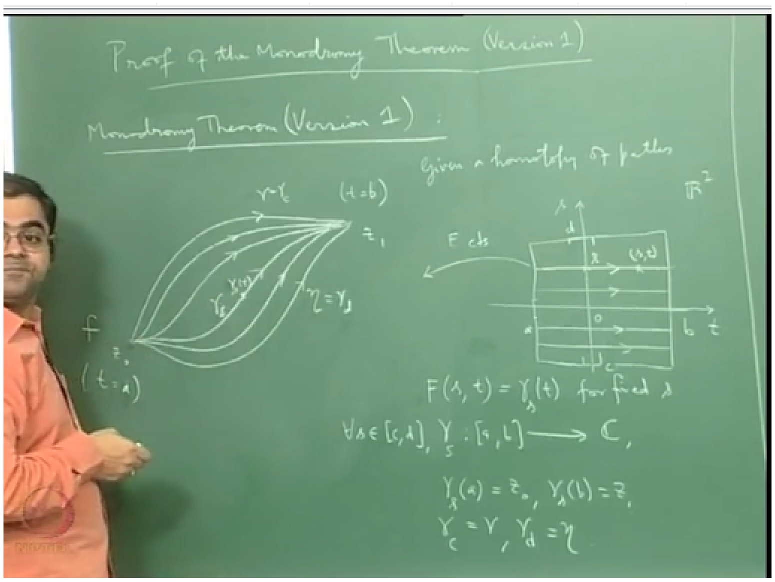

Ok so we have this rectangle which is product of the closed interval a, b on the real line with the closed interval c, d on the real line and we have a homotopy capital F which is the continuous function on this rectangle and it gives homotopy between the path gamma which is the beginning path in the homotopy and the path neta which is the terminal path the homotopy.

And of course any intermediate path in the homotopy is given by gamma s okay, gamma s is just f of s, t with s fixed and t varying alright and what we need to show is that if we know that there is an analytic function even at the point z0 which can be analytically continued along each of these paths. Then analytic continuation of on any path will again lead to the same function at the terminal point z1, that is what we have to prove okay, that is the Monodromy theorem.

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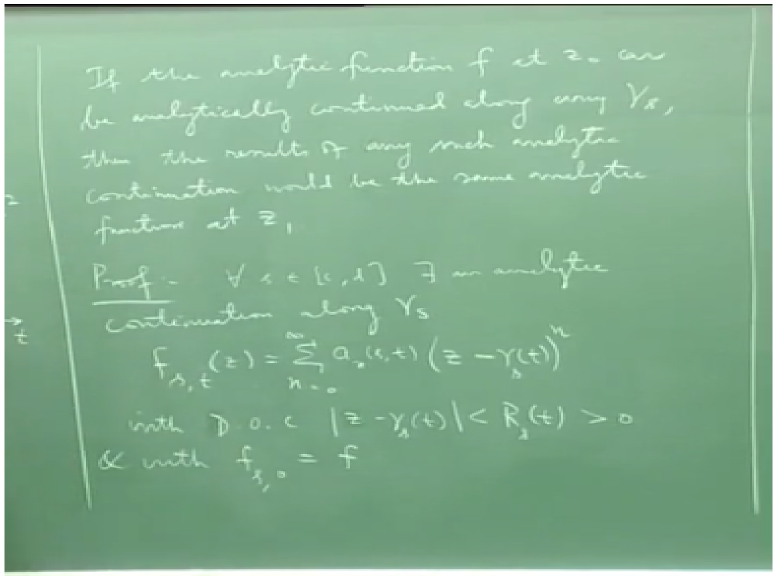

So, how do we prove this, proof is as (()) (02:14) so **so** what we will fo is we will do the following thing you know so what is given to me is that there is an analytic continuation of f along each of these paths okay. So, let me write down some analytic continuation alright.

So, it is so for every s in c, d there exist an analytic continuation along the path gamma s which is given by f of s, t of z sigma n=0 to infinity an of s, t into z-gamma s of t to the power of n with radius of convergence with disc of convergence mod z-gamma s of t is less than R sub s of t okay which is of course positive with of course and with f of s, 0=f okay.

So, for so what I have done is for each gamma s I have written this f of s, t which is an analytic continuation along the path gamma s starting with f okay. And it has a certain disc of convergence right. So, you know so on this picture gamma s is this path which is the image under capital F of this line segment okay where s is fixed and c I mean t varies from c to d, t varies from a to b, s is fixed value between small c and small d.

And t varies from a to b and the image of this line segment under this continuous function f is this path gamma s and if you give me a point t here point corresponding to a certain value of t between a and d. Then the corresponding point here in this rectangular have coordinates s, t and the corresponding point and the image of this point under f will be f of s, t which is s gamma s of t.

So, it is going to be this point gamma s of t I going to write a point here and I have an analytic continuation along this gamma s as t varies and that this analytic continuation f is t okay and at the point gamma s of t I am going to get a power series centred at gamma s of t okay. And the only thing that you have to remember is since there are 2 real variables or everything I mean f a, a n and gamma and also R they will all depend on 2 real variables s and t okay.

So, for the depending only on 1 variable, if you are writing an analytic continuation on the single path then you get only one variable which is the path variable. But now you are writing an analytic continuation on a family of paths okay which means that you have also a variable for different paths which is the variable s. So, there are 2 variables involved s and t.

So, everything is a function of the power series in the analytic continuation the coefficients of the power series the centres of the power series, the radial of convergence of the power series they are all depending on this 2 variables okay. So, less than t will appear in all of them that is how we write it okay and so this is given to me there is an analytic continuation like this okay, I do not care what this continuation is for the moment alright.

But it is given the there is an analytic continuation, now what I am going to do okay. So, the fact I am going to use is that this function Rs of t if you think of it is a function from this rectangle with s, t as of the variable of the rectangle. Then the claim is this R is a continuous function of that on that rectangle okay. In fact a n will also be a continuous function on that rectangle okay.

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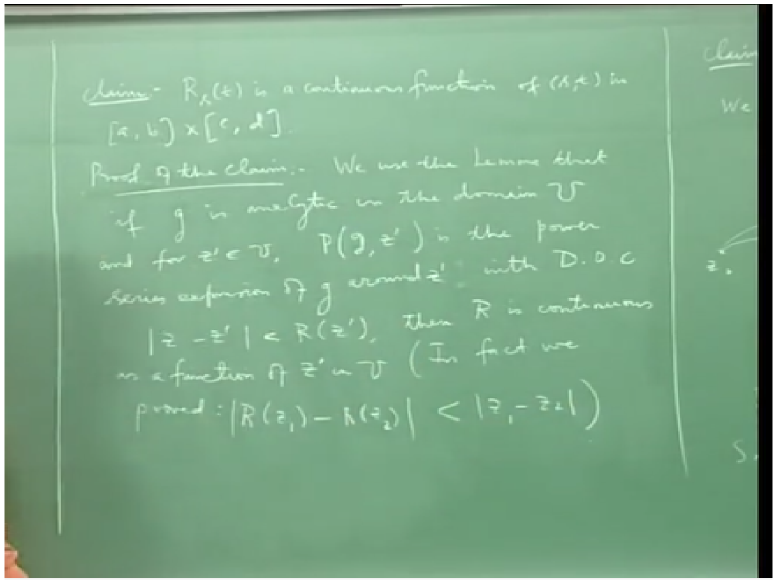

So, here is a claim R s of t is a continuous function of the point s, t in the rectangle a, b cross c, d it is a continuous function. So, this is the claim we use the lemma that if f is so if g is analytic in the domain d U and for z in U p, g, z , so I will let me put z prime for z prime in U p, g z prime is the power series expansion of g around z prime with disc of convergence mod z-z prime lesser than R of z prime.

Then R is continuous as a function of z prime in U. So, I am I am just using the fact that you know if you have an analytic function on domain and at various point he start writing it is power series expansion okay. So, at various points I will get various power series expansions and at corresponding to each of these power series expansions I am going to get radii of convergence.

So, as a change in the point I am going to get different radii of convergence okay depending on which point I am expanding the function of power series about, then the fact is that this radii as you change the point the radii of convergence will change continuously okay. So, we proved this, so in fact what we proved is in fact we prove we prove if you remember we prove R of z1-R of z2 mod is strictly less than mod of z1-z2, we proved this.

The difference in the radii of convergence of the power series expansion at z1 and z2 cannot exceed is smaller than the distance between the 2 points okay. So, this is the fact we need to use, so, let me draw another diagram.

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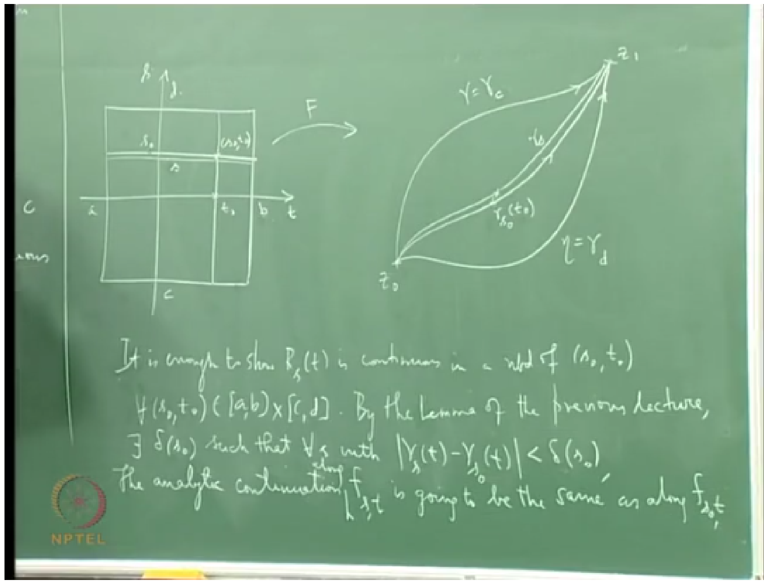

So, you know the situation is like this, so here is my rectangle so this is a, this is b , this is c, this is d and this is the variable t, this is the variable s okay. And here is my particular value s well if you want you know I can take s0 and then I can take certain t0, so I will get this point which corresponds to s0, t0 and I have this f, I have brought this f is going to give me as well it is going to map this segment onto of course gamma which is gamma of a rather gamma c .

This is the path from z0 to z1 and then I have this other path below which is neta and neta is gamma d okay. And corresponding to this line segment with y coordinate s0 I am going to get the intermediate path gamma s0, this is gammas of s0 okay. And this is the point that corresponds to gammas of s0 of t0 okay, this is what I am going to get .

Now see I have to show that I am trying to show that R s of t is a continuous function of s, t okay to show that a function is continuous it is enough to show it is locally continuous okay. So, it is enough to show that R s of t is continuous in s and t in a neighbourhood of each point s0, t0 in the rectangle alright. So you know fix so let me write that down it is enough to show Rs t is continuous in a neighbourhood of s0, t0 for every point s0, t0 in that rectangle it is enough to show this alright.

Because it is continuity is a local property to show that a function is continuous it is enough to show that on an open cover okay. So, you know I have those were frozen this s0 and t0 okay, now for the moment what you do is you see you just looked at this path gamma s0 okay all along this path gamma s0, there is this analytic continuation which is given by fs0, t there is an there is this analytic continuation fs0, t.

And it is starts with fs0, 0 which is f it is an analytic continuation of f along this gamma s0 alright. Now you see by the previous lemma okay for all t for all paths close enough to this path the analytic continuations are the same okay. We have see the previous lemma that we prove in the preceding lecture was a if you have analytic continuation on the path there along sufficiently closed paths the analytic continuation is going to exist.

And it is going to be and it is going to lead to the same function at the end okay, so what you must understand is on nearby path okay nearby means for s close to s0 okay. If you take nearby paths then the analytic continuations are going to be same as the analytic continuation on the path gamma s0 okay. So, by the previous by the lemma of the previous lecture namely the lemma that I have proved at the end of the previous lecture .

There exist delta of s0 such that for every s with mod gamma s of t-gamma s0 of t is less than delta of s0 the analytic continuation fs, t is going to be the same as analytic continuation along fs, t is going to be the same as along fs0, t this is what we, so you know I am saying that you know if you choose any s which is very close to s0 okay, see if you choose s close to s0 alright.

Then of course the gamma s will come very it will come very close to gamma s0, so this is gamma s, this is gamma s and this is gamma s0. If s is close to s0 then gamma s is close to gamma s0 that is simply because f is continuous. And gamma s is the image of this line segment that corresponds to s and gamma s0 is the image of this line segment that corresponds to s0 okay and nearby continuation function maps nearby objects to nearby objects is just continuity.

So, as you make s close to s0, the gamma s comes closer to gamma s0 but then if you have chosen s, so that the distance between the point gamma st and gamma s0 of t for each t is always less than this delta s0 okay. Then the analytic continuation among gamma is the same as the analytic continuation on gamma s0, this is what we prove in the in a lemma in the previous lecture in words to state that we prove to state what we proved was is that if analytic continuation exist along a path.

Then analytic continuation will also exist along sufficiently closed paths and all these analytic continuations will re-lead to the same function at the ending point, I am just using that lemma okay. The only thing is that this delta will now this depends on that path s0, gamma s0 okay. So, I am if I use that, so what I get from this if I use that lemma is that so in fact you know in fact what we if you go back to the proof of that lemma what we proved was that.

The analytic continuation at gamma s of t the analytic function you get at gamma s of t is the same as the analytic function you get the gamma as at a gamma s0 of t okay. Because what we did was we actually defined on a sufficiently closed path we defined an analytic continuation by simply writing out the power series expansion at that point okay. So, in fact the analytic continuation the function you get at s t is literally the same function that you get at s, s0 t for s sufficiently close to s0.

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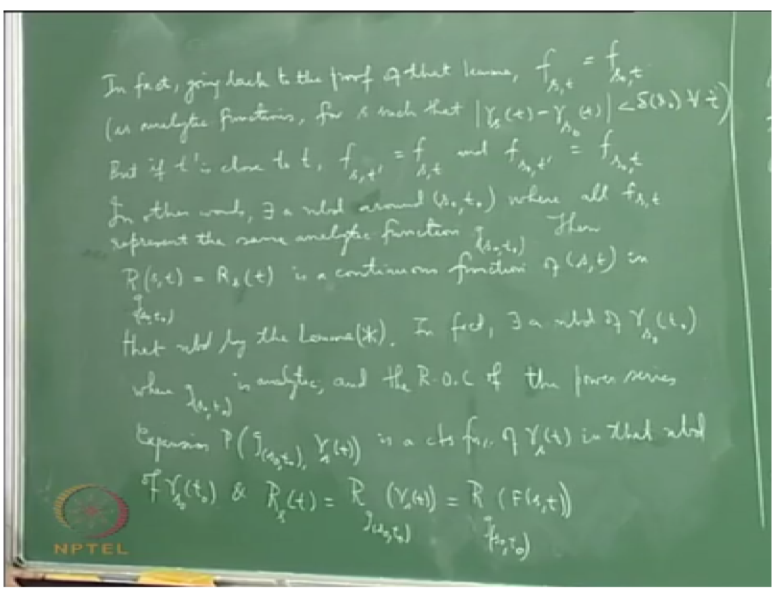

So, let me write this in fact going back to the proof of that lemma function fs t is the same as the function fs0 t as functions for s such that mod gamma s of t-gamma s0 t is less than delta s0 for all t okay. So, this is again going back to the proof of that lemma alright and so you know so what this tells you is that you know if you take s sufficiently close to s0 alright.

Then you are getting you are all for any value of t, the analytic function you are going to get is the same for a fixed value of t okay all for all these s which is close enough to s0 and for any fixed t the analytic function fs t is the same as fs0t alright. Now what I want it understand is that but you see if t prime is close to t of course fs t prime is the same as fst and fs0t prime is the same as fs0t that is also true.

That is because is the analytic continuations, analytic continuations require that as the t variable comes close to a particular value then the analytic functions given by the power series also coincide okay. So, what all these tells you is, it tells you that you know it tells you that there is this neighbourhood around s0, t0 where all the functions fst are a single analytic function okay.

So, in other words there exist a neighbourhood around s0, t0 where all fst represent the same analytic function okay and call that function as g s0, t0 okay. Then R of Rg s0, t0 of there is radius of convergence of the power series of gs0, t0 at the point s, t is just rst rs of t in our notation is a continuous function of s, t in that neighbourhood by the lemma.

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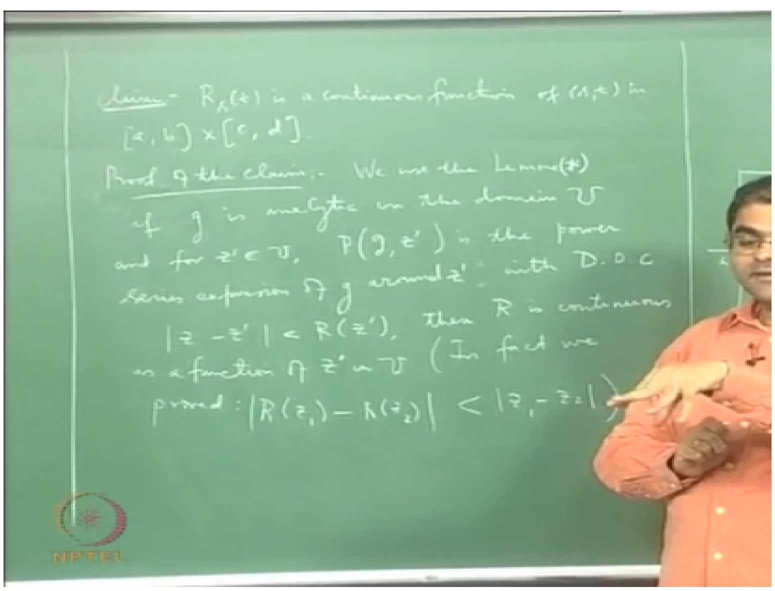

So, you know I will call this is I will lemma star so I will label this lemma star this is this lemma star okay not to be confused with the lemma of the previous lecture okay. This is lemma star is the lemma that if you take an analytic function in a domain then if you expand the analytic function as a power series at each point then the corresponding radii of convergence will be continuous will be a continuous function of the point.

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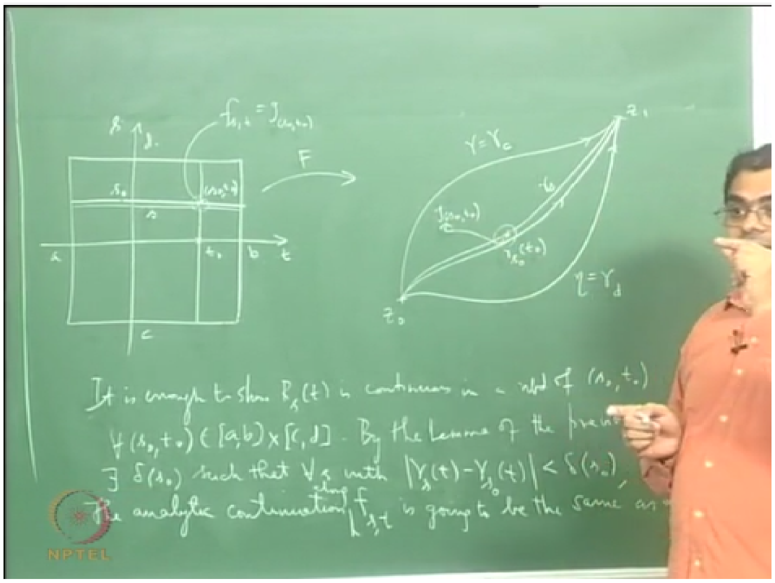

So, you know there is a small neighbourhood here where all the fst’s here all the fst’s represent the same function gs0, t0 what it means is if you take the image of this neighbourhood here okay you will because of the continuity of f I can find a small enough neighbourhood here into which the image of this neighbourhood goes okay.

And for all point in that neighbourhood your you are actually expanding the same function gs0, t0 in this neighbourhood which contains that neighbourhood okay, see you take this gamma s0 of t0 which is the image of the point s0, t0 under f okay. Then you take a sufficiently small neighbourhood of gamma s0 of t0 where this gs0 come at t0 live lives, see after all gs0, t0 is an analytic function which is it is not defined here, gs0, t0 lives here.

So, it is this so this is the point at which gs0, t0 is analytic okay, gs0, t0 is analytic at this point which is the point gamma s0 of t0 okay, it is analytic there and there gs if you in that neighbourhood you take any point and if you write the power series expansion of this gs0, t0 at

at that point and look at it is radius of convergence. Then the radius of convergence is a continuous function of the point okay.

So, the radius of convergence of gs0, t0 at each point in this neighbourhood surrounding gamma s0, t0 is a continuous function of the point gamma s of t. But gamma s of t is a continuous function of s of t because it is actually f , so R of Rs of t becomes a continuous function of s and t okay. So, in fact so let me write that properly so to in fact there exist a neighbourhood of gamma s0 of t0 where gs0, t0 is analytic.

And the radius of convergence of the power series expansion P gs0, t0 at gamma s of t is a continuous function of gamma s of t in the in that neighbourhood of gamma s of t0 okay, this is what I am saying right. And R st is actually and R s of t is actually the radius of convergence of the power expansion of gs0, t0 at the gamma s of t which is continuous function.

So, and mind you this is just R composed with I am here now I am thinking of R as composed with f of s, t. Because f of s, t is gamma s of t okay. So, it is a composition of f, f is continuous and R is continuous therefore composition of continuous function, so it is continuous. So, you know R is indirectly a function of s, t so I wrote it directly there but if you want it more explicitly I have written it here okay, this is the reason why R is a continuous function of s and t okay.

So, what I have proved is R is a continuous function of s and t locally okay but that is enough to say that is continuous globally because continuity is a local property right, it is a property that can be verified at each point in an neighbourhood of each point. So, I have proved this claim okay, so I have this claim that Rst is a continuous function of s, t in this rectangle okay. Now how do I proceed in the same way I simply take the image of that rectangle and under R and I notice that the image will again be a compact interval and it will have a minimum and I am going to call that minimum as delta okay.

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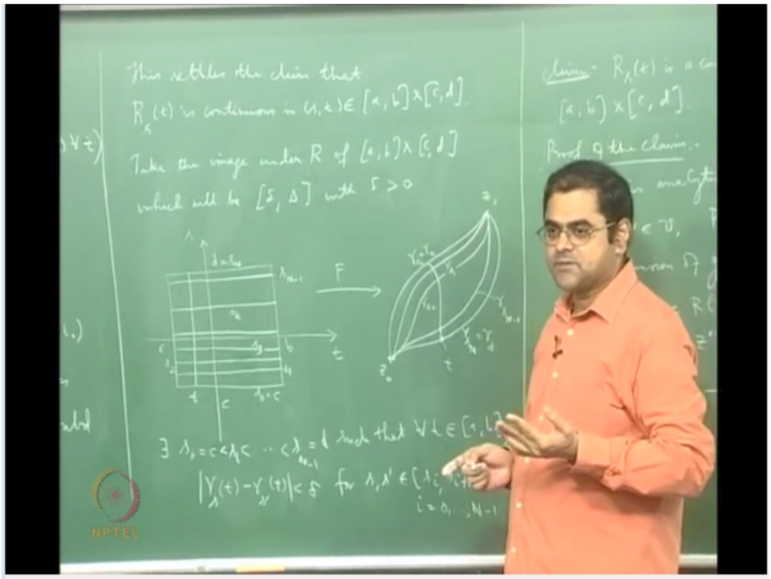

So, you know so Rs of t is so this settles the claim that Rst is continuous in st varying in this rectangle okay. And take the image under R of this rectangle okay, see this rectangle is anyway compact and connected, the rectangle on the plane is a compact and connected set. Of course it is a connected set because it is actually path connected any 2 points can be join by a path.

In fact even by a straight line if you want okay and so it is connected certainly and it is compact because it is closed and bounded. Because I have taken the closed rectangle, so it is compact and I have a continuous function R defined on this compact set okay. So, the result will be the image under R of this compact set will again be a compact subset compact connected subset of the real line.

So, it will again be a closed interval in the real line okay and of course R is always positive, so I am going to get a closed interval with minimum with left end point greater than 0 okay. So, which will be delta, delta with delta positive okay, so this is where I use a continuity of R, I need the continuity of R to say that the image under R of this rectangle is you know compact and connected and a compact connected subset of R is just a closed interval.

And of course these R are R refers to various radii of convergence they are all positive radii of convergence. So, the so R values are always positive never 0 therefore there is going to be a minimum value of R that is going to be delta small delta. And there is also going to be a maximum value of that R that is capital delta okay. Of course you know in all these situations .

I am really not worried about the case when at some point you know you get a power series whose radius of convergence is infinite okay, see you must always remember that see you always see me a writing this small delta, capital delta this capital delta tells you that you know the R is finite, the radii of convergence is a finite and you know radii of convergence of the radius of convergence is finite means that at the circle of convergence there is a singularity for that analytic function.

Because if there were no singularities the radius of convergence would have would become infinite okay. So, always when the radius of convergence is finite on the circle of convergence there is a singular point for that point, there is a point beyond which you cannot extend that function okay. So, there is a point at which you cannot extend that function, so there is a singularity, so you always see me a writing this delta here capital delta.

And I just wanted to make you this remark that you know, I am never looking at the case when radius of convergence is infinite. Because the radius of convergence is infinite means that one of the functions you are that occur in the analytic continuation is entire if one of the functions is entire then there is nothing to continue because an entire function can be continued everywhere.

So, you know you are not going to get you are going to not going to get anything you are only going to get that function no matter how you analytically continue it okay, it is going to be just direct analytic continuation just extension of that entire function to the whole complex plane. So, there is nothing to prove okay, so all these things become interesting only when the radii of convergence of all finite okay.

The radii of convergence become infinite even for 1 point all these results they become trivial there is nothing there is really no real question there to answer okay. So, that is the reason I am always thinking of R positive and finite okay, fine. So, now comes the now that I have this delta see now I am in very good shape.

So, you see how I use this delta is as follows what I do is I have this you see this rectangle that I have a, b cross c, d you see I can actually divide this rectangle into by a series of lines parallel lines you know s0, s1, s2 and so on some s k and so on. So, that you know well maybe I rather call this line as s0 that corresponds to c then I have s1, I call this as s2, I call this as s3.

So, maybe instead of writing it here, I will write it here, this is s1, this is s2, well and this is s3 and so on. Then I will end up with sk and finally I end up with s well some N capital N which is d alright and of course this value will correspond to sN-1. So, I can find these s is in such a way that you know if you take the image of each of these rectangular strips okay, you will get a piece of this homotopy leaf.

There is a piece of this leaf like this region in between these 2 paths, such that you know the distance of the points corresponding to given t is less than delta okay. So, let me write this so let me draw this diagram first, so here is how the diagram is going to look like. So, you know , so this is gamma s0 which is just gamma c, this is gamma s1 then I will have gamma s2 and so on.

Then finally I have this is gamma s sub N which is just gamma s of d and this guy here is gamma sub sN-1 okay, I can find these values starting from s0 to sN for sufficiently large N. Such that you know you give me any value of t then of course the image of something like this will be something like this well if I draw it, it will be something like this okay.

That this will correspond to a given t alright it will be this point this first end point z0 when t is a and this thing will collapse to the terminal point z1 when t=b but in between the image of this line segment will be something like this that will also be a path connecting a point, connecting this the point corresponding to t in the first path with the point corresponding to t in the last path okay.

And but the point is that you know if you take any 2 successive points the distance of those points is less than delta okay. So, you can find such a finite collection of points okay. So, there exist s0=c strictly less than s1 and so on lesser than sN-1=d such that for every t in a, b. The distance between gamma s of t and gamma s prime of t is less than delta for s, s prime belonging to any sub interval si si+1 I equal to 0.

And so on up to n-1 okay. You can divide this rectangle into small thin rectangular strips with this properties this is purely by compactness okay**.** So, is a compactness argument that you can further expanded and try to write down. But it is include to obviousness easy to write down. So, you can do this now once you done this once you realise that you can do this to prove the theorem is over.

Because you see what will happen is you see because the distance between all the paths in between gamma si and gamma si+1 is less than delta. They will all define the same analytic continuation okay that again the proof of that is again following the proof of the lemma of the previous lecture. So, what will tell you is that on each for each of these pieces the analytic continuation along the upper path is the same as the analytic continuation on the longer lower path.

And then you go by induction okay so, the analytic continuation along gamma s0 is the same as the analytic continuation along gamma s1. The analytic continuation along gamma s1 the same as the analytic continuation along gamma s2 and by induction finally you get that the analytic continuation along gamma s0 is the same as the analytic continuation along gamma sN.

In other words the analytic continuation along gammas gamma c which is gamma is the same as the analytic continuation along gamma d which is neta okay. And that proves the monodromy theorem okay. So, what you must understand is that it is a kind of a cleverly playing upon the ideas of the lemma that we proved in the previous theorem and also critically using the fact that the radius of convergence is a continuous function.

That is a very critical fact that keeps using and also let me again repeat the main idea in the proof of the lemma of the previous lecture was that you know if you take sufficiently closed paths then there is only there is a unique analytic continuation on that path and it is simply defined by expanding the if the relevant function on the given path into a power series okay. So, if you **you** take to nearby paths.

And if I have this analytic continuation along the path gamma s0 on along a nearby path gamma is the analytic continuation how is the defined it is very simple what you do is you simply defined the analytic continuation by simply expanding this function at gamma s0 of t0 at gamma s of t0. And you do this for every t0 okay, so the fact the whole idea is you know if a function is analytic at a point it leaves a neighbourhood.

And therefore that function itself can be used to define power series in paths in that neighbourhood okay. So, it in other words if you have a path and you give me a point on the path and you give me a analytic function on that point. Then there is a disc by definition of analyticity there is a disc where the function is analytic and whenever there is any other path which passes through the disc along the portion of the path which passes through the disc.

I can simply defined the analytic continuation to be the power series expansion of this function that leaves okay. And that is the crucial this is very very crucial idea okay so, that is the crucial idea that is being used and also the idea that the radius of convergence is a continuous function of the point okay.

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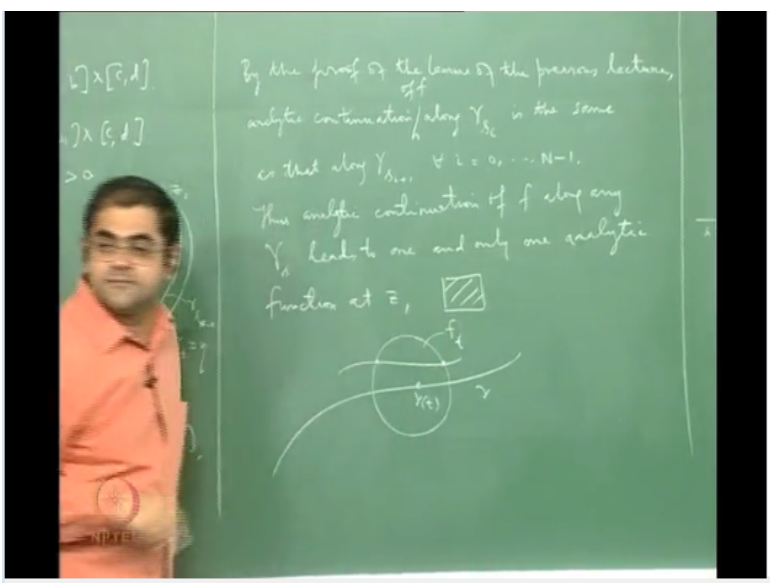

So, let me write down by the proof of the lemma of the previous lecture analytic continuation along gamma si is the same as that along analytic continuation of f along gamma si is the same as that along gamma si+1 for every i starting from 0 to N-1 thus analytic continuation of f along any gamma s leads to one and only one function at z1 and that is the proof of the monodromy theorem.

So, let me again at the risk of reputation let me against us the whole idea is if you have a path and at a point uh gamma of t if you have if you are given a analytic function fs I mean ft an analytic function here. Then if you have any other path which hits this disc where ft lives along this path along the portion of the path from here if here leaving out the end points there is this f itself has a trivial analytic continuation along this.

That is the whole idea that is being used again integrate and you are of course crucially using very very crucially the fact that the analytic continuation along a path is unique once you fix a parameterisation of the path the analytic continuation is unique for a given starting function okay. You cannot have two different analytic continuations with the same starting function for the same parameterise path that is one important fact.

The other important fact is the radius of convergence that varies continuously as the it is the continuously variable of the point where you are expanding or writing the power series about okay. So, these are crucial facts so, I will stop here.