Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-27
Existence and Uniqueness of Analytic Continuations on Nearby Paths

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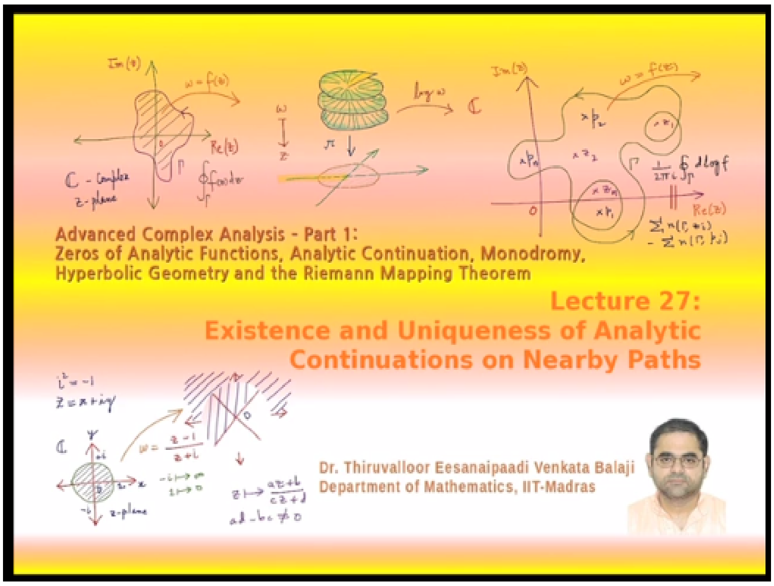

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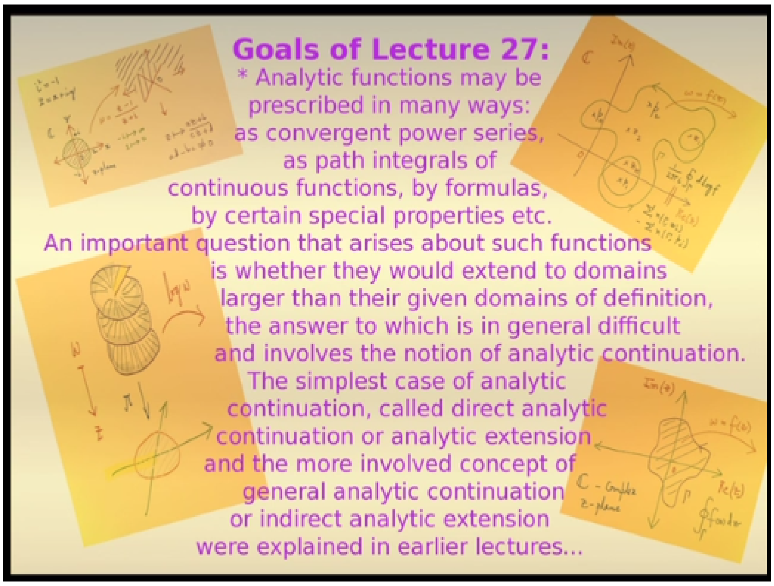

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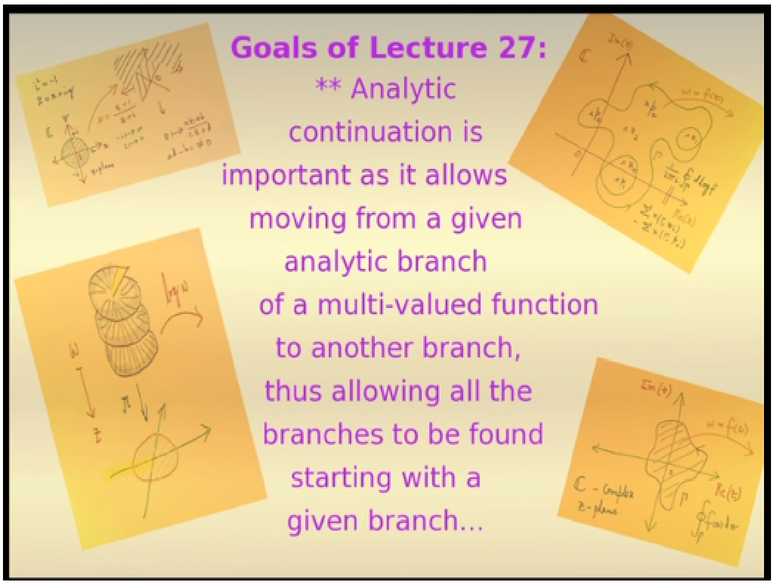

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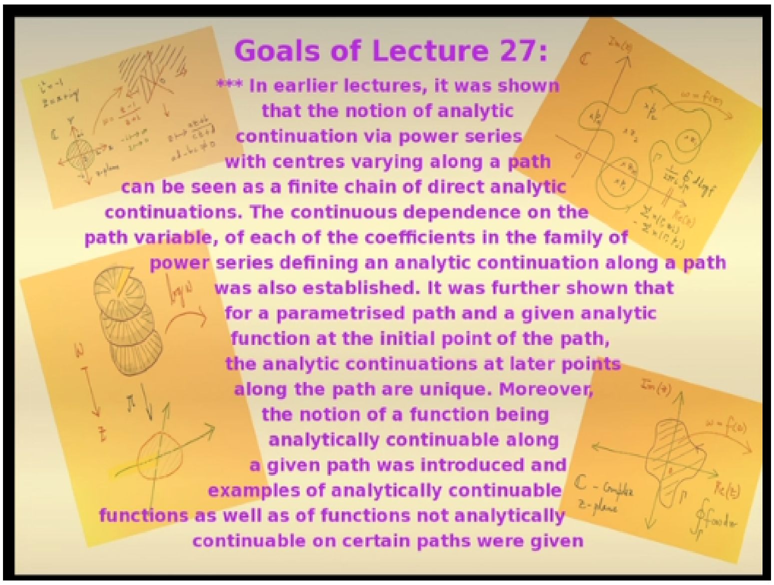

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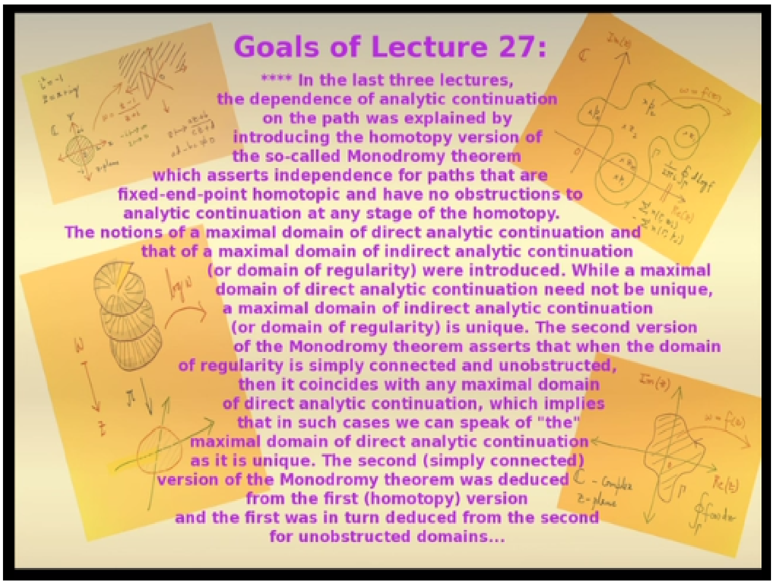

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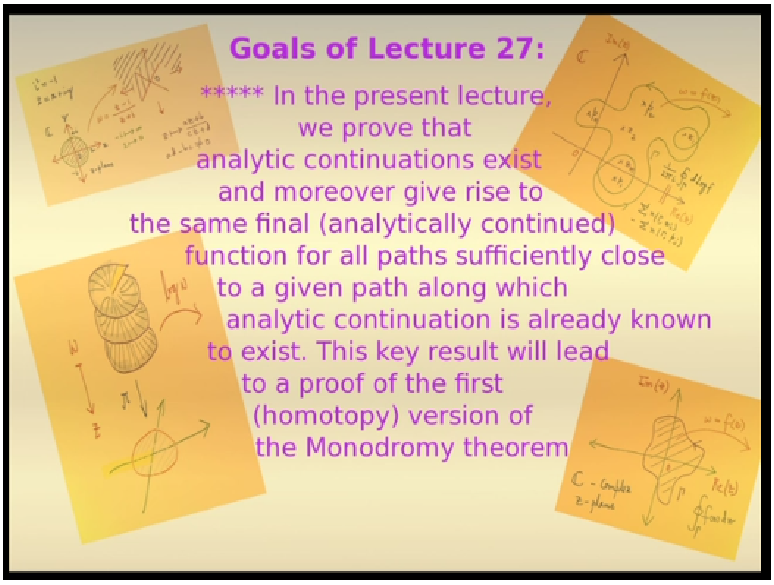

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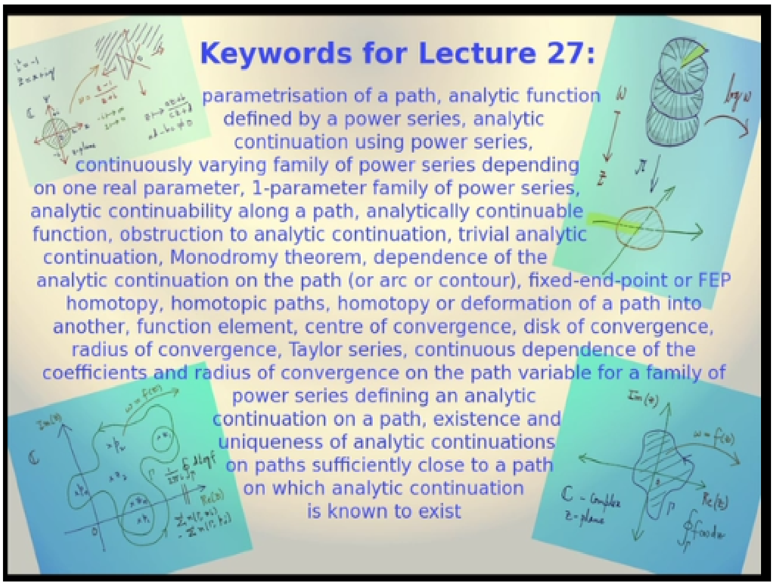

Ok so what I am going to do now is the time to discuss the proof of the monodromy theorem ok of which you have seen couple of versions right and so I am and of course going to prove version of the monodromy theorem, the first version of the monodromy theorem ok. So I begin with so let me put the title as proof of the monodromy theorem so this is version 1.

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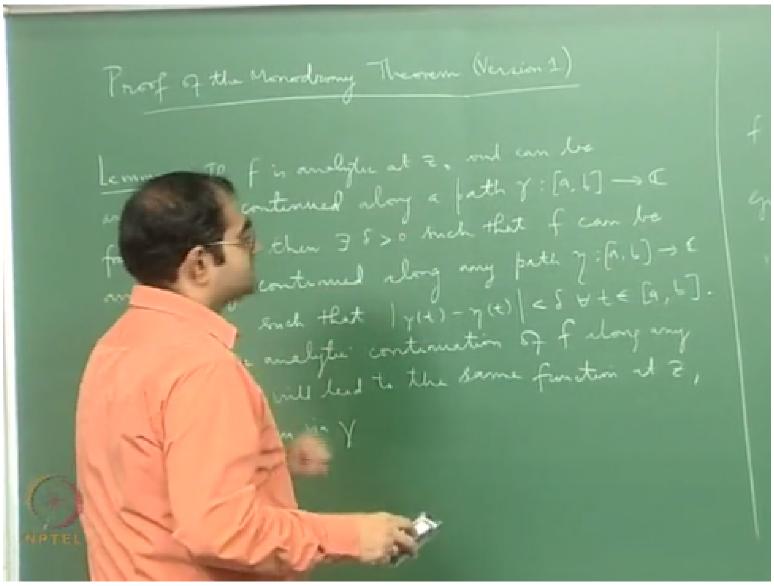

That is what I am going to prove, so we begin with lemma ok and what is lemma says is it says the following thing, it says that it suppose you have a path along which you can do analytic continuation of a given function at the starting point of the path ok then for all sufficiently nearby parts analytic continuations will exist for the same function ok and the fact is that analytic continuation at the that you get at the end of the path will be one in the same ok.

So here is a lemma if f is analytic at z0 and cab be analytically continued along a path gamma from z0 to z1 ok. Then delta greater than 0 such that f can be analytic continue along any path neta from z0-z1 such that distance between gamma of t and neta of t is less than delta for unit t in a b more over the analytic continuation of f along any such path neta will lead to the same function at z1 as the gotten via gamma.

So what is what is lemma says is that you know so you says that if you have a path along which analytic continuation then you know sufficiently nearby paths starting at the same point and ending at the same point ah also we had analytic continuation and these the result of all these analytic continuation are everyone the same ok. So you know so if I draw diagram is something like this.

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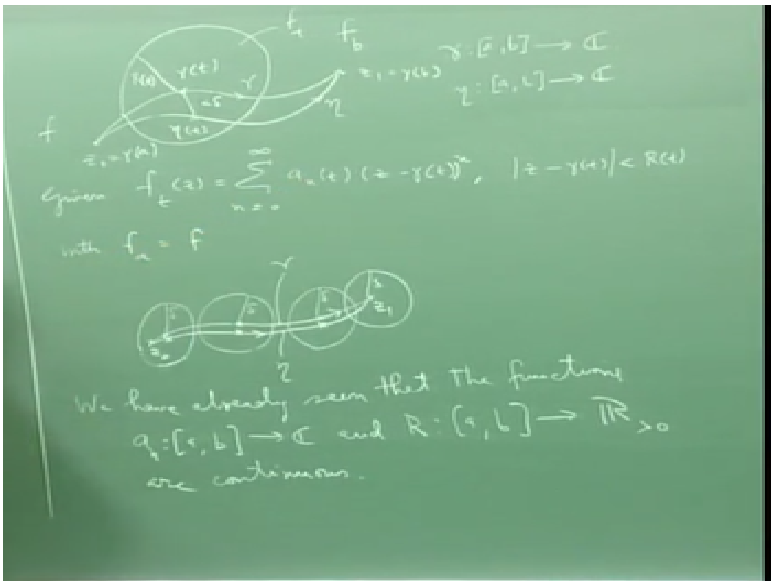

So here is the point z0 at which function f is given and I have this path gamma it goes to z0-z1 is function from the closed interval a b on the real line to complex plane, so it is a parameterized path and you know that f can be analytically continued along gamma, so which means that there is analytic continuation is given in terms of a power series, it is given in the terms of one parameter family of power series.

So so given ft of z=sigma n=0-infinity so probably a and ft z-gamma t power of n with modz with this power series converging in a disc under gamma t radius rt where rt is the radius of convergence power series which is assume positive and gamma t is the centre of the power series ok. So with f0 there is a faz ok and what is the final function you get it is.

So this correspond to t could a and this corresponds to t should b ok and this is gamma a this si gamma b and somewhere between we will get gamma t the point gamma t and at the point gamma t there is some there is a disc there is a disc cantered at gamma t with radius=radius of convergence r of t and you know that here ft the power series ft leads here ok it represent it converges here.

This is the substance, the power series ft is a power series centre at gamma t, so it is an expansion in terms of powers of positive integral power z-gamma t and a and ft are the corresponding questions ok and when you put t=a you get gamma a and corresponding power series fa which is the functionality analytic function f you started with and it after that the at the end of the path you will get new function here.

And that new function is fb is a is a analytic continuation corresponding to the analytic continuation with power series fb ok that corresponds to t to b ok and now what is lemma says is that you can find a delta such that you know if you take any other path neta which also is from a and the same coefficient will ab-c and it also from z0-z1. So you know the neta should do something like this.

So your neta is like this ok, so neta is also define from t close interval a b on real line and the point about what is the connection between neta and gamma for every t the distance between the points neta t and gamma t is less than delta, this discuss in gamma t and neta t has to be lesser delta this si the sense to be lesser delta, which means you know you actually looking at paths.

This so you can call this delta neighbourhood of a path gamma ok, so you know define so you know if you draw diagram let me let me read right here, so here is my gamma ok and you know at every point if you give me a point then you know I take this disk with radius delta. So this is my open disc with radius delta here and you know by taking here I will get another disc open that is delta.

You know and towards it will be like this, that is a fixed radius delta and here also even at including endpoints, see you now of course because of the because of the compactness of the path finitely many disks are enough to cover the path of course and the point is that you what the lemma says is that you if along this path gamma you can analytically continuous f then for any other path which starts from z0 and ends at z1 which is lying in this union of all these disc ok .

If you have any other path like this nearby path ok this is gamma and any other nearby part neta such that at each point that at each point t the discuss between the corresponding point of neta which is neta t and the corresponding point of gamma which is gamma t less than delta ok suppose you take path of that then the lemma says that on that path also the analytic continuation of f is possible.

And not only that if you say that the analytic continuation of f we need along that path also you give analytic continuation you will end up with the same function as you got in the case of gamma in the fb ok, so what this tells you it says it gives you existence and uniqueness of analytic continuation along nearby paths ok so you can refer to the lemma compactly as existence and uniqueness of analytic continuation along nearby path ok.

And how nearby that nearby is given by this delta which existence you have to show that is the path of the proof so what we do if proof is pretty easy ok. So you know we just use that we just use the following fact that you know see we have already seen that we have already seen that are the functions a and r from from closing to a b to see and r from the closed interval a b to r greater than zero or continues.

I want to prove this, I have already prove this that you know if you write a and if you writer analytic continuation like this an analytic continuation along path in terms of power series then the coefficient of power series they are contains functions of t each a an analytic continuation of t and the radius of convergence of power series is also a analytic function. So there is something I approve ok.

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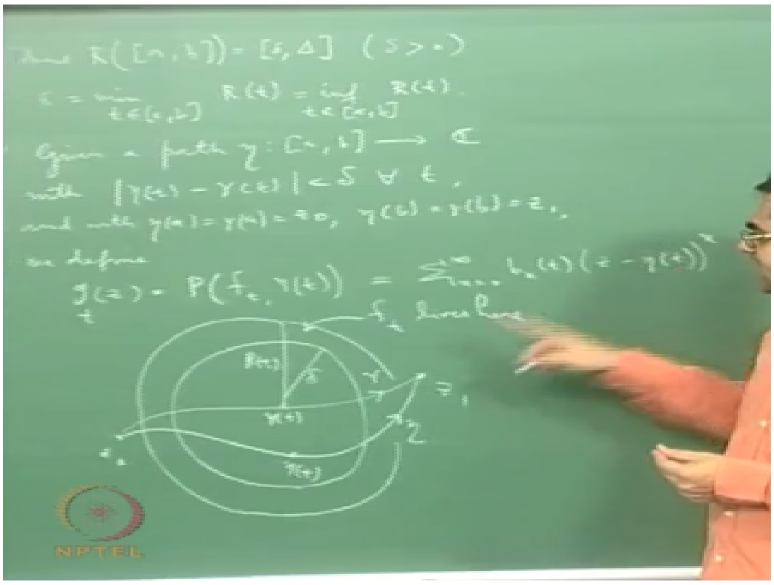

I am going to use that, so well so the if you take rf close interval a b what you will get is some delta, capital delta ok see mind r is continuous function ok, that is the continuous function so you know continuous function maths connect excel to connect sided maths a compact set to compact set. Therefore you see this close interval a b is going to be map to connected set on the real line.

So it going to be a real interval and is also going to be map to compact set, so you are going to get an interval on the real language is connected and compact ok and of course every value that are taxes positive that for this delta is positive. So you know another word Delta is the minimum of all the radii of convergence of all the power series, it is small delta little delta is the minimum of these Rts.

So delta=minimum t belonging to a b of all, it also write infimum of t belonging to a ab Rt and of course you know infimum and minimum will be one and the same because in this case a variable is is defined on a compact set ok and the compact set all contains its boundary. So fine so this is the delta that I want, I think this is the delta that we need to prove the lemma ok.

Ok so now so the question is if for this delta what I have to show is suppose they give you another path eta satisfying this condition that the corresponding distance between for each t neta t and gamma t is less than delta I have to show that on that path neta also I can define and analytic continuation and have to show that that analytic continuation will also lead to the same function as f as f b.

Then you reach the point is z1 for the parameter value t=b ok. So how does one prove it very very easy given a path neta from a b to c with difference between neta t and gamma t less than delta for all t and with neta a is gamma of a to z1 z0 neta of b=gamma of b=z1. We define gt, so I am going to define a analytic continuation. So it is very very simple, so let me draw on the lone, you know how I am drawing and diagrams.

So here is z0 is a z1 is some t at and this is the path gamma, so this point is gamma t I have a nearby path which is neta and the corresponding point on it neta t and the assumption is that with respect to gamma t I mean with respect to gamma of t you draw circle centre at gamma t radius delta then this circuit contains neta. So this neta t and gamma t is the centre that is assumption.

So what you do is very simple what you do is just you define gt of z b=sigma b=the power series expansion of ft centre at neta t. So what we have understand see this if you take the point gamma t, then I am having you know the add gamma t I have the power series expansion for I have the power series expansion of Ft, so ft is already the power series, it is expanded at gamma t.

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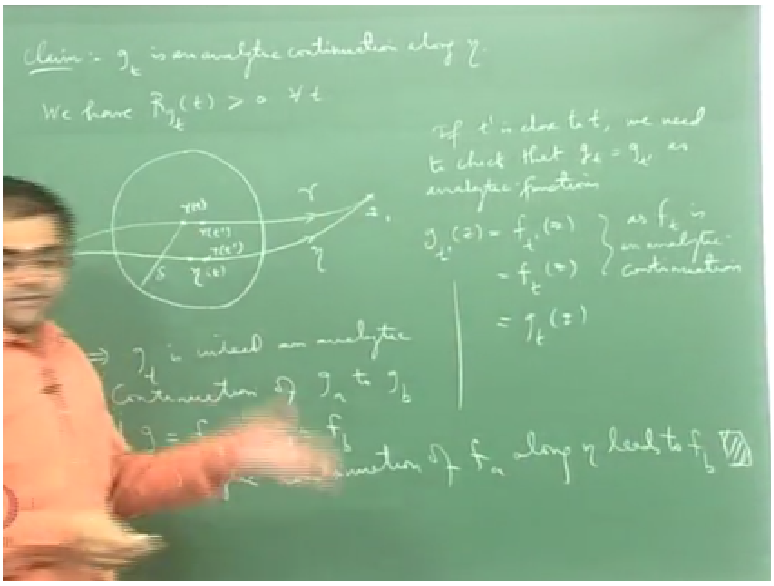

This is expanded at this point and it has got release of convergence Rt ok and you know this Rt is this Rt is always greater than delta. So you know if I take this point gamma t and if I draw the circle with if I draw the disc of convergence for ft I will get bigger disc, I will get a bigger disc ok, centre at gamma t ok, if I consider the function ft the power series of ft, it will represent an analytic function the power series will represent analytic function.

Whose Taylor series will be this power series itself and that that where is it valid it is very the disk of convergence and the disc of convergence is at the this centre at gamma t ok and the radii is Rft, but you know that Rt is greater than delta because delta is minimum of all the Rt and now what you was understand is that this bigger disc I have drawn that is where Ft leads, the analytic function ft leads in that bigger disc ok.

Since the analytic function ft is the bigger and this neta t is inside that bigger disc I can write the power series of this Ft at eta f ok I can do this. So what it will what it will give me a z i will get some power I will get some power I will get some coefficient sigma n=0 to infinity some bn of t z-neta t to the power of n, this is what I do. So this is I am just writing the power series expansion of the function f t at this point.

So it is an expansion in terms of integral positive integral powers of Z-integral ok. Now my claim this that will be define like this my claim is that this gt is analytic continuation along the path neta. So the claim is gt is an analytic continuation along of of I mean analytic continuation along along the path neta and why is why this are true, see what is the condition for our what is the condition for an expression like this to give rise to a analytic continuation.

So the first condition is that for each t you must get the power series of radius of convergence that is the first condition and the second condition for for nearby ts the functions that are given by the power series they should be one in the same analytic continuation ok. So you see now for each t which is very clear that this function this power series of positive radius of convergence.

Because what is this convergence of this radius of this convergence of this at least ah this difference between neta t to end of this ok because as far as the ft leads ok. So what you must understand is actually let me write this the radius of convergence is you know so I should say at least it is at least this distance Rt-the distance between gamma neta t and gamma t.

This is at least this true ok, so you know if I draw this line like this, this whole length is going to be Rt alright and this this length is distance between gamma t and neta t and at least this much should be the you know at least here this power series will lead because the power series are Ft. So so the power series is at least radius of convergence is this much which is the distance because the distance of Rt and this distance will be neta t and gamma t ok.

And that is the that is what it is so for each t the radius of convergence is positive ok that is the first condition, so each t I am really giving you a proper the power series I should not give you power series zero radius of convergence that is not a log ok for each t I will ensure that I get the power series has a positive radius of convergence, that is the first condition, second condition for nearby t is the power series should represent the same analytic continuation.

That is what that is the second condition for an expression like this to give to define an analytic continuation. So does one see that if I use simple C so we have let me write this fact you have so it is a I think so this is the radius of convergence Gt so let macro environment write the Rs of gt of t , so this is the radius of convergence gt, this is radius of convergence ok that is the first condition.

So the second condition is so let draw the diagram which is like this, so here is my gamma here is my neta this is z0, this is z0, so you know so I have this point gamma t and I have this point neta t and of course I have taken a disc centre at gamma t radius delta . So this is how I chosen a neta ok now you see if you take t prime close to t ok.

Of course neta t prime will be close to neta t and gamma t prime will be go to will be close to gamma t and that is continue, so you know if you take a t prime very close to either on left or right let me take it on the right so this gamma t prime which is rt prime close to t and here is neta t prime Rt prime close to, so you see if if t prime is close to t we need to check that ft that is gt=gt prime as analytic functions.

You have to check this ok so one of the continues of the analytic continuation is that is given by power series but the power series are all expansion of analytic function at different points ok ok. The points are varying along the path but the condition is if you go to nearby point then the power series will be different but you will say analytic function ok. The power series of course will be different change the centre of the power series ok.

So what I have to check is that gt for t prime close to t analytic function define by the power series gt and the analytic function depend that the power series gt prime around this have to check that only then this will be analytic continuation and that is very easy if you check r you see so it is just the matter of so you know what will happen you will see that gt.

So you know gt prime of z we calculate this will be ft prime of z ok as analytic function it was I define it because gt prime is define to be the power series of ft prime, so it will be equal to analytic function ft prime that is why I have define it here ft is the power series of ft therefore the function represented by gt is just ft ok. So gt prime is ft of z, but you know f is an analytic continuation.

So for t prime close to t this will be same as analytic function ft of z, so this step is because f is ft is analytic continuation, this is ft because analytic continuation and t prime is close to t ok, but then ft of z as an analytic function trial this is because ft Zayn and coordination entity function is the same as gt of z it is also gt z, gt is define to be the power series of ft centre i needed.

So so that therefore so this implies that gt prime gt is Indian analytic continuation and analytic continuation of ga to gb ok. So you have proved existence of analytic continuation along the path neta along any path neta which is with delt neighbourhood of the path gamma ok. Now what is ga but you see what is ga, ga is f, fa because gt is after all the power series expansion of ft ok.

So Gi is sgt f and you see the gi is fa and gb is also equal to ft by definition, so the analytic continuation of fa, so that is the analytic continuation along the path neta of F-fa and meaning to fb ok, now use a fact that if you give me a parametric path ok then the analytic continuation is unique if you fix the starting function, so because along because I have proved that along the path neta there is one analytic continuation which starts with fa and ends with fb.

Uniqueness will tell me any analytic continuation along the path neta which starts with FA will always end with FB ok, so I will get the same analytic continuation ok, so so any you know here I am using this fact that you know if you have a path give me if you have a path and parameterized the path and along that path if you have an analytic continuation of a starting function then there is unique.

The analytic continuation is unique, you cannot get different analytic continuation in fact analytic continuation along each point is uniquely determine for every point other than the starting point. So it is uniqueness of the analytic continuation along a fixed parametric path like that I am using ok. So so any analytic continuation of fa along neta leads to fb.

So that definition the proof of lemma ok, it is a it is a pretty simple lemma and nothing complicated about but the power of lemma it tells you that if you have an analytic continuation along the path then along nearby paths also analytic continuation will exist and they will be the same they will give rise to same analytic continuation ok and of course I should again let me again repeat the risk of being overly repetitive.

But we always keep using this fact that if you give me a parameter path and if you give me a starting function is analytic function the starting point then there is only one analytic continuation of that function along the path, this is essentially because of the identity ok. So that is something that we keep using alright. So this is the lemma that says that sufficiently closed path along which you have an adequate continuation will also lead.

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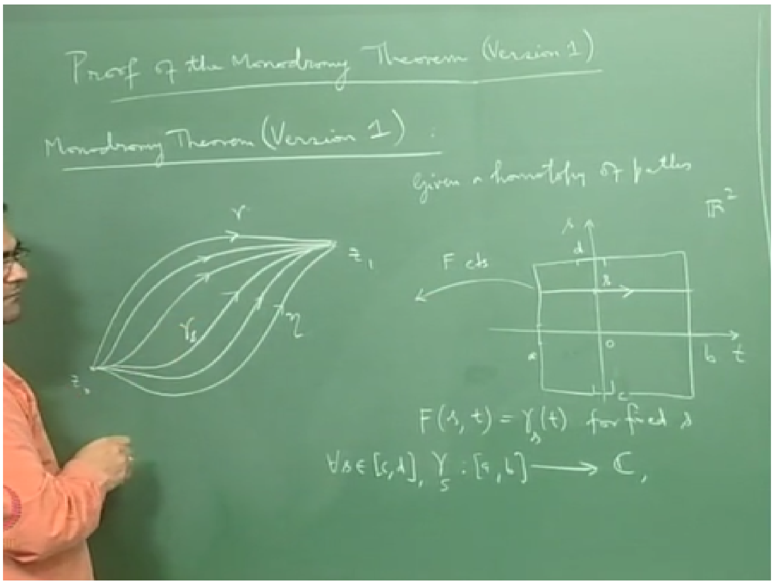

We also have an analytic continuation but you will get the same result ok. Now I am going to use this then prove use these ideas and prove the first version to the monodromy theorem. So let me write it down so given so let me let me draw the diagram so you know the given point z0 and z1 of course today I am drawing Z0 and Z1 seem to be distinct point but there is no no restriction they could be one and the same point ok.

So you use that case also then so z0 and z1 are 2 points ok and you have 2 paths that is the path gamma from z0 to z1 and another path neta from z0 to z1 which are homotopic, so there is a homotopic from gamma to neta ok. So there is a homotopic from gamma t to neta by intermediate paths which means I can continuously deform the path gamma to the path neta.

So I have this of course because complex plane is simply connected any part like this can be the form to that ok and how do I write this homotopic so we are so give an homotopic of path from drawing diagrams the other way I am drawing to the right side but does not match, so let me do it like this. So you know you have you have this real line having a real plane R2.

And I have t parameter and I have here the S parameter and this is the unique square ok and I have a continuous function f from here to here ok and what is this continues function does is so this continues function is written as f of s, t ok and what it does is that if you freeze an s then it gives rise to the path gamma s. So f of s, t is gamma s of t ok. So for fixed s of course I have put I have you know so we can use using unit square.

You could have used ABCD you could have use produce of two closed intervals but I am not doing that also has we can do that and it is alright rather gammas are all our paths are define ab so let me also use that let me do this maybe I can take some a here and b there and my homotopic could be defined on CD could be like this and in the argument is going to be no different like can I still have it like this.

So I will get this, so this get this rectangle which is given by a b cross CD ok t lies from A to B and s lies from C to D this show the picture crab images what happening is that gamma s is for every S in give for every s if you give me certain s value which lies from C to D I have this gamma s which is different from a b to the complex plane giving, so you know if I fix the value of s from C to D.

And if I take the class and if I led t to vary from a to be I am getting going to get this line ok and the as I move as t moves along this line this is going to trace the curve gamma s ok, gamma s is a path alright and then t= s=0 I have gamma then s=d sorry s=c I have gamma and s=d I have beta. So you know so all these intermediate first they all started time T=0 this time t=0 this is not t=a and this is time t=b ok.

You think of the time parameter and you think of interval a b as time, it is it is prime as a parameter as you moving along the path ok. So so what is happening here is all these path gamma s of a is always z0 gamma s is always z0. So this is what is called fixture and point that is all the paths involved the same point and end at the same ok and gamma a gamma c is gamma d is there ok.

So this gamma is gamma sorry c this implies gamma of t and you have the intermediate for intermediate value of s between c and d you get various intermediate paths. So actually what is happening is that you know you are having you can think of all these lines you can think of various lines like this and image of all these lines are going to give rise this, so the image of this whole square and this whole rectangle is this information of this.

But except that your collapse this whole end to the point z0 and collapse that end to the point z ok, so you know if I take this thing and collapse it I will get something like this ok, this capital f is just continues. Now so this is the this is so monodromy theorem says that you are given paths gamma and neta which are homotopic and what else are you given, you are given an analytic functions here which can be continued along gamma.

And not only on gamma it can be continued on each of these paths ok and I am starting analytic function at this point F which I can analytically continue along any of these paths and the question that monodromy theorem answer is about is what is the function you will get it this end when you go along different parts. The monodromy theorem says you will get the same function you will get different function.

If you could expect that if you go alone gamma you got one function but maybe if you go along neta you may get another function and if you go along some intermediate path you may get another function ok. So you would expect that if I start with S and analytically continue along gamma s and I will end of the function f s and you would expect that fs could change, but the monodromy theorem says no.

It says that would not happen says always you will get back the same FB which is one in the same function that you get so you can analytic continuing any of these ok. So monodromy theorem says that if you have homotopic like this and there is no obstruction that means the given analytic function at the starting point can be analytically continued along all these paths, then the function you get at the end along any of this path is there are one and the same.

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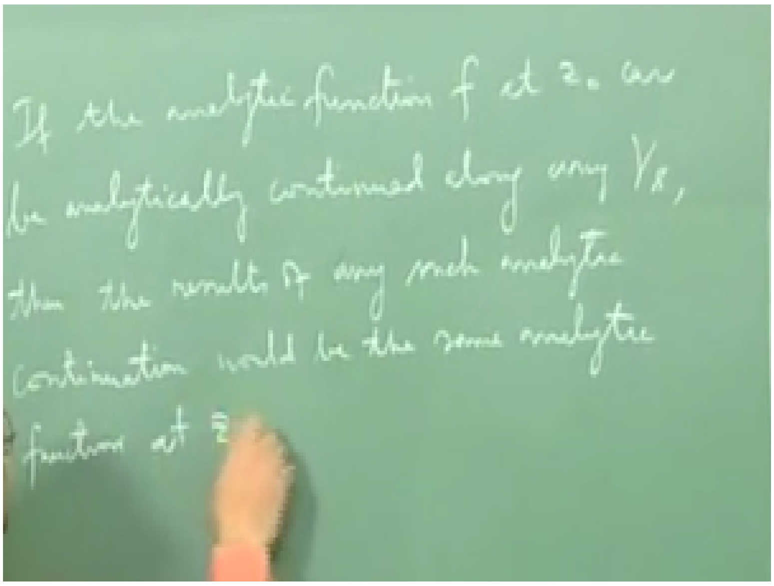

You are going to get only one analytic function right, so that is what we are going to prove, so let me write that down that if the analytic function f at z0 can be analytically continued along any gamma s then the result of any such analytic continuation would be the same analytic function at z1. This is the monodromy if at the starting point z0 I am given analytic function.

And suppose I can analytically continue along it any of these intermediate paths intermediate paths of course you also include starting point gamma and the ending path neta ok. Then no matter along which path you continue analytically the final function that you will get at the other end point at the terminal point z1 it will be the same, that is what monodromy theorem says.

So if you want to state it in compact language monodromy theorem says that well analytic continuation of of a analytic function along homotopic path then analytic continue the result of analytic continuation of a given function will be the same that is what it says, but of course the technical point is that you must make sure that the analytic continuation is possible along every path in the homotopic.

That that should not be some there should be some whole here there should not be for example ti should not be the we cannot be like this you know that for example the originals here and here you are starting the branch of the logarithm and certainly a path which crosses the origin that is the path along is the branch any branch logarithm cannot be continued analytics.

So such a things should not have that should not be a point in this leaf like region that I have drawn that should never be a point where there is obstruction to analytic continuation, there should not be a point where I cannot continue a certain there the function that worry about, there should not be, so there should be no objection to analytic continuation.

So well let me say a few words about how you going to prove it, the method is very very simple what we want to prove is it really is in this Lemma that you know you can find delta rather we have seen lemma which is nearby paths have give rise to the same analytic continuation ok. So we have what we going to do is we are going to show that you know you can .

And how and nearby that was given by delta so we are also going to find the delta here ok, it is different delta ok, but that delta will give you know distances so that will break up these homotopic into you know strips like this and along each strips the analytic continuation are going to be the same ok. So by breaking this whole thing just like a done in this diagram into finitely many strips of thickness delta ok.

And by using the fact that along each of these successive path of path the analytic continuation along the top path is same as analytic continuation along the bottom I mean you do this finitely many times you will get the analytic continuation on the first part will be the same as analytic continuation on the second path and then analytic continuation on the second path will be the same as analytic continuation along the third path.

And you go by induction and finally went off seen the analytic, the analytic continuation that they are equal to the analytic continuation along the last ok. So this is the idea of the proof, so the idea of the proof is be able to exactly draw diagram I mean diagrammatically to get something like this breaking the leaf into smaller leaves of thickness delta ok, that is the idea of it, the idea of the proof will be same, so I will explain that in next lecture.