Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem
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Lecture-26
Deducing the Second (Simply Connected) Version of the Monodromy Theorem from the First (Homotopy) Version

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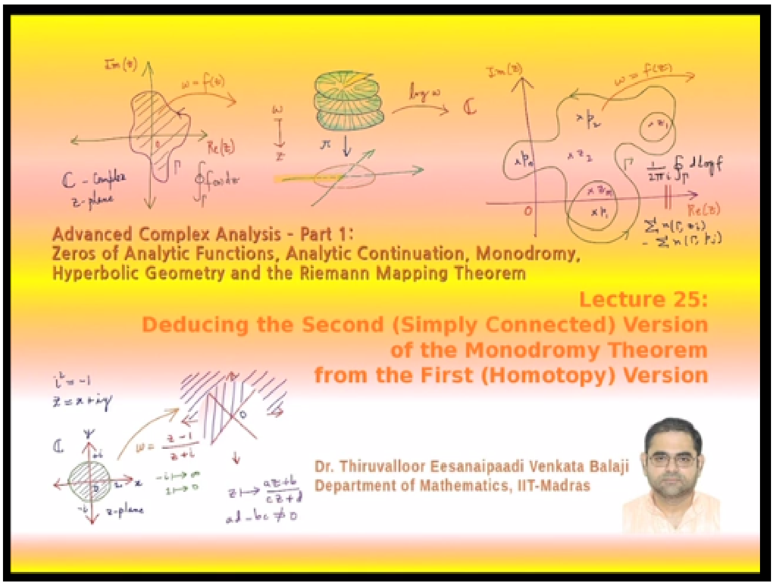

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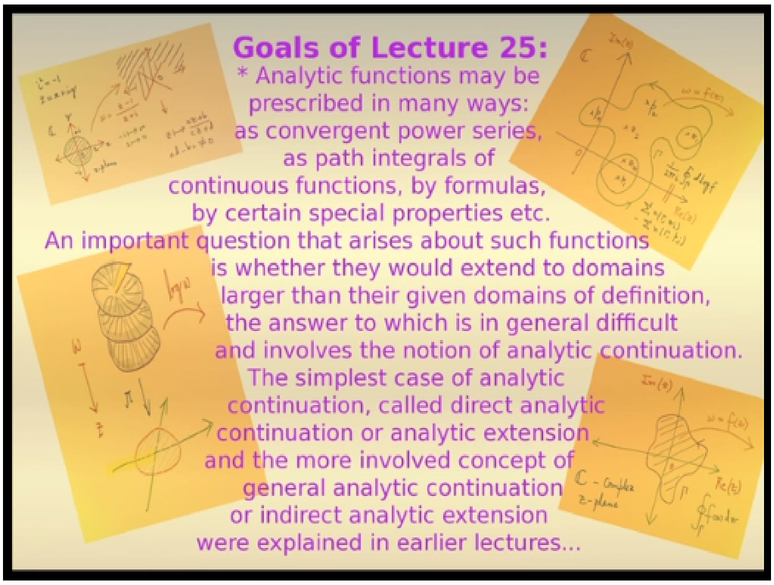

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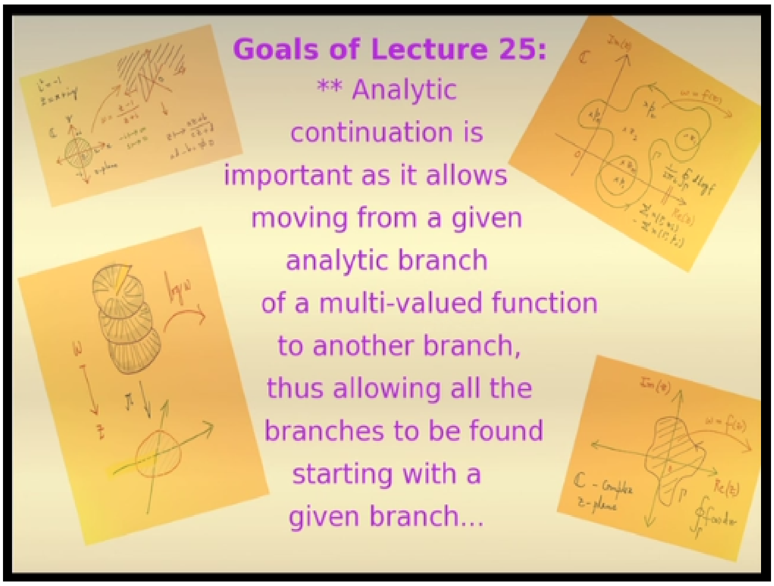

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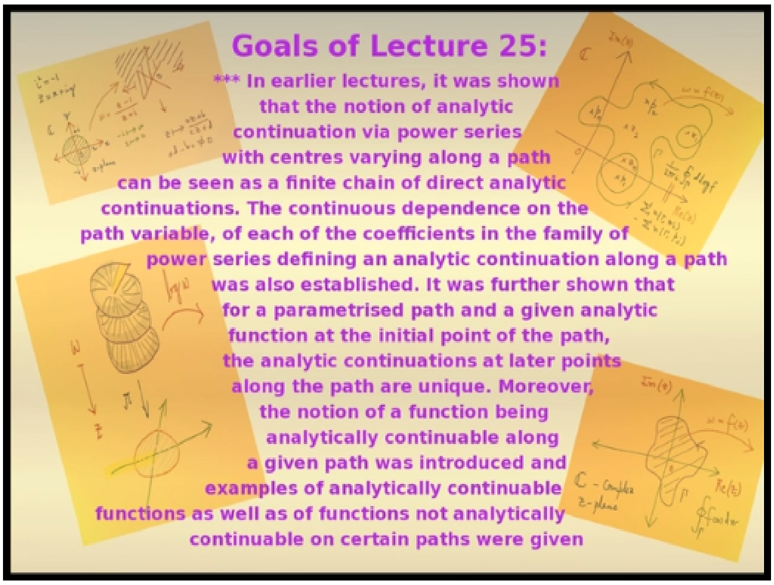

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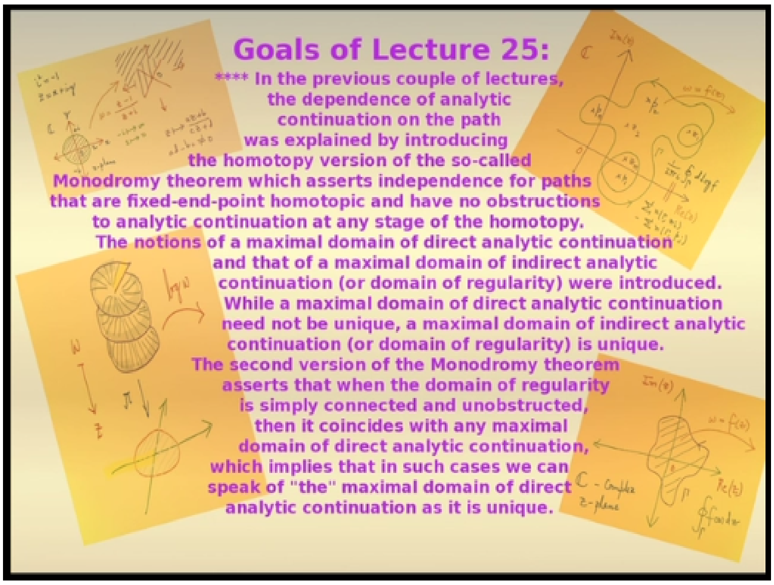

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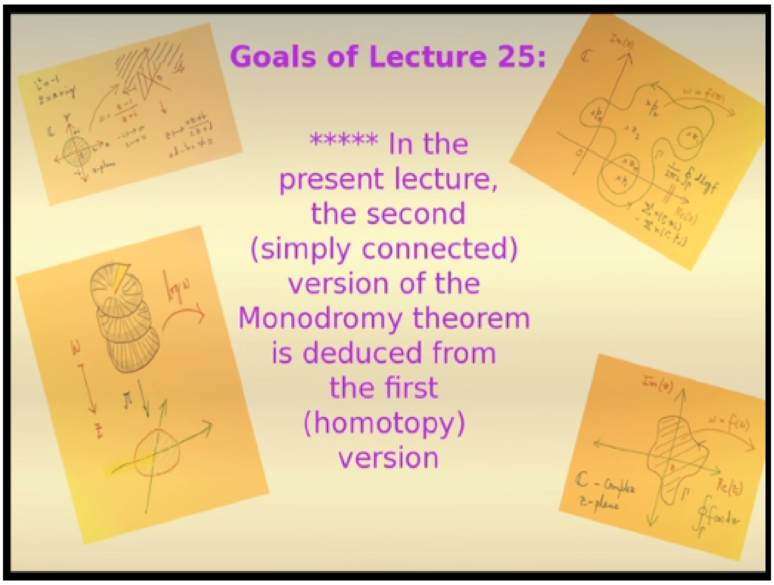

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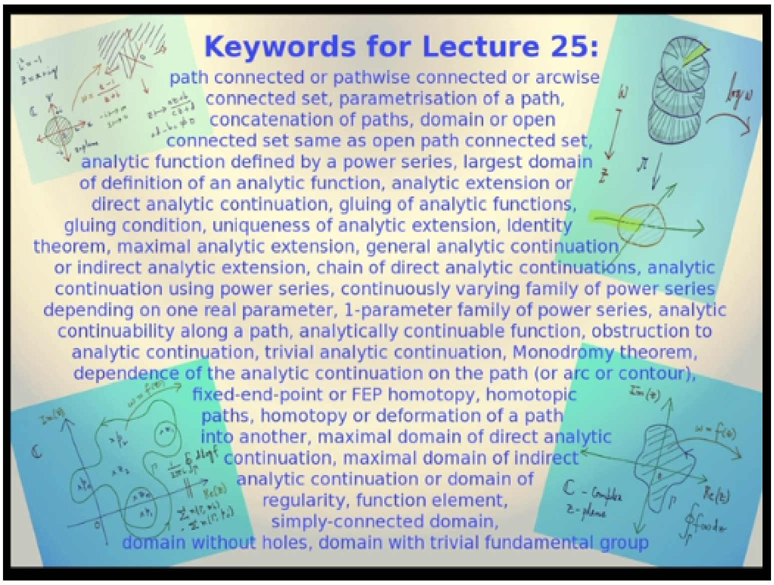

Ok so we are looking at these two versions of the monodromy theorem ok and trying to prove that their equivalent ok. So basically so we have monodromy theorem version 1 is so the diagram is like this.

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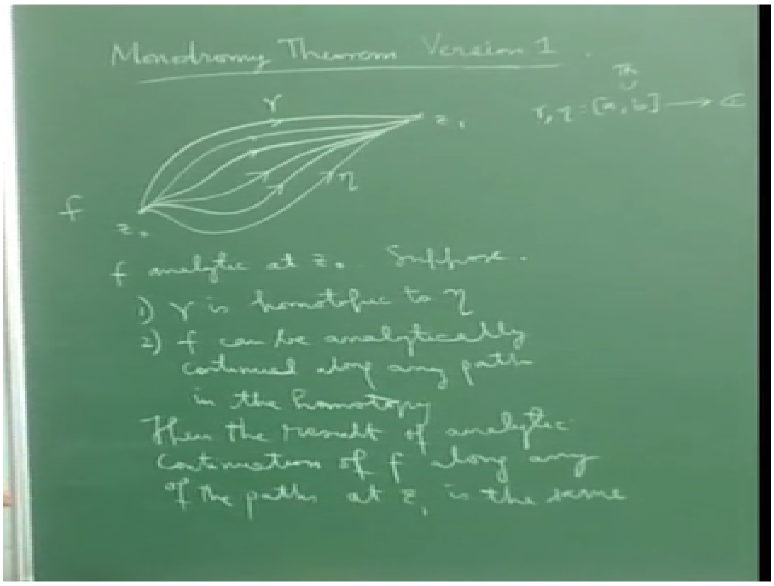

Suppose you are having two points z0 and z1 and you have 2 paths gamma and neta both of both parts thought both paths are defined on close interval b in the real line and taking a listener contact numbers and suppose and you assume that you are given function f analytic at z0 ok. Suppose following these poles under one the path gamma is homotopy to the path eta ok gamma id homotopy to neta ok.

That which means that you can start with gamma and then you can continuously d form gamma into sequence of sequence of intermediate paths which leads from gamma to neta ok like this. So that is gamma=neta and then number 2 f can be analytically continue ah not only on gamma but on each of the intermediate paths the homotopy including neta.

So f can be analytically continued continue along any path in the homotopy, in another the homotopy is given by continuous succession of path which start at gamma and end at neta ok and the assumption is f the function f which is analytic at z0 can be analytically continued on each of these paths that occur in the homotopy. So another f saying this as a headset earlier.

There is no obstruction to the analytic continuation of f along any intermediate path ok all these intermediate paths you can an analytically continuous ok. So you must remember that first of all what we assuming is that along each part they can be analytic ok in other words it means that direct just some analytic continuation f along each 1, but then already I told you the earlier lectures.

You know if you have a path ok and if you have a function analytic function at the starting point then the analytic continuation f along the path is unique we have already seen this. In other words what this means is that this hypothesis mean that all on his path that is only one analytic continuation of f, that only one analytic continuation possible and its path and it is there ok.

But the only question the monodromy theorem answer is what happens after you analytically what is the function at the other end point z1 and the monodromy theorem say that all the functions you are going to get after analytic continuation along any path at z1 they are continuation ok then the result of analytic continuation of f along any of these any of the paths at z1 is the same ok.

See you analytically continue this function f along any path it will includes the starting part gamma the ending path neta and any path in between ok and when you come to the point z1 that function that you will get the analytic function that we will get that will be one and the same, that is what the monodromy says that is what monodromy theorem says ok. So in way short if you analytically continue function along 2 homotopy paths.

Then the result of the analytic continuation is going to be the same at the end point it will get the what are analytic function you get by continuing one path it will be the same as the analytic function you will get by continuing on another path, the only requirement is that these 2 paths should be homotopy to one another ok and of course you have assume that to every intermediate path in homotopy analytic continuation exits ok.

So ok it is a it is a kind of statement which assumes the same existence and uniqueness ok, so you can assume the existence of analytic continuation on all the paths intermediate paths ok, the starting function is the same the analytic functional is starting with that the starting point in the same and what is the uniqueness of the analytic function that you will get at the ending point ok that is the monodromy.

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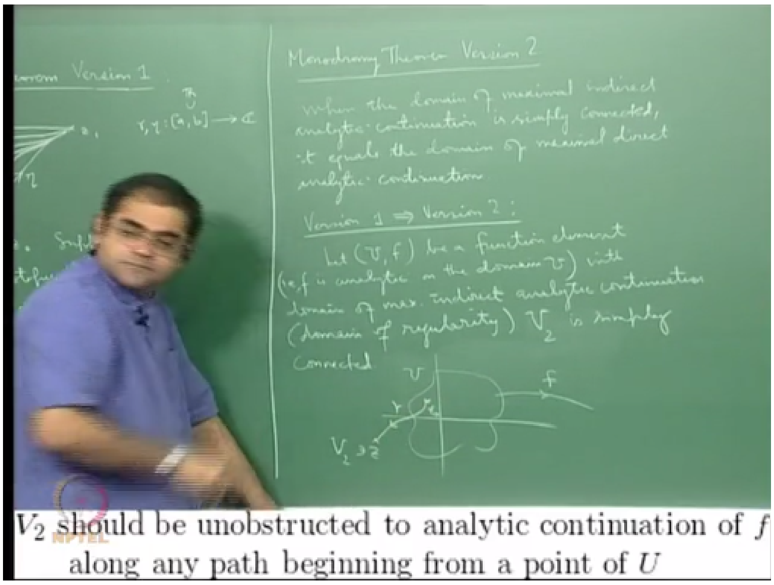

So this is version 1 ok and then what is version 2 version 2 it is the following statement suppose you have a suppose you have function element ok namely consisting of a pair of analytic function and domain on which it is defined then if you take then for any such pair you have 2 you have define two sets, one set is called the maximal analytic extension of the given function ok.

It is called the region of maximal direct analytic continuation of the given function ok and then the another set which is called the region of regularity of the given analytic function or also called the region of indirect analytic continuation of the given function and what monodromy and already we have seen example that you know the region of regular ok. That is the reason of indirect analytic continuation of the given analytic function can be bigger.

It will be bigger than its region of maximal direct analytic continuation for example the at any principle back logarithm, then the region of maximal analytic continuation will be a slip plane you will have to prove the negative relaxes where ass if you take the region of regularity it will be the plane. So the reason of regularity is bigger than the region of maximal direct analytic continuation.

In the same set the region of regularity also includes points on the negative realisation, the reason is because across points on the negative relaxes you can continue the log function any branch of the logarithm in analytic way and an analytic continuation exist along across any path which goes through points on the negative real axis ok. So what the monodromy theorems question to say that situation that you can say that the religion of maximal director analytic continuation is equal to the region of regularity.

There is a reason of maximal direct analytic continuation is equal to the region of the domain of maximal analytic continuation=domain of maximal indirect analytic continuation they are one and the same and the answer is that it will happen when the domain of the maximal indirect analytic continuation namely the domain of clarity is simply correct ok. So you state that when the domain of maximal indirect analytic continuation it equals the domain of maximal direct analytic continuation.

Going to remain the maximal indirect analytic continuation is simply connected then that domain also becomes the domain of maximal direct analytic continuation ok. That is version 2 of monodromy ok and now what I want to say is that these 2 are these 2 are equivalent and then I will say this version. So I will have to prove this version. So this version you can proof of yeah let us assume version 1 and version 2.

So what I do is we assume the monodromy theorem version 1 ok and we will prove version2 ok. So what is will obtain to version 2 prove that you means you have hypothesis of the version 2. This means you will assume you have a function whose domain of maximal indirect analytic continuation simply connect. So let f let u, f be a function element as f is analytic on the domain U with domain of maximal indirect analytic continuation.

We call as domain of V2 is simply connected**.** So there is this I do not know what notation I used probably I used V2 alright. So V2 is the domain of regularity of a f in the see remember

V2 is which is union of all those points in complex plane, such that there is a path from a point U to that point along which from analytic continuation.

So you know V2 diagram something like this you have so this is if you want this is U and this the function f define U and if I if mean to a point z in then complex plane when do I put it in V2 if you can find a point in U z0 and you are able to find a path gamma and z0 to z and you are also able to analytically continue a f along that path to z, all that z you put in the set t and it is clear that you know what you told you that whenever a function is analytic on a domain.

Inside the domain if there is any path then you can always analytically continuous that is true well because you needed to be an analytic continuation alright analytic function can always be continued to any point in its V2 ok and that you just reality confirmation you are actually getting back the same function ok, but the whole point about indirect analytic continuation is that you might go around a loop.

And you end up with the new function like what happens if you go around the origin once start with the branch of a logarithm and you will end up with the next branch of the logarithm ok. So the point is it is always trivial that if you have a analytic function or domains is that the path inside the domain on that path you can always be analytic continue namely it will be you will get back the same function at every point on the path alright.

But the question is if you are able to find the path that goes out of the domain along with you can still continue the function. All those terminal points of all such paths put together you get another set V2 and that turns out to be again open and connected okay and therefore that will be come to do in this is called a domain of regularity. So V2 domain of regularity of f.

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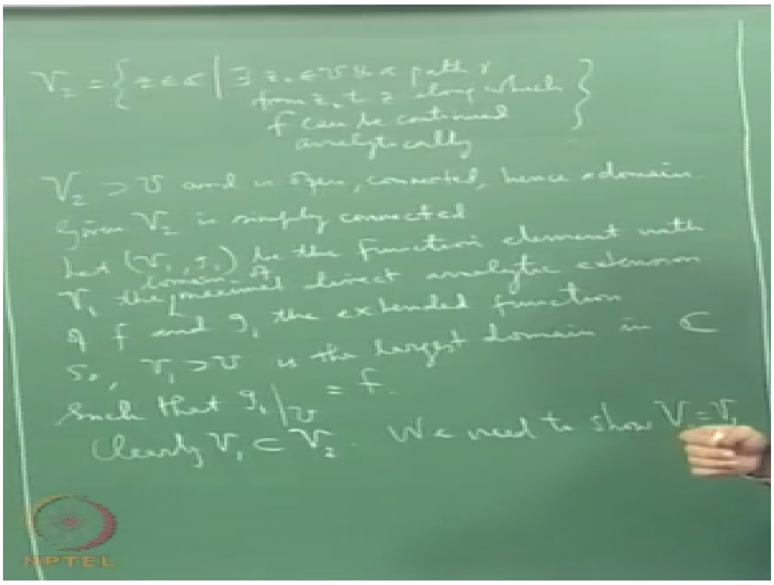

And that the point that given you is said V2 is simply connected but so you know so let me write let me just write down what I said V2 is set of all z you see set that exists z0 belonging to U and a path gamma from z0 to z along with f can be continued analytically. This is a definition of V2 and V2 by this definition itself by we check last time it is very easy to check the V2 visa both open and connected switch to domain V2. V2 contains U and is open connected .

Hence domain and as the domain will add and what is given is that V2 is simply connected is given V2 is simply connected, what you have to prove I have to prove that V2 is equal to V1 which is the domain of maximal analytic extension V2 is equal to V1 see we also define V1 to be the domain of maximum director analytic continuation may be it is the largest open subset of open connected subset of C where the function f can be directly analytically continued to give analytic function ok.

And we if we call that is V1 we want to show V1=V2 ok, so of course but you now it is it is obvious that V1 contained in V2 alright because it is always obvious that V1 is contained in V2 because V1 contains a direct analytic extension of f and you know direct analytic continuation is also an indirect analytic continuation you can treat that also as an indirect analytic continuation namely the trivial analytic continuation alright.

So let V1,g1 be the be the pair with be the function element with the function element with V1 the maximal direct analytic extension the domain of domain of maximal direct analytic extension or continuation of f and g1 the extended function. So you know that means V1 is a domain, V1 contains U g1 is analytic ob V1, g1 restricted to U is same as f ok.

It is direct analytic continuation of f aright and V1 is the largest possible to make, it is the largest possible domain to which you can you can exchange f alright. So let me write that down V1 contains U is the largest domain in the complex plane such that g1 restricted to U is equal to f ok. This is the maximal domain of direct analytic extension or continuation of f ok.

And of course you know since g1 exchange f it is very clear the V1 is contain in V2 ok clearly V1 is contain in V2 why because you see after all j you know if you take any point in V1 ok then that point in V1 is connected to U by a path ok. That is because you know see any two points of V1 are connected by a path, why because V1 is a domain, it is an open connected set.

And then open connected set is also path connected ok an open connected set is path connected. So V1 is a open connected set therefore any two points of V1 are connected by a path therefore if you take a point of V1 and point a view their of course content by path inside V1 ok. Now along that path at the starting at the point of U you have the function f which is g1 restricted to U.

And out at the but throughout that path g1 is defined, so you can think of g1 as a director analytic continuation of f along that path ok. So what is means is that you can analytically continue along any path from a point U to a point in V1 and analytic continuation of FZ given by this function g1, therefore all these points in V1 will also getting we will also be points in V2 ok.

The only problem the only point with V2 is that it may contains points at which you will not be able to directly analytically continuous f ok but at which you may be able to only indirectly analytically continue f ok, that is why wV2 could be bigger than V1 ok. So certainly V1 contains V2 alight. Now what here what the monodromy theorem version 2 says is that if V2 is simply connected then we will have to be equal to V2.

That is what it says, we are given that V2 is simply connected and we have to show V2=V1 so you have to show that V2 contains union ok, we need to show V2=V1 you have to show that alright, that is what you have to show. So how do you do that is pretty easy.

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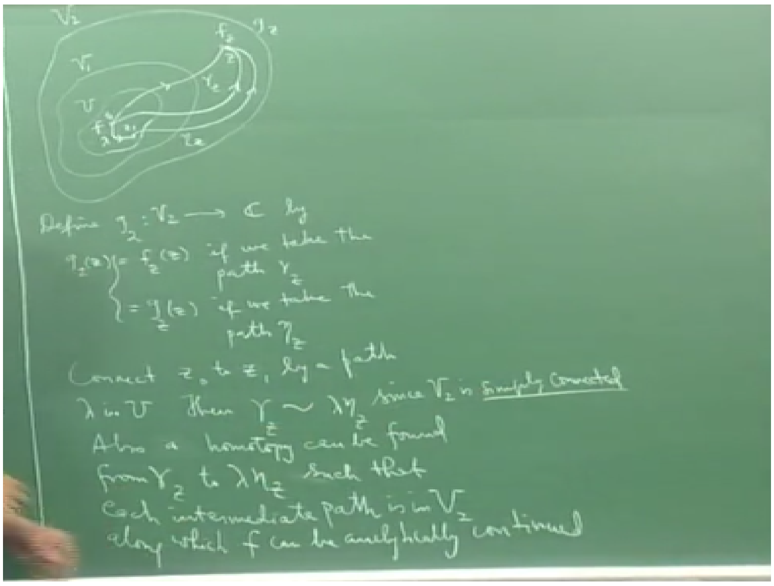

So what you do so what we do is we do is it going to equal to V1 then at every point of V2 also you can directly analytically extend f and that extension is going to be g1 because g1 is a maximal analytic direct analytic extension of f to the subset to this largest possible domain V1 ok. So in other words you know what is happening what is happening is that V2.

In V2 what is happening is you are getting various points to which you can directly you can indirectly analytic continue f along paths, but we do not know what the function is that you are going to get at the end of the path ok. But trying to show V2=V1 is the same as trying to show that even after analytic continue f the function any function you are going to get is actually g1 which is just a direct extension of f ok.

You are trying to show that for every point z in V2 ok whenever your point is z in V2 whenever you which means that you have path from a point in U to that point whenever you analytic continue f whatever you are going to get is you are going to get just g1 you are not going to get anything else, that is what we are trying to show. So what we will do is we will we will define.

So you now the picture is something like this you have U of course you know the way of I am drawing the pictures and I am always drawing bounded domains and I am drawing simply connected bounded domains ok, but they need not be like that, what is, so this picture is only to help you think alright, but should not take it to be accurate alright.

So the sequence of things is that you have U the set on which f is define the domain on which f is define that is contain inside V1 that is the you know maximal domain to which you can extend f and the maximal extension is g1 ok and then there is V2 and V2 consist of all those points to which you can analytically continue f you know that you can analytically continue f, but you do not know what is the function we are going to gt.

And the claim is that if V2 simply connected that V1 is same as V2 that is the claim. So what you do is you do the following thing take a point z in V2 by definition there is a point z0 in U and there is a path let me call that path as gamma of z alright and there exist an analytic continuation of s so I start with f here and I get a function f sub z ok. So now define g2 from V2 to c by g2 of z g2 of by g2 of z=fz of z.

So look at this definition look at this definition so very clever definition but anyway very simple definition, the definition is very simple you give me a point z of V2 small z V2 then by definition there is a path which ends at z starts at a point of U say z0 and alogn that path your f can analytically continued and you know once you know we already know that once you analytically continue f along a path.

Then the analytic continuation is unique as far as that path fixed path is concerned along the same path you cannot get two different analytic continuation there is the uniqueness of analytic continuation along a given path ok, that we already seen. So the ending function which you are going to get is going to be some function I am calling the function of f subset z because it is at the end point z alright and my what is g2 of z it is f subset z of z.

And I am defining a function like this ok. This function is well defined function, there is no problem all is well defined why so here is very simply connected hypothesis will come, see the point is this z is connected to z0 by a path gamma sub z ok. Now what do you understand is you know I could have taken instead of taking z0 I could have taken z1 ok, so let me draw it so that I can draw so instead of z0 I could have taken z1.

So this is z0 I could have taken z1 you know and I could got another path and this path well I can call it as neta z right, so try to understand what is happening when I so this is the this is where I am trying to say that this function G2 is well defined ok, I am trying to say that g2 is well defined see what I said earlier was f sub z is well defined f sub z is well defined because it is an analytic continuation of f along the fixed path gamma sub z ok which starts at z0 is the point of view.

But then for the same z I might have another point z1 in U I mean have another path starting neta z starting from Z1 and ending at the same z along which also f can be analytically continue it can happen after all it can happen and as well as far as the path neta sub z is concerned if I continue the same f along neta sub z I might end up with another function ok that tells me that this definition seemingly is not well defined.

Because this is f sub z depending if we take the path gamma sub z and this is also equal to well if you if you go along the path neta sub z you might get some let me call give me some other name to g sub z of z if we go if we take its ok you can have can you see if you want G to Oz to be well defined this to have to be the same otherwise it is not well defined to have to be the same the point.

Now the point is that you know well the point is you know z0 and z1 they belong to U anyway ok, so what you can do is you can connect z0 to z1 by a path delta ok I think delta is not a very good let macro environment use some other symbol let me say lamda ok. So you connect z1 to z1 by a path lamda ok and now watch start with f along lamda take the previous analytic continuation because after all lamda os a path inside U.

And along the path inside U you can always take a trivial analytic continuation which means you you analytic continuation is same function f along each of the path that means along each point of the path you are simply writing the power series of the same function f at centre at that point, you have the analytic you have trivial analytic continuation of f along lamda ok followed by the analytic continuation start f at z1 and leading to gz along neta z ok.

So if you them together you will get the analytic continuation of f to gz along the path which is gotten by lamda followed by neta sub z alright on the other hand you also have the analytic continuation fz which of f starting at the point z0 why are the path gama sub set alright. Now you see the path lamda followed by eta z is homotopy to the path followed to the path gamma z.

That is because both are because both points end points are inside V2 which is simply connected and both paths start at z0 at end at z1 see what is the path simply connected region if you have 2 points any 2 paths any 2 paths starting at these 2 points at starting at fixed point and ending at fixed point ok, any 2 path are homotopy to each other ok. Therefore what will happen is that the path lamda for eta z is homotopy to gamma z.

But now I assume monodromy theorem version 1 which says whenever you have 2 paths which are homotopy to each other and along all of the intermediate paths the there is no option 2 the analytic continuation along see both these paths or region between the paths is inside V2 and inside V2 there is no obstruction to analytic continuation because V2 consis all those points where you can analytic continue ok.

So both condition are the first monodromy theorem first version are satisfied ok, so that will tell you if you put them together you will tell you that if fZ is equal to z z just because of version 1 ok. So that will tell you the G2 is well defined ok, so let me write this connect connect z0 z1 to z0 to z1 by a oath lamda U then gamma z is homotopy to lamda neta by lamda neta z I mean lamda followed by neta z continuation of 2 path to give a new path ok.

Also the homotopy also be any ah halos a homotopy can be found from gamma z to lamda inter z via such that each intermediate path is in V2 along which analytic continuation of f exists. So you see I can find I want homotopy gamma power by neta z and f, so I can intermediate pass like this and all the intermediate paths are lying in V2 ok, why why I am getting this homotopy because it was in eta ok.

Since so here is where V2 is simply used, so this is where I am define those of V2, any 2 paths starting at same point and ending at the same point of homotopy one and another for points and paths lying in simply connected domain ok. So because simply connected to V2 these 2 are homotopy and the homotopy can be chosen such a way that every intermediate path is also lying inside V2.

But what is the property of V2 any along any point for all points in V2 you can always you have indirect analytic continuation. So along all the intermediate paths of this homotopy ok, also you will have indirect analytic continuation. So there is no obstruction to analytic continuation of f along each of these paths. So you know all the continuation of version monodromy version 1 are satisfied.

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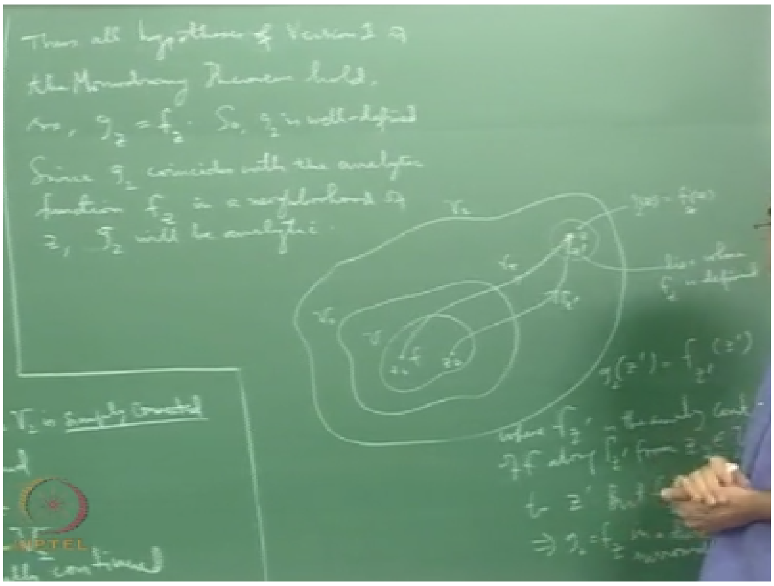

And therefore what this will tell you that fz=gz, so let me write that down let me draw line like this thus all condition all hypotheses of version 1 of the monodromy theorem, so gz will be equal to fz. These two functions of the c ok, so what this tells you is G2 is well defined. So I have managed to define a function g ok, so let me again emphasize you give me a point in V2 and I am going to define function I am going to define function I am going to define a value at that point.

And what is the value I define I will do the following things, since you are at pointing V2 I can choose the point of V1 and if you get into the point of U and a path along starting from that point and continuing and I continue f analytically I can continue f analytically along the path and I will get a new function at Z. I take that function taking its value at z, I call the new function of f sub z I take its value at z.

That is how I am defining g2, the only problem is that I could have gotten analytic continuation along some other path starting from some other point in U and it because a simply connectedness and of V2 and it is because of the assumption first monodromy theorem the first version of monodromy theorem and that I am able to show the G2 is well defined ok.

Now g2 is well defined g2 analytic is very easy because G2 in the neighbourhood of point z ok g2 is simply analytic function fz, fz is of course the analytic function ok at z, so g2 is just f in the neighbourhood of z ok. So g2 is locally analytic and personality because I have this it is a local property to check that functionality got just check that at every point is analytic ok. So since G2 coincide with analytic function locally is analytic ok.

So let me write that since g2 coincides with the analytic function fz in neighbourhood of z g2 will be analytic ok. So what you understand is probably this also requires a little bit of a little bit of more thought maybe what you must understand is probably this also requires little bit of more path namely what you was understand is you know so let me draw one more diagram.

So you know you really understand what is going on, so you know this is so this is my U then I have this V1, on V1 I have this f which is defined in extending to the maximum extension G1 and then I have V2 which is assume to be simply connected, you see I take the point z at this point I define g2 to be fz eta of z to be f of fz of z. This is my definition and how do I get that fz I start to the point z0 here.

I chose a path I mean gamma sub z along which f can be analytically continued and at this point when I reach this point z I get a new function analytic function locally there I call that f of z and take this values fz ok. Now what I want you to understand is well if you take a small if you take this neighbourhood where this f sub z of small Z is defined, that will be some disc surrounding the point z.

So suppose this is a disc where you know fz of z is defined alright suppose this is certainly fz is analytic function I just show it is going to live in it going to be analytic in this surroundings, now what I am telling you if you choose any other point z prime in this disc ok, then the G2 at z prime will be the same as fz at z ok, g2 of z prime will be the same at fz at z prime ok and why is that.

So and that is the claim g2 need to coincides with FZ in neighbourhood of z ok. So what is going to happen, so your situation is going to be that in I am going to have another point say z2 is the point of U ok and z prime is here alright and you know by definition z prime using this disk where fz is define, but the point is that z prime is in V2, so which means that there is a point of U z2.

And then there is a path like this from Z2 from along which f can be an analytic continue and you and you know this path can be called something let me you something so let me capital gamma, gamma z prime. So capital gama z prime si the path starting from a point z to a new along a continuous f and then I get as a function fz z prime whose value of gz prime is what I define z2 of z prime.

So g2 of z prime is just fz z prime at z prime, where f sub z where f z prime is d analytic continuation of f along capital gamma sub z prime from z to U to z, this is how I define, my claim is this fz prime is same as fz, in other words I am saying g2 is always the analytic function of fz is only one function that is decline and what is the what is the reason for that, the reason for that is very very simple see.

You see the reason is you know I do the following thing you see I connect q1 I am just showing where z with centre z prime some point if I can actually take radial line I take a line from z to z ok. Along z to z prime I can analytically continue FZ trivial ok and now if I take this point from z0 if I take this path starting from z0 along gamma of z and following by this line from z to z prime.

I will get a path from z0 to z prime ok and along that path you can analytically continued only get fz prime and after all I only get z if analytically continue at along gamma z I am going to I am going to add z when I ended up at Z I got fz ok, when I move along the radial line come to z prime and still keeping the same and simply trivial analytically extending the fz at z and fz at z prime.

Because the radial line is in fact the disc where fz is and wherever a function is well define and always trivial analytic, so from z0 I have another path to z prime along which my analytic continuation gives fz to prove, it will take 2 different paths starting from 2 different points of U and you analytically continue f and end up with particular point the function you get the same ok.

So the moral of the story is that by connecting z to z prime by a trivial path and taking trivial analytic continuation of f of z along that along that along that radial line ok and putting these things together you can see that fz prime is the same as FZ ok, fz prime=fz yeah the same ok and therefore so this implies that g2 is equal to fz in a neighbourhood of z in a disc surrounding z ok.

That is the statement I need here, since g2 coincides to the analytic function fz in neighbourhood of z g2 is analytic that is locally analytic at any point z g2 is analytic that is all. So if I produce a global function on V2 which is analytic and what is the function restricted to U that function is same that function is same. So what you have to do is that you have this global analytic function g2 yeah g2.

So we define on V2 and which is directly extending your analytic function f on U, so you prove that V2 also contain V1, it tells you V2, G2 is a direct analytic continuation of U,f so it means that V2 and V1 is suppose V2 domain of maximal direct analytic continuation, so V2 continue to V1. So we already told V1 contains V2, so you get V1 to V2 and we ger the external analytic continuation to be g2, but g2 what it has that V=0 ok, so so that means this is the proof that version 1=version 2.