Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-25

Maximal Domains of Direct and Indirect Analytic Continuation_SecondVersion of the Monodromy Theorem

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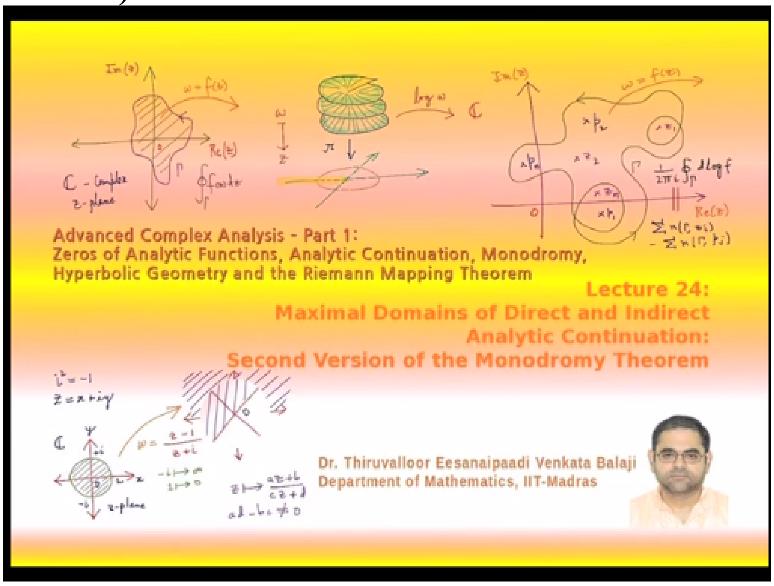

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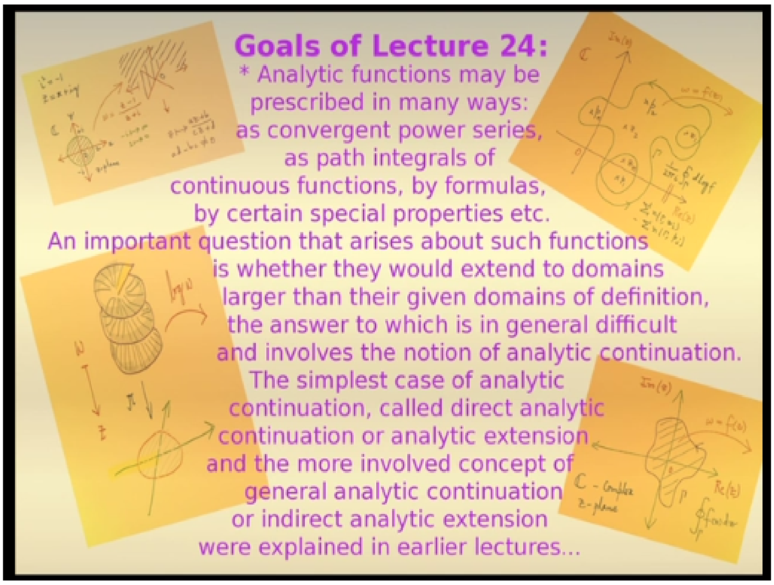

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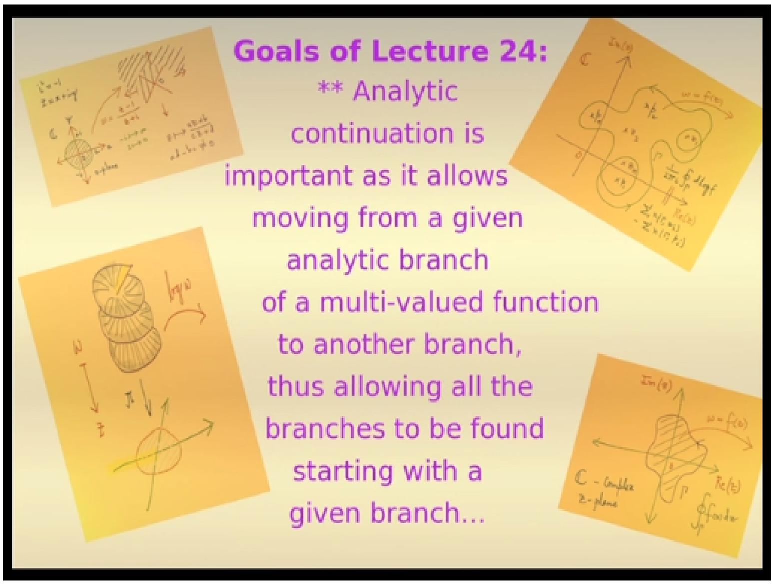

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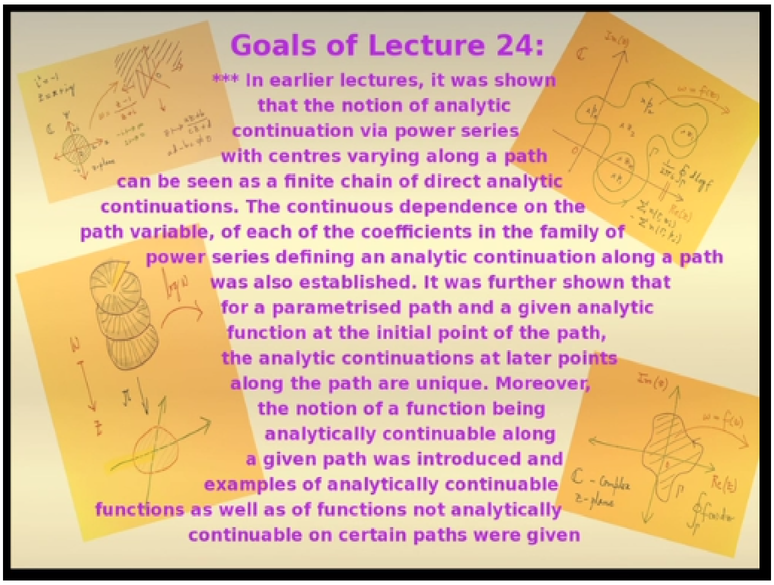

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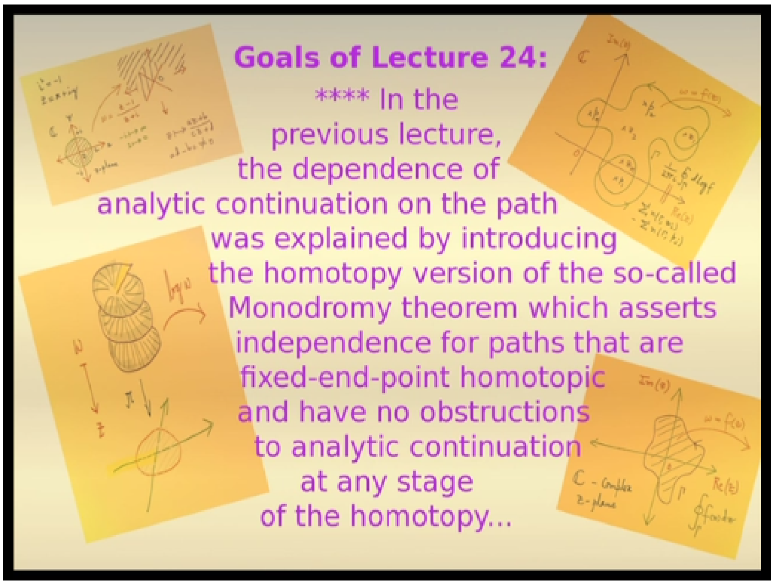

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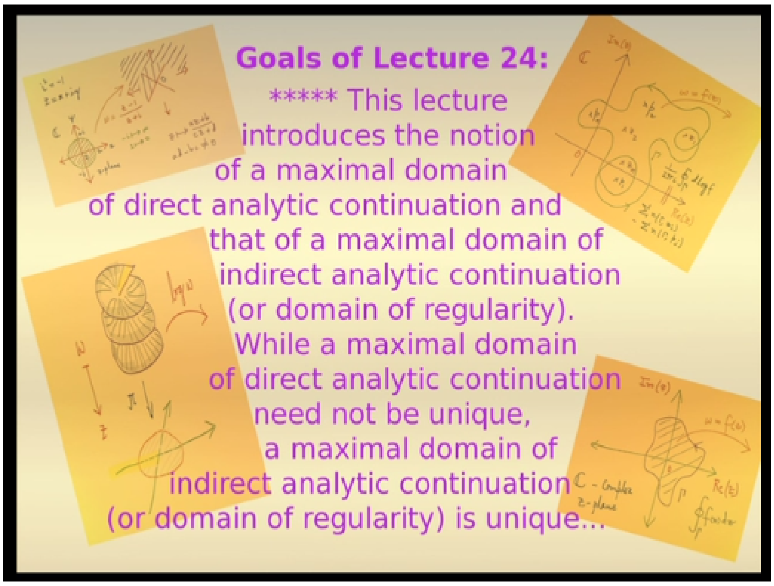

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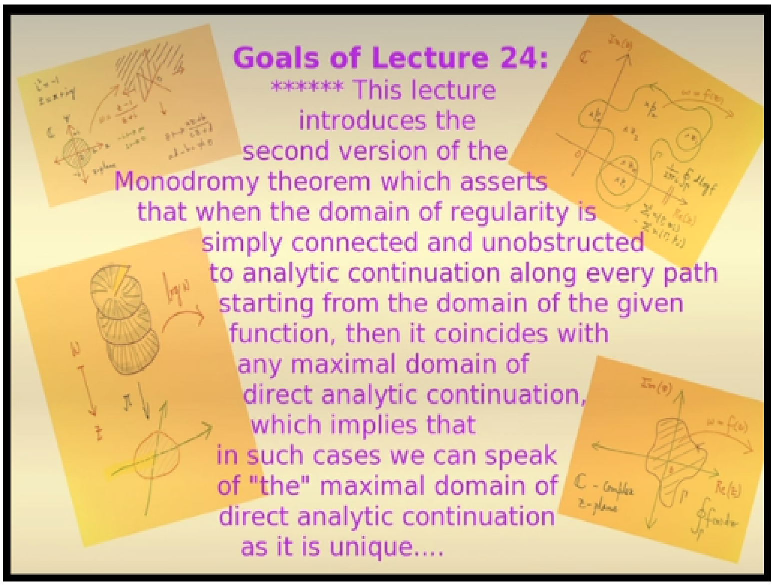

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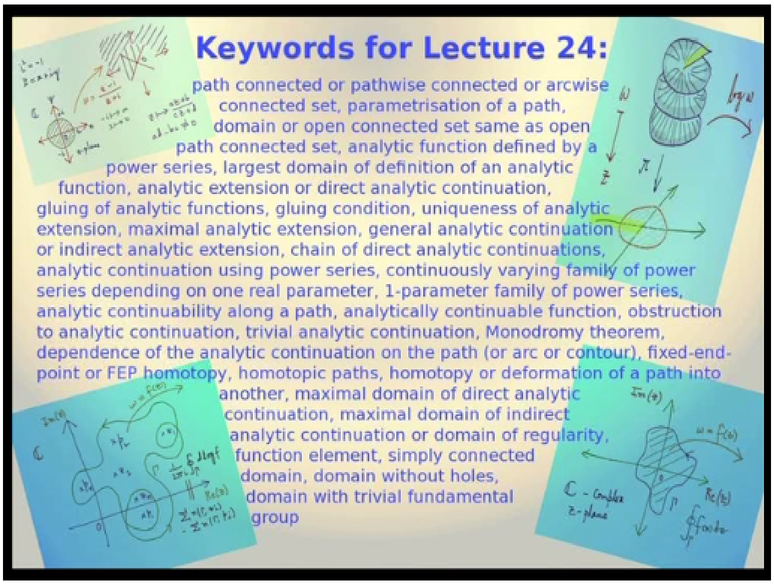

Okay so let us continue with the discussion of the monodromy theorem. So what we have is so here is a monodromy theorem.

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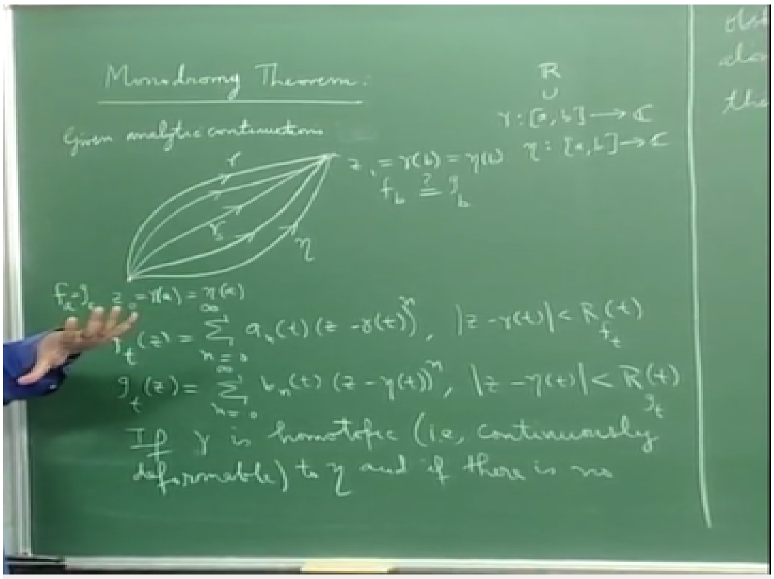

So the situation is like this, you are given two points Z0 and Z1 and you are given a path gamma and you are given a function, so this path gamma is defined on close intervally b and it takes values and it traces out geometric path okay, it is given by this function and defined in a close intervally fb0 and you are given a function fa which is analytic at this point okay.

This point by the way is gamma of a, the initial point and you can continue with analytically along this path through a function fb analytic at this point and z1 is gamma of b, okay and this analytically continuation on gamma is given by analytic continuation . So you know the analytic continuation are given by a specifying a family of one parameter family of power series. So you have ft of z is equal to sigma n equal to zero to infinity a n of t z-gamma t to the power of n.

We find in the disc of convergence of this power series which is given by modz-gamma of t is less than r of e . So I will put the subscript r sub ft okay r sub ft is the radius of convergence of this power series ft okay and of course we always assume all the radii of convergence are positive because they are all analytic functions when t equal to a you get fa the analytic function fa which is which is analytic at this point.

And when t is equal to fb you get fb which is analytic function at this point. So fb is an indirect analytic continuation of fa along this path okay and suppose you are given another path like this, which is say neta and suppose along this path also you have an analytic continuation of fa okay and . So I again start with fa which is the same as ga okay.

So gt is another analytic continuation it is another analytic continuation on another path, the other path is what is common between the other path and the first path is that both of them have the same starting point and ending point okay. So z0 is also neta of a and z1 is also neta of b. So you have another path neta defined on ab with values in c okay and the same starting point and the same ending point.

And I again start for this for the analytic continuation along this path and given an analytic continuation along this path of the same function fa, but it is now given by another one parameter family of power series which I denote by g sub the So this is sigma n equal to zero to infinity bn of t*z-eta of t to the power of n and this power series has disc of convergence z-neta of t is less than radius of convergence of gt as a function of t.

So you are given two analytic continuations like this ok and the question is I started with fa equal to ga and at the end I am getting fb if I go along if I analytically continue along this path I am getting gb the question is are these two equal and the monodromy theorem says and that it is they are equal under certain conditions. So what are the conditions, the conditions are first of all that this path gamma should be continuously deformable to this path eta okay.

So you should continuously be able to deform the path gamma to the path eta okay which in the language of topology or the language of homotopy is said as follows, they say that gamma has fixed end point homotophic to neta okay , it is gamma and neta has same end points and you can continuously deform gamma to neta okay. So that is the condition and the other condition is.

Of course that along any of these intermediate paths through which you are deforming there is no obstruction to analytic continuation of fa ok. So here let me write it down if if gamma is homotopic that is continuously deformable to neta and if there is no obstruction to analytic continuation of fa along any intermediate path gamma s aa then fb is the same as the gb. This is the monodromy theorem.

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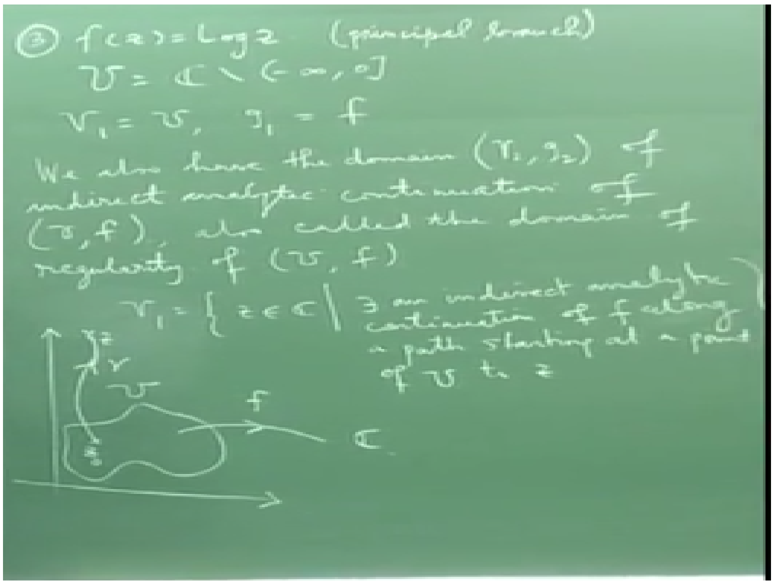

So so an intermediate path is something here okay it is something it is one of the paths that occur in the continuous family of paths which started gamma and ended neta ok. So this is the monodromy theorem okay. So the important thing the crucial thing is of course that along you at the starting point you have an analytic continuation, at the ending and at the starting path you have an analytic continuation.

And at the ending path also you have an analytic continuation and in the intermediate along each of the intermediate paths also you have an analytic continuations okay. Analytic continuations should exist. The function fa should be analytically continuable in any along any intermediate paths okay. We express that by saying that there is no obstruction to analytic continuation of this function along any intermediate path ok.

In other words, give me any intermediate path, the function fa can be analytically continued along that path. You make that hypothesis and then the monodromy theorem says that the final function that you get when you go to the end point you are going to get the same function okay. This is the monodromy theorem. Now so you know so how does, so let me expand little bit more on the statement .

I have to tell you what I mean by gamma is homotopic to neta and there is no obstruction to analytic continuation of fa along any intermediate path. So this is a definition this involves a definition of homotopic which you might have seen in a course as a bit of policy but nevertheless it is very easy to understand. So you see the idea is like this. So what is happening is that you know you have you have this on or to okay.

And you have you have the axis and you have this interval it need not be of the real line it could be some ab and it could be some cd , so ab is a close interval on the real line where all the paths are defined okay and cd is the parameter is a parameter s. So you know this is the parameter t and this is the parameter s. So what you get is you get a square like this or a rectangle like this.

We will get something like this okay. So the x coordinates where from x equal to a to x equal to b which I am calling as the So if t coordinates vary from ta equal to a to t equal to b and the y coordinates which in this case which I am labeling as variable s. The s coordinates vary from c to d ok and what is happening is that you have f is a continuous function from this into the complex plane okay and what is this f doing it is doing the following thing you see , you know when s is equal to c and t varies from a to d alright, you are getting the path gamma sub c which is the same as gamma okay.

You start with so the diagram is like this, so you have gamma sub c which is gamma, it starts at z0 ends at z1 okay and then and you know as you as t moves along this line the path that trace this gamma which is gamma sub c okay and then if you take any value of s in between ok any value of the s with the coordinate in between, then what you get is an intermediate path, gamma sub s okay which is what I wrote there.

And when s becomes t is equal to d and t varies from a to b you get the path gamma sub d which is neta. So you get this which is gamma sub d which is the path okay. So this function f is a continuous function it is a continuous function of two variables two real variables. So we write f is as a f, d ok and we write gamma s to be a path of f of s gamma s of t we call gamma s of t to be f of s , t for fixed s is also a path which starts at z0 and ends at z1 okay.

And so what you are seeing is you know see, try to imaginate like this if you have a square here or if you have a rectangle like this if you take continuous image of this rectangle on the complex plane you will get something like this, you should put something like this with you know if we make continues need of complex plane I should get these 4 ends to corresponds to these 4 ends of it.

This started rectangle alright and the image of this line segment well be this path the image of the line segment with that path, the image of that line segment will be a path like this and the image of this line segment will be a path because image of all these line segments are continues need of interval. So they are going to pass, so you are able to get something like this, but then you know if I put the condition that at the point t=Armstrong

All the all these needs of the ah each of the path gamma is the all started the same point and so then it will, it will amount to actually you know for all these interval point ok and at the point t=b, if I uses that all the points end at the same point all this will get collapse. So you know all this gets collapse to z0 and all these points from this arc on this path, they all collapse to z1.

And then you know if you collapse should have said this is the resulting diagram, this is what is happening and this is an intermediate part here ok, so if you collapse this diagram this is what you get ok and that is and we say that capital f is a fixed end point come out from gamma=gamma c to gamma neta= t, so this is what, this is what I mean by saying that gamma is homotopic that is continuously deformable think ok.

So this is this explains this statement in a very precise be alright, so that is why that is one thing and the second I should tell you that there is no obstruction talent along any intermediate path. So you I know that fa can be analytic continue to along the path gamma, which is the gamma c and I know that fa can also be analytically continued along the path neta which is gamma circle.

But I need also that fa can be analytical continued along e intermediate part ok I need that condition that path that is the part of the hypothesis and then monodromy theorem says that in fact that any of these intermediate part you continue fa the final function are going to get we have to be the same. So it has to be fb which is what it was for the first ok. That is monodromy theorem alright.

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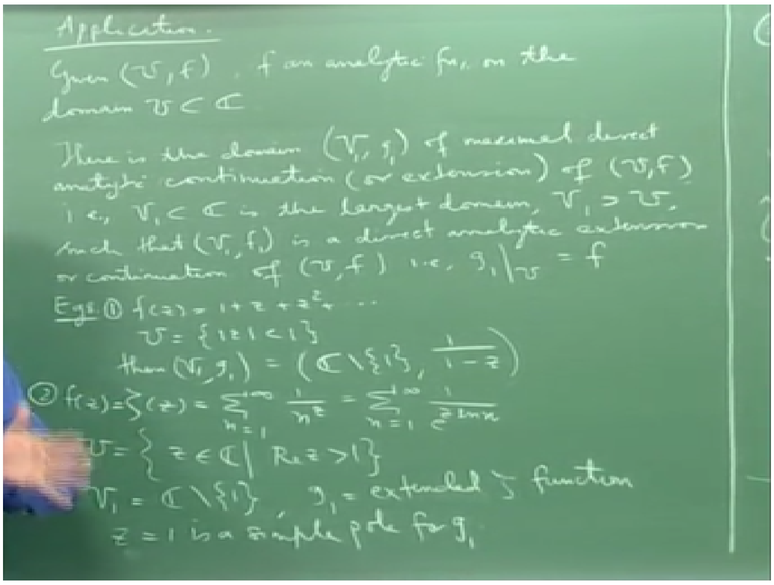

Now the point is postpone this game but let us try to see how why it is important, why you choose, so you know I have told you that are given so applications one of the application is the following see suppose you are given you, f which consists of a f f an analytic function on the domain you suppose you given a pair like this. We have define two types of domain connected with this we define two.

We have define 2 and we have defined and we have done that earlier there are two types of domain sets define with respect u,f, one domain is called the domain of maximal and the called the domain of maximal analytic continuation of ok. So there is the domain v,g of maximal v1,g maximal analytic maximal direct and analytic extension or extension of U, f.

There is there is there is the domain of maximal analytic I mean in other words what this means is that D1 is the largest open set in the complex plane which is the largest domain in the complex plane which contains U ok you too which extends to an analytic function on all of need and find I am calling at a given ok. So let me recall that is V1 Uc is the largest domain V1 containing You such that V1,f1 is an direct analytic extension extension or continuation of U,f which means this is just trying to say that you know if you take g1 is continuing you will get F ok.

So this is the largest this is the domain of maximal analytic continuation ok. This a largest domain to which you can continue analytics function ok. So you know you seen you seen example I will just recall if you take F of Z to be 1+Z+z squared and so on geometric series and you to be the domain mode Z less than 1 unit disc then we have seen that V1, G1 is simply is simply the pair that consists of the whole complex plane -2.1 and the function is 1/1-z/

This is there the domain of maximal analytic continuation is complex plane puncture at z=1 and corresponding analytic function 1/1-z ok and of course this analytic function cannot be continued to z=1 because z=1 has a it has a singlet it is a simple port, it cannot remove that you cannot continue it is because if you continue if you are give you can can you read means that that is why it is removed at at least local not removing act and take the second example is that is a zeta function zeta of Z=n=1 to infinity 1/n power z.

So this is defined to be sigma n=1-infinity 1/e power z along n and this is the Riemann junction and you know the domain on which have to be zeta and I am taking U to the right half plane to the right of the vertical line real part of Z=1 ok. So that the fall complex plane real part greater than 1. So this is zeta function you all know that it represent analytic function in right half plane we proved that earlier.

And then I told you that it was his theorem, this theorem is not trivial it is ok but in this case V1 is actually again like the geometric series the domain of maximal and configuration is just the complex plane puncture at the point one that means this zeta function extension the whole complex plane except the point 1 and what happened and of course be extended function hh is called g1.

So G1=extension or it is called extended zeta function or extended Riemann zeta function which is the extension of this function you actually from the right half plane to the whole complex plane but extended to. Z equal to one where Z equal to one you cannot be extended because it can be a simple, so Z1 Z=1 is a simple pole for g1, ok this is example and then there is of course there is another important example.

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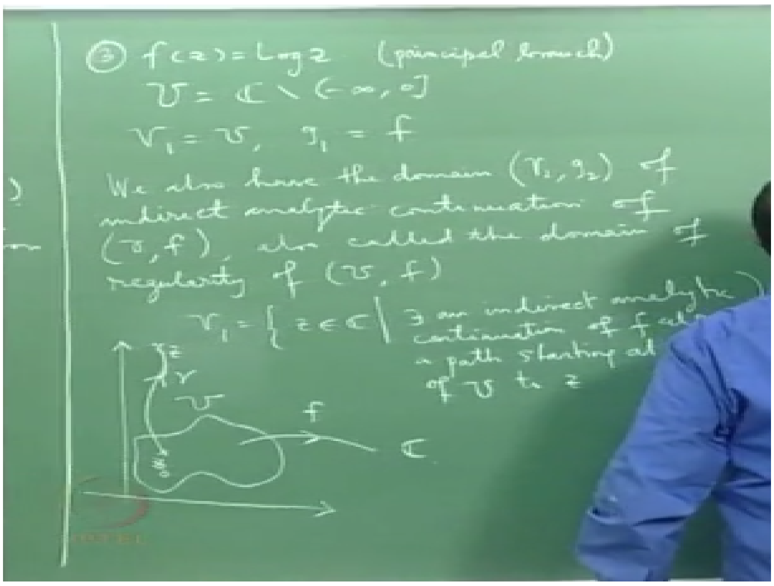

You take f of z to the principle logarithm of z principal branch and that in this case you take that the open set to be the slit plane. So this is just complex plane-you remove that negative areas including arc ok and you know the principal log analytics here alright and in this case what happens is that the if you take the maximum if you can if you take the domain of maximal direct analytic continuation it will simply be the same.

In this case V1 will be equal to U and G1 will be same, so in this case you simply cannot directly analytically continue it across the you have made by deleting the negative real axis ok. So this is the this is the situation with respect to this is the 3 examples in giving respect to the domain of maximal analytic continuation ok, but the domain of maximum director analytic continuation is called the domain of maximal analytics of function ok.

Then then what you have defined, you have defined analytic the given function analytic functions also define that domain is called the domain of regularity si call the domain of indirect analytic function ok. So we also have a domain V2, G2 of indirect analytic condition of U, f ok b also called the domain of regularity of regularity of U,f.

You also have this, now what is this domain, see this is the domain which consists of all those points to which the original function can be indirectly analytically continued along a path ok. So so ah now what is the definition V2, g2 is so v2, so whenever I say the domain and also included whole domain and the analytic function, so that is why pair like this called a functions element with mainly consist of domain and analytic function define on the domain holomorphic define on the domain is called a it is pair called a function element ok.

So so what is the domain of regularity V1 is a set of all z domain to c set of there exit and indirect analytic continuation of a f from a path along a path starting at the point of U to z. So drawing something like this you have so you have this is my complex plane and here is my domain U and here is my analytic function F with valence C and what am I doing collecting all those point z with the property that.

You know whenever there is there is a point Z0 in the domain and there is a path there is a path gamma starting at this point z0 in the domain U and such that along the path you can analytically continue indirectly continue as to get U f analytic function points and you put together all these points ok, you put together all such that ok, so you see you put you put together all the points.

And the result is the result will be an open connection because a point to see if you can if you after all you can analytic continuity along this point then you know I am waiting I am a function at that point. So for every other point in that small disc I can extent that analytic function itself extends along a smaller radial path ok. So if I can extend from z0 to z analytically indirectly analytically f ok.

Then I can do so for every point in its small disc surrounding z ok and mind you an analytic function on a domain is of course analytically extendable along a path trivially on the domain. So the moment I say that I can extend analytically f to obtain a new analytic function here at this point it means that it is an analytic in a small disc surrounding that point and it means that for every point in that disc it is analytically continuable for a path starting from the centre of the disk to any other point.

So the moral of the story is the set of all set z is open ok and it is by definition path connected okay, therefore it is domain, so this V1 is a domain by definition ok and what is happening you are getting a new you are getting a new set, but the only problem is that in this case at various z you will get various analytic functions ok.

You started out with analytic function f on the domain U but then if you got different points you do not know whether the final functions z you get weather they are indirect analytic continuation or whether they are direct analytic continuation you do not know ok and so the so the fact is that so you can look at these three examples with respect to example 1.

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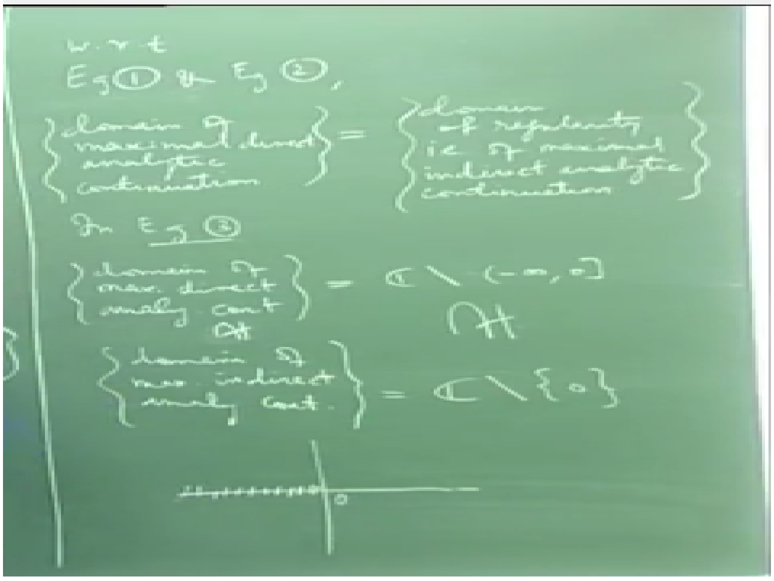

You see that if you take a domain of indirect analytic continuation that is the domain of regularity of of U,f you will stay get the same domain because you cannot continue it to the point z=1 because it is a simple projects, so the only point that left out, the same thing will happen to example 2 ok. So in both examples 1 and 2 the domain of direct analytic continuation the domain maximal domain analytic is the same as the domain of regularity for 1 and 2.

Because the only the point that is left out is the point z=1 but z=1 you cannot even indirectly analytical continue the function because of the fact that any indirect analytic continuation will amount to direct analytic continuation because every point every hole any deleted neighbourhood about one is already the function is already analytic there ok.

So any indirect analytic continuation will amount to actually a direct analytic continuation you cannot have a direct analytically continuation you cannot continue this two functions directly to the point z=1 because z=1 is a pole ok. So in example 1 and examples 2 domain so you have domain of a domain of maximal analytic continuation is the same as the domain of regularity that is of maximal indirect analytics continuation.

So this is what happens in these 2 examples, but something striking happens in this case. So in this case what happens is that the domain of maximal directory continuation so of course here I should say maximal direct analytic continuation will be strictly slightly smaller than the domain of maximal indirect analytic continuation. So in example 3 maybe if you take fundamental branch of the logarithm.

What will happen is that the domain of maximum direct analytic continuation will be of course the slit length which is the complex plane-say negative real axis along with their origin removed and this will be properly contain in what is going to be the domain of maximal indirect analytic continuation it will be just a function ok. So the whole negative real axis except the point zero whole negatively real axis will also come.

So it will be this is properly contain c-0 punctured plane which will be the domain or maximal indirect analytic continuation, so this is properly so this properly contains ok in other words if you take the principal branch of the logarithm what is happening is that you know it is analytic to make it analytic I have to throughout the negative real axis and the origin ok.

But if I want to analytically continue it indirectly along path I can always analytically continue it across any point on the negative real axis so long as the path does not go through the origin ok and therefore the domain of maximal regularity ok the domain of regulatory domain of maximal indirect analytic continuation will also included the whole negative real axis ok and therefore it will be just c-0.

Of course 0 can never be remedy because at the point 0 a logarithm is not defined, logarithm is not defined ok, so you see there is a big difference so you know all these examples so there are 2 definitions, 1 definition of maximal direct analytic continuation there is another definition of maximal indirect analytic continuation is regularity and questions how are these two related ok.

And we have seen that this could be because of that ok. Now so what the monodromy theorem says, the monodromy theorem says the following things. So you can ask the question when are these two, when can you say these 2 are equal ok. So the monodromy theorem in another version ok which is actually equivalent to this version says that suppose your domain of maximal indirect analytic continuation is simply connected ok.

Suppose your domain of maximal indirect analytic continuation is simply connected then it is the same as the domain of maximal direct analytic continuation ok, that is the reformulation of the monodromy theorem. So monodromy theorem says that if you start with a function element namely a pair consisting of a domain and holomorphic function on the domain analytic function on the domain.

Then and if the domain of regularity of that element of that function maybe the domain of maximal indirect analytic continuation of that function, if that is simply connected then that has to coincide with the domain of the maximal director continuation. In other words both these domains will be equal and on the domain of maximal indirect analytic continuation the domain of maximal indirect analytic continuation will actually become a domain of maximal direct analytic continuation which means that the function will extend to single value function on the whole domain of regularity ok.

Whereas you know you do not expect it here in this case for example log cannot you cannot extend log to single valued analytic function on the punctured disk you cannot get a single valued analytic function of the logarithm on the punctured disc ok. That is a if u want that is a basic exercise is a simple explain the first course in complex analysis right you can never find a single valued analytic branch of the logarithm function ok.

So so the moral story is that in this case the domain of regularity is bigger than the domain of maximal direct analytic continuation and the problem is this domain of regularity is a puncture plane is not simply connected ok. So the problem is that you are not able to get a single valued function, you are not able to get a single direct analytic continuation to the function to this domain of regularity.

The reason is that this domain of regularity is not simply connected it has a has a pole ok and the monodromy theorem says is that whenever you have a domain of relativity simply such a thing cannot have ok. So let me so you know let me let me state that version with theorem that we keep it as it is so and let me state that version of theorem and trying to convince you why the monodromy theorem implies that.

So this is the very important questions o f the monodromy theorem answer it says that whenever you are in a situation where you can you know that the domain of regularity is simply connected then you can for sure say that the domain of regularity is the same as the domain of directional analytic continuation ok. That is what the monodromy theorem says.

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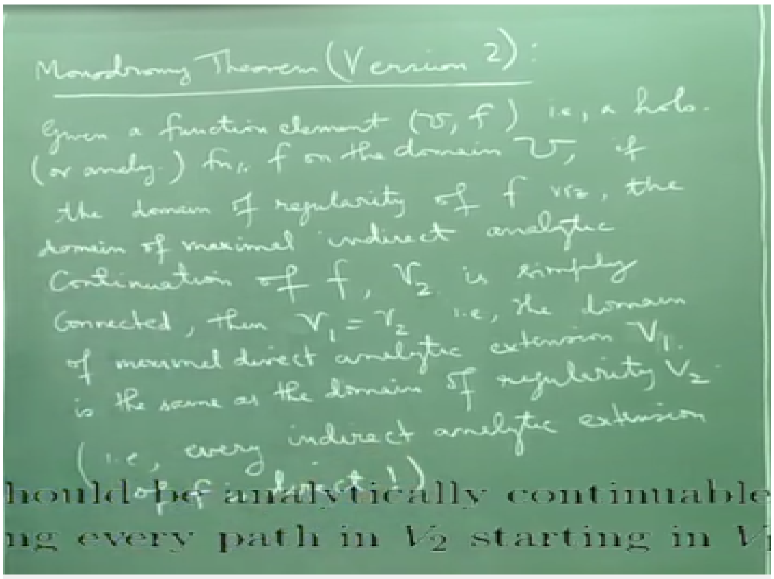

So here is monodromy theorem version 2 so you know I therefore let me call this version 1 is monodromy theorem version 1 and this monodromy theorem 2, so what does it says given a function element U,f which means that is a holomorphic or analytic function f on the domain U ok. If the domain of regularity of f namely the domain of maximal indirect continuation of f V1 is simply connected.

Then it is equal then V1 so v2 what I view for let me use V2, so if we read this at the moment given a function element U,f namely the holomorphic or analytic function f on the domain U if the domain U domain of reality of f mainly the domain of maximal indirect analytic continuation of f we which is which we call V2 is simply connected, then then V1=V2 that is the the domain of maximal analytic extension maximal direct analytic extension is the same V2, V1 is the same as the domain of regularity V2.

In other words what we are saying is that wherever you can analytically extend f indirectly no matter your extension may be indirect analytic extension, so what it says is every indirect analytic extension of F has to be a direct analytic extension ok. So this is also same that is every indirect analytic extension of f is that is what, you this is the conclusion you can get you can come to if you are seeing the domain of a log simply connected ok.