Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-24
Analytic Continuability along Paths_ Dependence on the Initial Function and on the Path - First Version of the Monodromy Theorem

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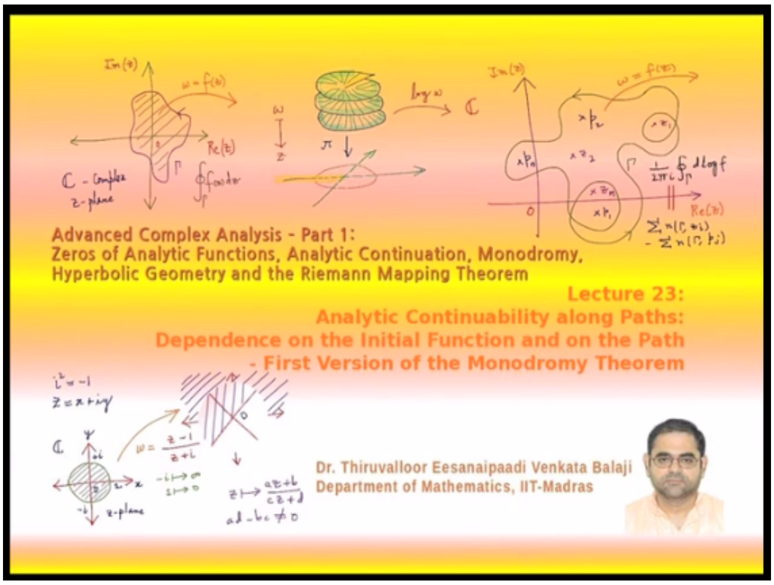

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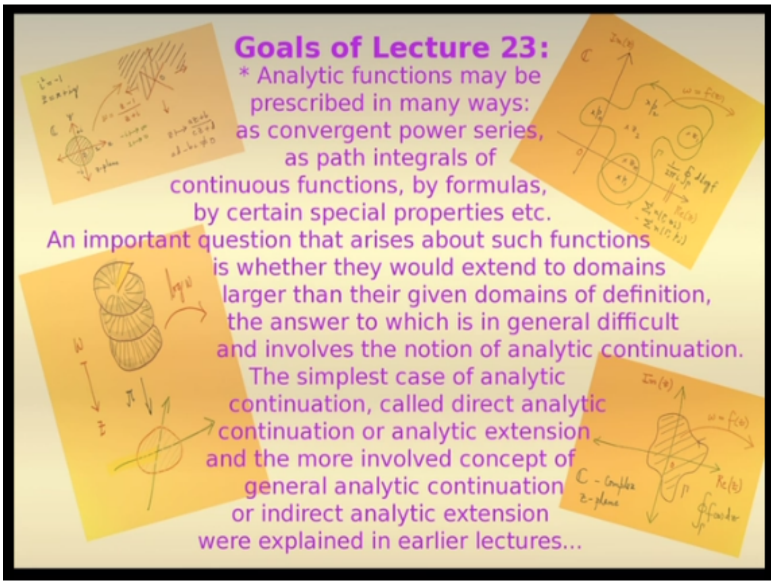

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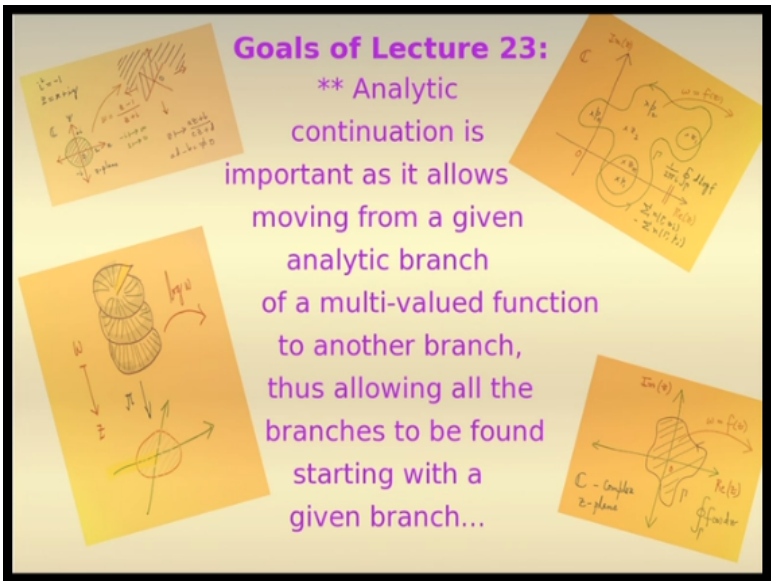

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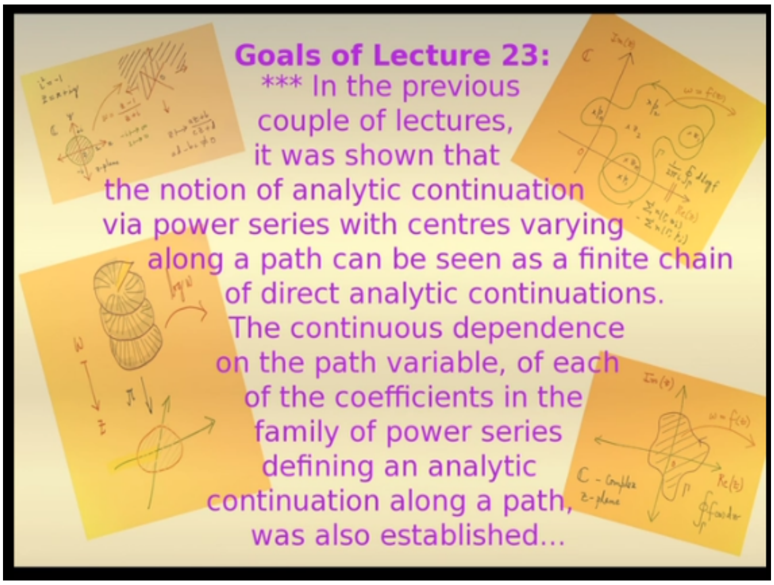

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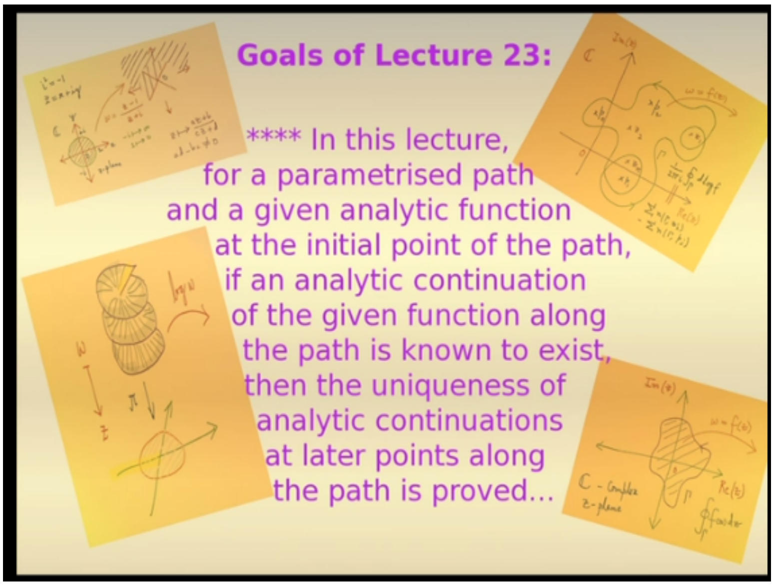

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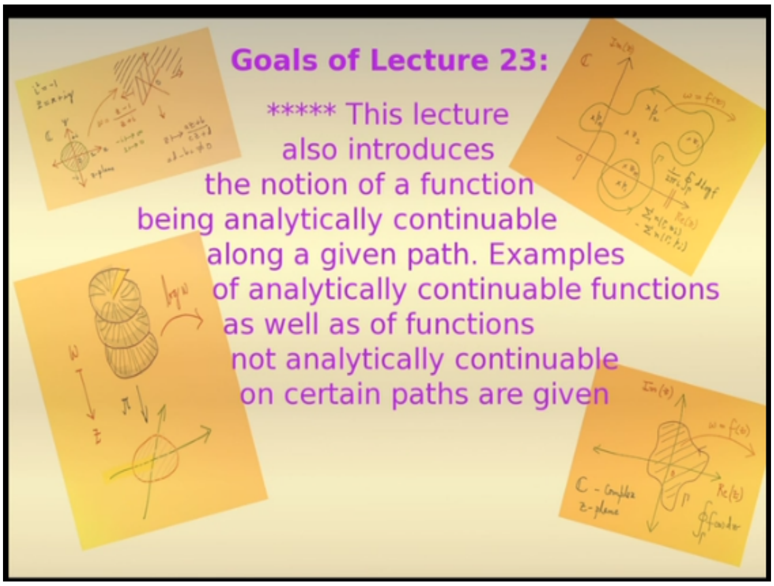

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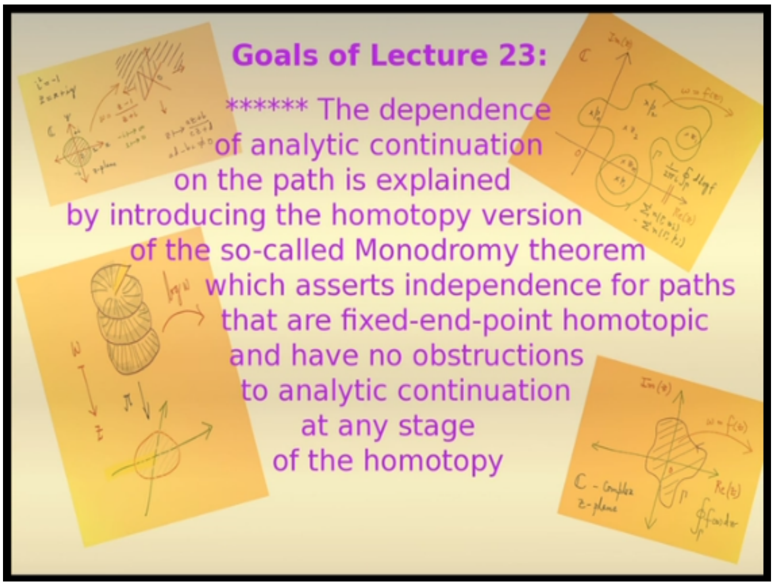

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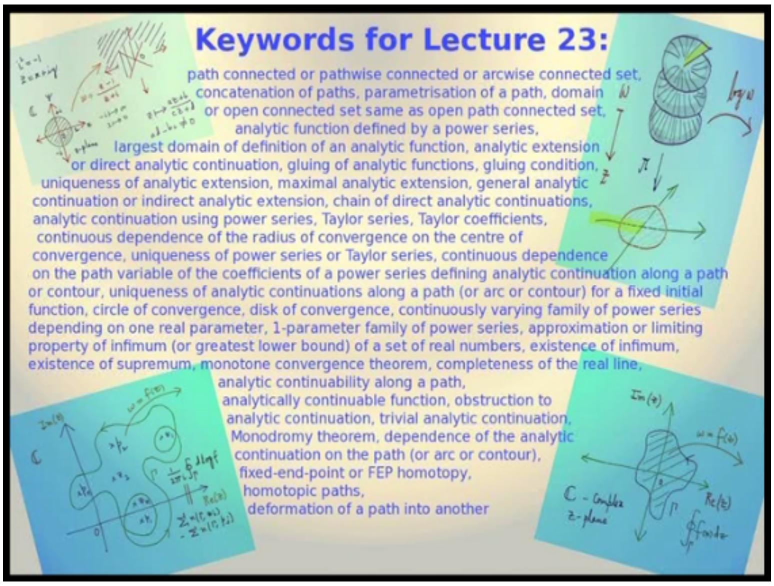

Ok so this is a continuation of the previous lecture, see what is on lecture was that if you have an analytic continuation along an arc or a path.

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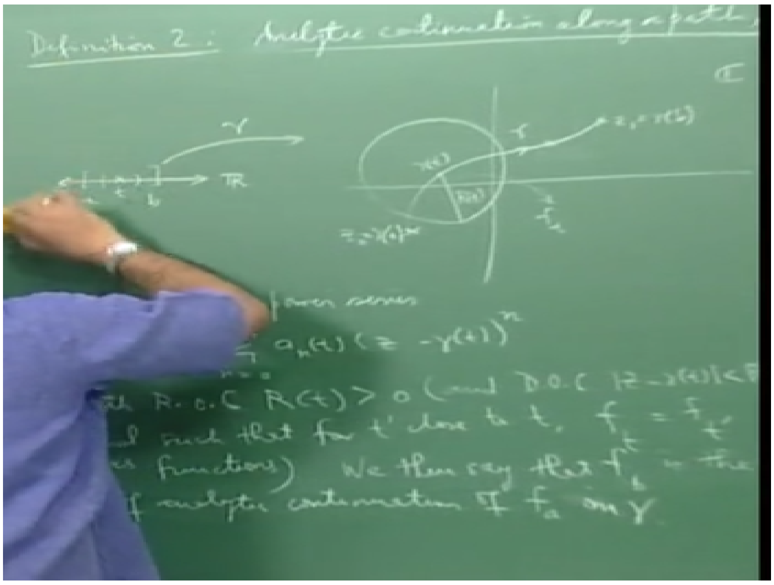

Then which is given by a family of power series also parameterized by the path variable ok. Then the coefficient an as well as the radius of convergence R contains function for the path parameter ok. Now what I am going to do now is going to tell that because of this I am going to tell that the whole analytic continuation all the function ft greater than a, they all completely determined by fa namely the initial function.

So in other words I am saying if you have a path for which affect the parameterization and if you have if you have an analytic continuation of a function along this path then the analytic continuations the function, the analytic function at the starting point the path, it remains all the other analytic function that correspond power series later on in the path ok.

It is a so what it says if you give me a particular parameters in our path and start with an analytic function at a initial point then all the analytic function that we will get at various point by analytical continuation. They are all uniquely determined ok. In other words for the same parameterization of a path you cannot find two different analytic continuation for which the power series at t are different with the initial power series being set ok.

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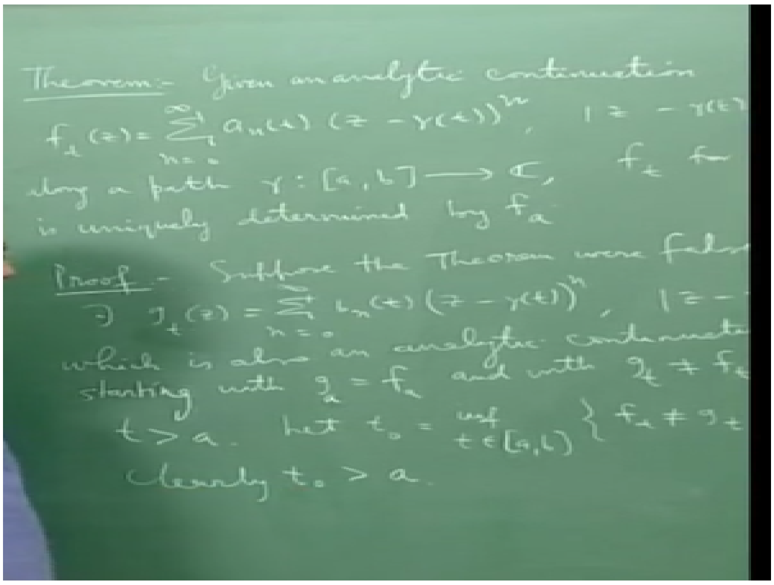

So let me write that down so here is a theorem given analytic continuation ft of z sigmam n=0 to infinity an of t z-gamma t to the power of n md z-this power series valid in this disk of convergence it is not that –zgamma of t centre at gamma t less than t along a path gamma , so the segment is ft for t greater than a is unique determined by ga.

So the starting to the power series t=a namely the initial point of the path, the value of the parameter t when is equal to a it is initial value the parameter and the corresponding power series fa is initial power series starting and I am saying that that initial power series that analytic function that determines all the other all the analytic function get in terms of power series along as you continue analytically around the path ok.

So then I am saying if you give me a path parameterization and if I start with an analytic function at the initial point ok do any indirect analytic continuation on the path, then at the end of the path I am going to get one in the same analytic function, not only at the end if I am going to get even another point of the path the analytic continuation of that function I'm going to get at that point, it is going to be unique.

So analytic continuation completely controlled by the initial function and the path that is that is what I am trying to say ok, so what is the proof the proof essentially actually you know if you think about it it's just tautological to the lemma that a and R(t) are continuation functions of t ok, but let me explain that so you know you need to proof by contradiction alright.

So suppose the when the theorem that falls that falls then it means that you start with fa fixed you could have that could be a certain t for which you could get different you could get different family is a power series ok with both start with the same fa, the same function fa but it later at some point for some value t parameter greater than a the corresponding power series for different analytics ok.

It is that is what it means the contradict to this theorem ok, if I write it there exist gt of z=sigma n=0 to infinity bnf t z-gamma t to the power of n modz-modft, so you know I will use r till of t will be the radius of convergence of gt of z of the power series of ft z centre at gamma t ok, so I am using a different r ok and such that which is which is also and an analytic continuation along gamma starting with ga=fa.

And with gt0-ft for some t greater than a ok, so the claim is that the claim is that you give me one analytic continuation like this on the path, then that is the only thing you can get. So long as you fixed the initial function fa ok and how do I oppose this claim, I oppose this claim in based on the another analytic continuation ok, with the property that it also begins at a to the same function of ga is fa.

So this power series at equal to a starting point is the same as this power series as function they are same ok, as a starting point, but for a certain value of t greater than a the corresponding function I am getting at different term ok, that is that is how I am contradicting the statement with theorem, now I get a contradiction, now it is a now contradiction very evident because you see .

You see you know the contradiction will just come in a very easy way let e0 infinity of t in a in a, b such that ft is not f yeah ft is not equal to g ok, so there is one t beyond a for which ft and gt are different functions ok. So you can look at the smallest such you can take the infinity you can leave ft then of course you know fa-ga therefore t0 is greater than a clearly t0 is greater than a, no doubt about it.

T0 cannot be a because fa is equal to 0 and right when you take infimum of all these ts ok, you know infimum is also it has also a limiting property, the infimum of set of real numbers which is minded below is a limit of the set of of a suitable sequence in that set ok. So there is a whenever you study infimum and supremum in a first course in analysis you always have the so-called approximation property of the infimum which says that the infimum of supremum is actually a limit ok.

So this limiting copy, so this infimum is certain limit of all ts in the certain property and it exist because all these numbers all these ts are bounded, so this is you know of you recall the first course in analysis fa analysis you would come across this statement that you know at every subset of real numbers which is bounded below has an infimum is equals to same at every subset of real numbers is mounted above has a supremum.

That equal to the completeness of real life ok, it is a very deep set, so it is also equivalent to the mountain convergence theorem that any sequence of real numbers that decreasing and bounded below converges are equally any sequence of real numbers that is increasing and bounded above also converges ok all these statements are equivalent to the completeness of the real line it is a deep fact ok.

So the infimum exist because this all this is a bounded below by a ok and infimum has the infimum if you call it as t0 it has a limiting property, so it is a limit of a sequence and since the close interval a, b is close the infimum also belongs to that infimum ok, and the infimum cannot be a it has to be greater than a right. Now I will get easily contradiction for anything close to t0.

And I know lesser than t0 of if you compare see because I have to use the fact that you know ft is a ft is a ah analytic continuation therefore you know for t prime for unity for t prime sufficient close to t, ft prime has to be equal to ft similarly for al t prime close to t ct prime has also to be equal to ct. So what you get is for all t prime very close to t0 and lesser than t0 ok, you will get ft prime=gt at ft prime is also .

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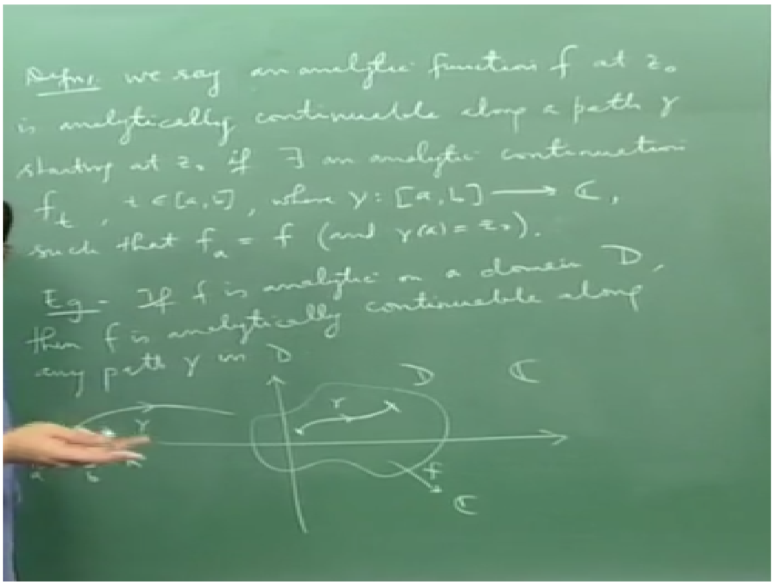

So you will get ft prime=ft0 and you also get ft prime=gt and will it will conflict the fact that ft0 is not equal to gt ok. So if I write it down we have ft prime is same as ft0-epsilon ok and you will have that equal to f that is=gt0-epsilon which is g or equal to gt prime ok, you have see you will have this because ft is a analytic continuation so for t prime close to t0-epsilon.

Ft prime has to be equal t0-episolon and but ft0-epsilon has to be gt0-epsilon because t0 is the smallest of those values for ft is not equal to gt. So any anything lesser so the infimum is called the greatest lower bound anything that is less than that is not robot ok. So t0-epsilon is lesser than the infimum, so it is not going to be a so this condition is not going to be true for t0-epsilon ft-epsilon should be the same as gt0-epsilon right because t0 is a least if you go to the f(t0) ft and gt giving a same right.

So these 2 are equal because the definition of t0 and these 2 are equal because gt is an analytic continuation ok. But also ft0-epsilon is the same as ft0 and gt0-epsilon is also equal to gt0 for epsilon small enough, these all true ok alright probably this is maybe even the previous statement is not important maybe this is the statement is important see if you have choose epsilon small enough.

Then ft0-epsilon should be the same as ft0 and gt0-epsilon should be equal to gt0 ok alright and so you will get, so this will tell you that ft0 is the same as gt0 ok and you see at the same argument will also instead of -epsilon you have put +epsilon also it works, see because here I can put instead of – I can put +/- ok, here also I can put +/-, that is because f is an ft is an analytic continuation.

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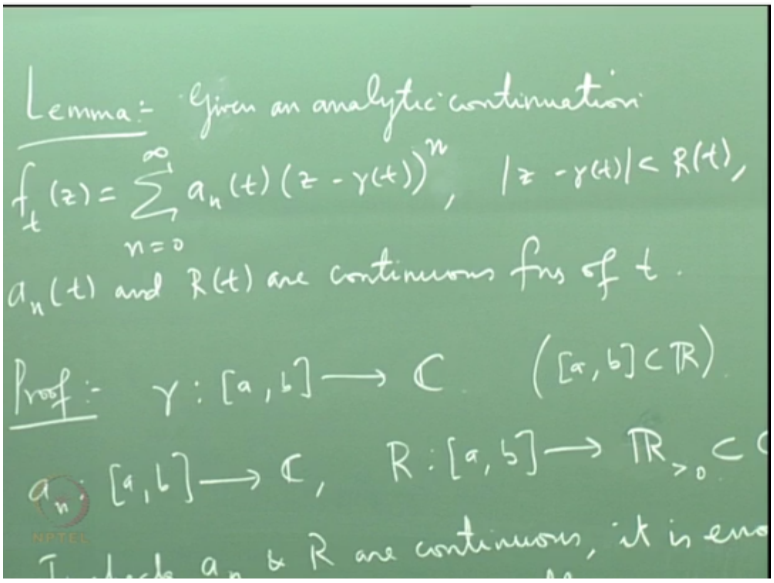

If you take t0 then you take a small neighbourhood around t0 ok, then al the ft prime should be the same as ft0 ok and similarly since gt is an analytic continuation it you take gt0 and you take a small neighbourhood surrounding t0, all the gt prime should be gt0 ok and so what this will tell you this f small, so I can put plus or minus and then you will of course if you take you get a contradiction.

If you compare these two alight and if you compare these two alright what you will get is that ft0+Epsilon and gt0+epsilon will be the same for epsilon small. So the index ft0+epsiloin=gt0+epsilon for epsilon small enough a contradiction to the definition of to the definition of t0. See I am just looking at the neighbourhood t0 ok, if I look at values before t0 then all the ft prime and gt prime are same aright.

And they should also represent the same function to the right of t0 also ok, but then to the right of t0 ft0 and ft the ft and gt suppose to be they are suppose to be points the right of t0 as close to t0 as I want where ft and gt are different because of the approximation property of being human and that is a contradiction ok. So let me repeat this if you look at the point if you look at the point t0 at the point t0 if you took the ft0, that ft0 has to be equal to the power series ft0 +/-epsilon.

So sufficiently small and gt0 is also similarly has to be equal g t0+/-epsilon for f0 sufficient is small, but if you take – if you look at t0-epsilon ok then the ft and in the gts are the same because the definition of ft. So what this will tell you is that fts and gts are also the same for values greater than t0, which is not for all values in a neighbourhood to the right of t0, but that is not suppose to be because given over small and neighbourhood to the right of t0.

There is a value t for which ft is not equal to gt, that is approximation property of t if ok. So so this contradiction which says it a contradiction uses some very basic analysis, so this contradiction tells you that your theorem 2 ok, so this is that is the proof, so moral of story is the following, the moral of the story is you know if you give me a parameter parametric path if you give me a path and your parameter is it.

And you started the functions at the starting point of the path then that then any analytic continuation along the path if it exists then it is unique determinant ok, see in particular so everything depends on the parameterization of the path and this initial function everything depends on that fine. So any way the moral story is that if you give me a parametric path and if you give me an initial function.

Then the then the there is only one analytic continuation which starts with that function, they cannot be more than 1 ok, but of course it may happen that there is no analytic continuation at all, there may be a path along which you cannot continue functionality that could happen. So what I do next is I want to tell you about 2 things the first one you could have a path on which along which you cannot have analytic continuation at all ok.

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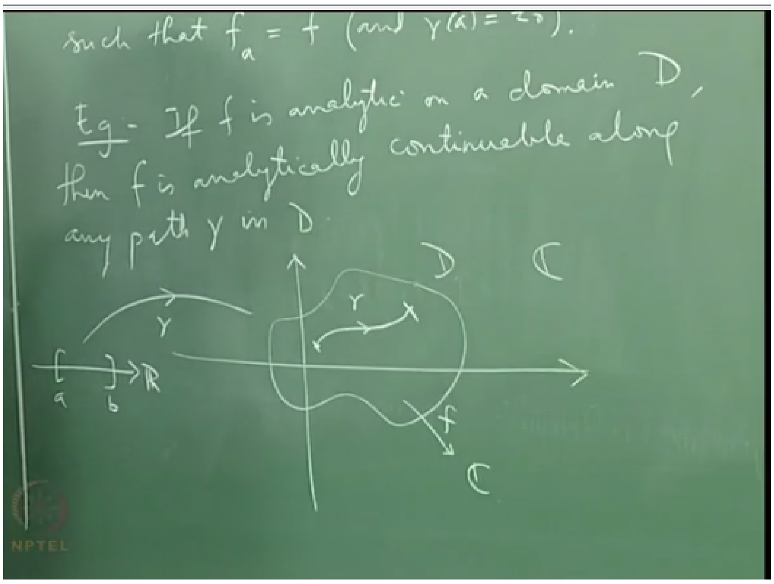

And it will happen for example you need a function which has a non removable similarity at a point ok. So you know for example so let me make this definition we say f analytic function f at z0 is analytically continuable along a path gamma starting at z0 if there exist an analytic continuation ft t belonging to a, b where gamma is path from a, b gamma is the path parameterized.

That the parameter variable in this interval, such that f a is f and of course and of course gamma of a is z0. So I am saying that if you have a point z0 and you have a path starting from z0 and your function with is analytic in a neighbourhood z0, if you we say definitely continue along the path if you can find analytic continuation given by a family of power series parameterized by the path parameters by the interval which also parameterized as path.

Such that the beginning function is the given function ok, so the fact is so the question is so our aim is to start the analytic function for the point try to look at all possible paths along which you can continue it and try to find out what is analytic function you are going to get at the end of ok. so that is the end the philosophy is that if you do this then you will get all possible branches of analytics ok.

So this is by so this analytic continuation will give you all possible branches of a given analytic continuation right. So so that can be analytic function cannot be continued along the path. So for example you know so here is an example if f is analytic on the domain D, then of course f is already continuable along any path in D. Then f is analytically continuable along any path in D.

So important Karma it's very simple right because you know if you have some you have some DJ dominant you know if you have a path super pages then you know if defined here India function within its domain of all district can be continued along any path in the domain, it is very simple right because you see so you know if you have some D here in the complex plain and it is a domain and you know if you have the path.

If you have path like this gamma then you know so f is define here the value is see, any analytic function is the thing it is a domain of analytic and the continued along any path in the domain here, it is very very simple what you do is we just define at each point of the path just define the power series to be the power series just Taylor series expansion of f at that point that is all ok.

So it is real that if the function of analytic in the domain then it can be analytically continued along every path in the domain and all these all these analytic continuation I am going to produce anything they going to give you back the function f ok. So you know if you have this situation you just define ft of z is equal to power series that is Taylor series of f centre at gamma n that is all.

If you make this definition then ft will give you a automatically to give you a analytic continuation of your following that path and what is that analytic continuation is a trivial analytic continuation, it is simply you are going to just get back the same function along the whole path ok. All the fts ok all the fts the power series will be different because the power series will change as the centre of the power series changes the power series which is the coefficient will change ok.

And finally what function to the converts to they all converts to the same function, so all the fts are as a function where the same f ok, but only thing is if you write them as power series you get different power series and the power you get different power series because you change the centres ok. So this is this statement that if a function analytic on the domain then and you have previously analytically continuable on any part of the domain ok.

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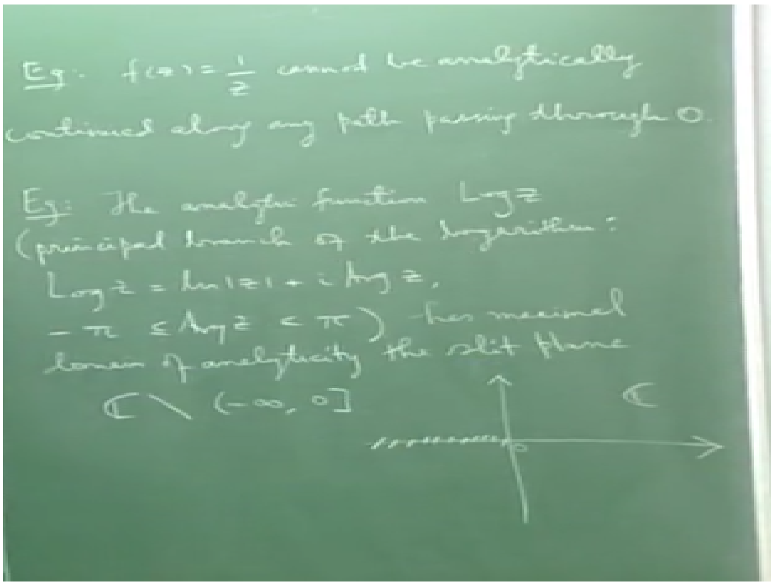

Then the other important example is of function which you cannot analytic and that is the case when you enter point which is a bad point for function just to a similar point of function so example you know if you take f(z)=1/z cannot be continued analytically continue alone any path passing through o that is because origin is there, you know simple hole ok. So you cannot analytically continue it along the path which causes origin.

Because at the origin you cannot find of tral the function 1/z is defined as domain of analyticity is a punctured play it is a whole complex plain may its origin, so the only point where it is not define is the origin where at that point it has a simple port alright, now at that point you cannot extend it ok. So so if you have any path passing through the origin you can automatically continue it.

So so you now this gives you a hint that you somehow if you keep track all possible pass along which your function can be analytically continue, then you will get points which are good points reasonably good points of function ok. So this leads to a definition, so let me make a statement, here is one more example the analytic function you know log z which is principal branch of the logarithm.

This is given by logz is lawn 1z items principal argument of Z and z is taken from –pie to this pie, this has maximum domain of analyticity the slit plane may be complex plane-you remove the line segment from minus infinity to zero ok, so it is the slit plane we just throw out is 0 margin and negative in access on the rest of it is analytics ok, that is analytic function and you cannot extend it cannot extend it as analytic function to any point of the axis ok.

Simply because at any point on the equatorial axis the argument function is discontentment the argument function is being is imaginary part of your log ok, so you cannot do anything, so you take this analytic function. Now the amazing thing is you know you take a path that does not cross the origin you know if it crosses this negative real axis, we can still analytically continue it ok.

So even though this whole negative real axis is full of bad point, it is points they are all non isolated similarities, we are all that is this whole continuous way, so the continues line full of similarities for the function ok. In spite of that you know if you draw an arc which crosses the negativity real axis you and then you can really analytically continue log and you know what will be you are going to continue log to the next branch which is actually the condition is happening on their famous episode logz which is being affected here ok.

So log z can be analytically continue on any arc which D-0 summation, you know that so you need the path crosses the analytic axis, even if the path crosses the negative real axis you can you can analytically continue it ok, so you are able to so here is an example of analytic function is can be analytically continue across the point along the path is passes through a very bad point and here is one that cannot be continued.

Along the path passes thorough that, so you see so this is so you know analytic continuation is one more tool of time to distinguish between analytic function in terms of properties ok. So I will try to explain this one more detail later, but I will like to check that this is the case ok, so in particular I would like you to try it as an exercise you take a path which goes like this.

You started logz here ok, started logz and at this point you take logz you can write, so here I will take logz+2square+2pie r ok and this logz+2piei is certainly analytic continuation of logz along with you can make it ok. So I want to try that makes this, so that will along this path which crosses the negative real axis, logz can be analytically continue to logz is 2pier. So you know the little thing is even though you are crossing this way.

There is another avatar of there is another branch of the of the log function which continues which leaves ok across that that negative real axis is full of bad paths ok. So proofs as n axis, so ok so that if you say if you take up arc like this going from the upper half and the lower half plane crossing the negative real axis is not the logz about and you end up with logz+2piei to different successive branch of the logarithm.

So that you can actually you can find analytic continuation that goes to get here, ok that is our aim is you know you have if function of analytic continuation and then you want to know what is the final function if you get at the end of the analytic continuation ok that is the aim. So the monodromy of something called monodromy which says well the finial analytic continuation is going to be the same even if you change the path ok.

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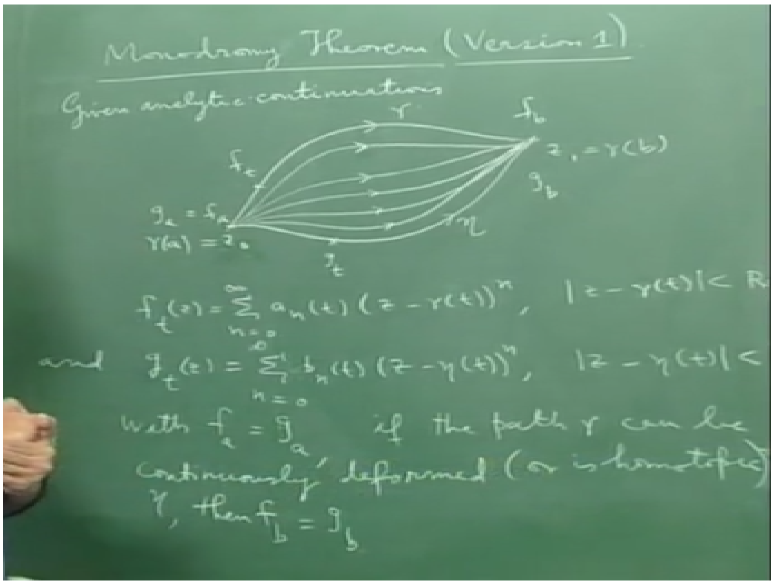

So there are many there are many version of so called monodromy theorem, so I give you one first question of monodromy theorem, so here is the monotony theorem so the idea is like this, you see I start with of point z0 I start with the point z0 and I end with the point z1 ok and I have path ok. Now and along with path gamma I have an analytic continuation given by a power series ft of z sigma n-infinity nt z-gamma t to the power of n.

Mod z-R(t) lesser than z-gamma t less than r, so this is one analytical continuation of fa to fb, ok where a is this is just z0 is gamma a and z+gamma, you know that fb is which is completely determined by fa we have seen that for this particular path, now suppose so as far as you see if this path is fixed, starting point is fixed, then I know that every analytic continuation I am going to get latter on each of the function is fixed.

That you have already proved that was given ok, the question is how will the ending function change if you change the path. So you know it is gamma suppose I give you another path theta for theta is another path which also starts with Z0 and ends at z1 ok and suppose you have another analytic continuation along theta ok.

Such that n tz is given by something sigma n=0-infinity a so some b and t z-theta t to the power of n mod z-thetat is less than R so here may be for you put Rf gamma of t and here R(f). So suppose I have another analytic continuation ok and assume that theta a is same as fa ok with the initial functions are the same fa is same as sorry I refuse not meet up it is gg so let me use g.

So fa is ga ok, so I am having on this all the fps that starts with fa here and end with fb here and along this path I have gts that starts with ga which is same as ft and I am going to get gb, what is the relationship between fb and gb. So the question is in monodromy theorem one monodromy theorem says when you can guarantee that fb adn gb are the same and the answer is it is a topological answer it is when gamma will be continuously deformed to b.

Ok if you can continuously be from gamma then fb and gt will be the same in other words if you start with a given analytic function and the analytical continuity along a path that analytic continuation you get at the end of the path is going to be independent path so long as the path the path of the same up to a deformation of one another which is not in topology is called as homogony ok.

So so that monodromy theorem version 1, and there are other versions and I will explain, so given analytic continuation this and this with a fa-ga, if the path gamma can be continuously can be continuously deformed or is homotopic t, beta then fb-gb ok. So if you start with the function and so long as you deform it along as you and click continue so long as you analytically continue it along the path.

The final function that you get is not going to be it is not going to be change, if your path is going to change only up to continuous deformation mainly ok, so this is this has a very nice statement in terms of covering space theory also it can be rephrase that the action of the fundamental group the action of the fundamental group on the zones of analytic functions on the covering space ok.

So can be refaced, I will try to explain that also in a little later ok. So the point I want to make is that if you have gamma and if you have another path eta which are which are homotopic and this is called fixed end point homotopic which means that you know you can find you can find paths like this continuously you can find continuous sequence of paths or continuous family of paths which start with gamma and end with it like this I deform this gamma to the passing time I can do that.

Then the analytic continuation along theta and analytic continuation and the gamma and analytic continuation on eta along eta will be same, in fact it will be the same for everything, you take any anything in between this is the deformation. So all the cases the final among any of these paths you know the function you get there up on analytic continuation it will be a same, it will the same as fb. It is only one ok, that is that is the continuity of monodromy theorem ok. So I explain the proof of that in the next lecture.