Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem
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Lecture-23
Continuity of Coefficients occurring in Families of Power Series defining Analytic Continuations along Paths

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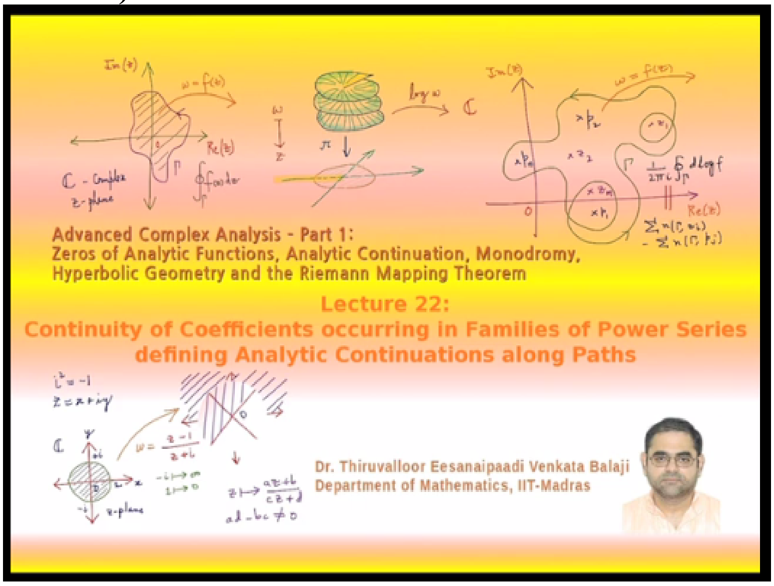

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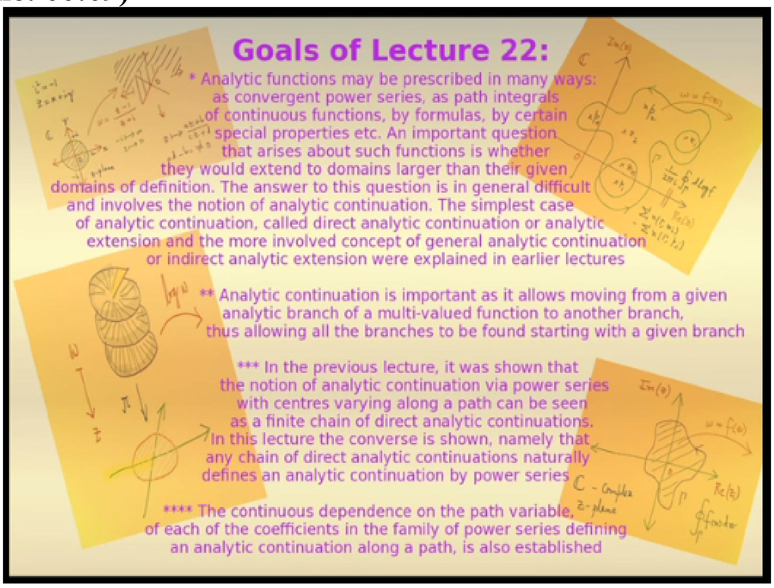

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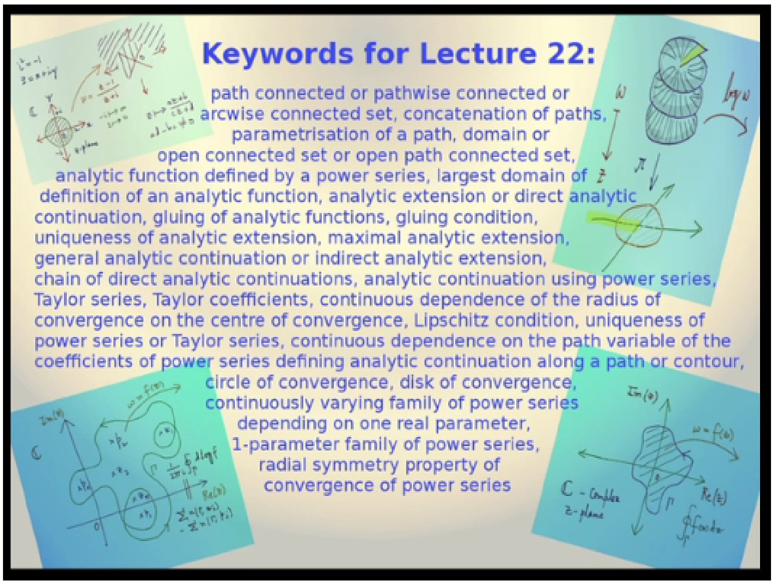

We continue with our discussion of analytical continuation, so you see last time I told you that there is so I do recall the various function of analytic continuation.

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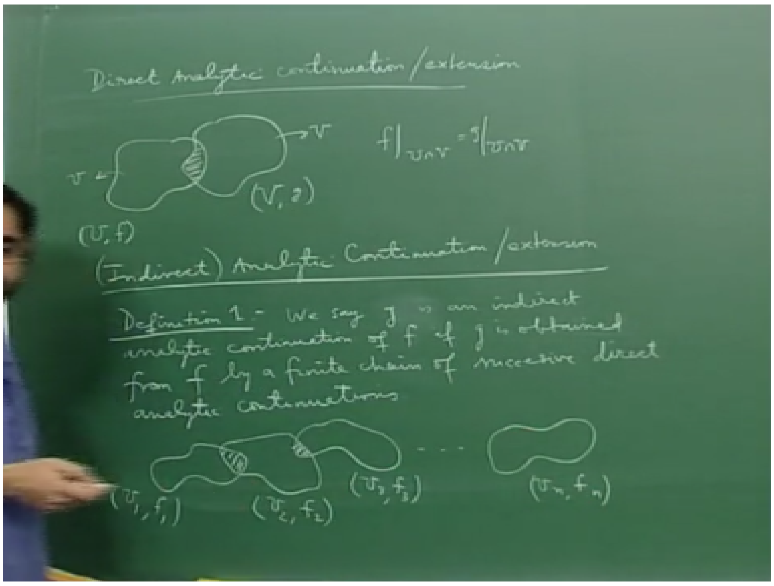

So that is this notion of direct analytic continuation which means direct analytic continuation or extension/extension. So this just involves pair of higher of data may be an open set which is connected open setting the domain and function defined on that and another set another open set which is function depend on that, so this is the open set U.

This is open set V in fact U will be supposed to be domain and you are not just open the suppose be connected and then the requirement is that we say that f is direct analytic continuation g is a direct analytic continuation of a f or f is a direct analytic continuation g, if on this intersection f restricted to this intersection is equal to g restricted to this intersection ok.

So f is intersection b intersection g, so we say that the pair we say that the analytic function f has been continue to the analytic function which is directly and you can also say symmetrically that the analytic function g has been continued to the analytic function f directly and the point about direct analytic continuation is that on this union ah the both functions can be due together become single analytic ok.

Define a function which is equal to f on U and is equal to g on v and that depends function properly on the union U union B that is because on the intersection f and g coincide, so it defence well defined function which is also analytic ok. So but then what we are interesting is introduction the notion of direct analytic continuation. So this direct analytic continuation are you can also say direct analytic extension.

So usually in the literature people do not use this adjective indirect they just say the analytic continuation and even expense even the word extension is not possibly used ok. But and I am stressing on this on using this word because an indirect analytic continuation can be defined in two ways basically and what you are doing in the last couple of lectures was trying to show that you know both definitions are equivalent alright.

So there so there is definition 1, definition 1 is we say v,g we say analytic function we say g is an indirect analytic continuation of f if g is obtained from f from f by a finite chain of successive direct analytic continuation, so in other words what happening is that you have got too 2 pair, but you have finite number of pairs such that.

So such that each successive pair is consist of a direct analytic continuation ok and the first pair is f, the last that the functions corresponds to the first pair is f and the function corresponding the last pair is f ok. So it is like this, so the picture is something like this, so you have U1, F1 which is define on this domain which is U1 that intersects with this domain U2 on there is analytic function F2.

And this U2, F2 is direct analytic continuation of U1 and F1 means that F2 and F1 coincide on this intersection ok and then this goes on that is one more pair U3, F3 and U3, F3 is direct analytic continuation of U2, F2 which means F3 and F2 coincide on this intersection which is U2 intersection U3 and then goes on like this and finally you end up with a last pair Un, Fn, this is Un which is a of course all the U are domains.

The way I am drawing them I am drawing them as if they are bounded domains but they are not be bounded ok, they just need to be open connected sets and drawing a picture like this very recently so this called you to imagine so you know this will be the intersection with U3 and U4 and this will be the intersection with U and OF Un and Un-1 ok and so U2 F2 is an direct analytic continuation of U1, F1, U3, F2 is direct analytic continuation.

U2, F2 mind you this does not imply that U3 F3 is a direct analytic continuation U1 F1 unless and until U1 U2 and U3 all intersects ok, it may happen that U3 never intersects U1 alright. So all you can say U3 is an indirect analytic continuation of U1 because it 2 step chain ok. So in this way we have finite many steps and you have F1 is the option f you started with and Fn is a function g.

Then we can we say Fn is g is an indirect analytic continuation of F1 and f ok. So indirect analytic continuation according to this definition is just given by finite chain successive direct analytic continuation and of course you may wonder why worry about indirect analytic continuation. So this is so whenever we do something you should ask the question why we do it at all ok.

So find the answer is that it is rather funny and it is rather they involve on teh one hand and it is rather funny on the other hand that you know if you start like this and you have a chain and assume that the last domain is the same as a first domain ok, assume that Un is the same as U1 ok. Then what could happen is still you could get different function ok, so you can even think of it like this.

You can start the function on the disc and then you can start moving that disc to get a successive to get successive direct analytic continuation and then we come back to the disc you may end up with the different function, that is the whole point and then the question is what is the different function . So even if U1 is same as U1 F1 need not be the same as F1, F1 may be different from F1, that is the whole point.

And then the question is what is relationship between this F1 and Fn, the answer is that they are branches of the same function. So in other words the whole point of studying analytic continuation is to try to get hold of all possible branches of an analytic function. That is the whole thing right. So well you know this is one definition which is given like this.

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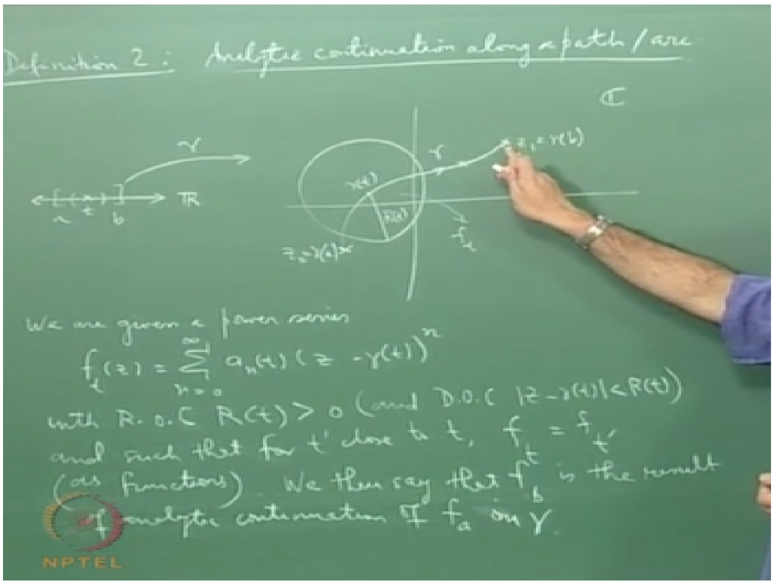

There is another definition adn this definition is what is called this definition also gives a definition of an indirect analytic continuation but it is called analytic continuation along a path ok using a power series. So so this definition is called indirect analytic continuation, so I use I simply call it analytic continuation along a path or arc and what is this, this si the second definition of analytic continuation well .

So you see so what I am having is that I have this open interval and have a closed interval a b on the real line and then I have this map gamma continuous function with maps this close interval and to a path on the complex plane.0 So I have to complex plane here and you know I have a path gamma, so for any point P I get the corresponding point gamma of t and of course this is a z0 which is gamma of a.

The initial point of the path and this is Z1 which is gamma of b, so I have path like this and then what and suppose so does it mean to have an indirect analytic continuation on this path that is a analytic continuation on this path. So we are given a power series ft of z given by sigma n=0 to infinity an of t z-gamma of t to the power of n convergence with ad of convergence r of t greater than zero .

And disc of convergence modz-gamma of t is less than r ok, so it means for every point you give me any point gamma give me any point gamma t so let me call the gamma here if you give me a point gamma of t then you know that is this radius rft so I have this disk centre at gamma t radius r of t which is the disc of convergence of this power series ft of z ok and for every such t I am given such a power series.

So the point about this power series as t changes this power series changes, you how does it change the coefficient change ok and the centres of the power series also changed. So the centres of power series of the power series of moving along the path and their parameterize and since the parameters parameter by t the centres are also parameterized by t just the point in the path.

And the coefficients of the corresponding power series of the parameter by t ok so see if you have given power series like this and assume that the radius of convergence always positive alright and you know such that you know for every for such that for t prime close to t ft=ft factor as functions ok. So in other words see this ft is the power series which vary as t waves even ok.

But the way we wanted to very is that you wanted to locally represent the same function ok. So that means if you give me t then for all t prime in a small neighbourhood of t the corresponding ft prime should be the same as ft ok, by that we mean that the power series you see ft prime is equal to ft is same as function ok, but it does not mean that an t prime is the same as a and the,

It does not mean that, that is because a and t prime will be different from a and t because this centre has changed gamma t prime is different from gamma ok, if t prime is different from y then gamma t prime and gamma t are different points on the path and therefore the of you therefore the power series at t and t prime are certainly different power series ok, but our requirement is that the functions there a present ok the same ok.

So you must not make the mistake of thinking that if t prime is close to t ft=ft prime means actually that you know this a n t is equal to ant does not correct because for ft and ft prime t and t prime different that gamma t and gamma t prime are different ok, they are usually different ok and therefore the power series are difference, right the questions are also different, but what we want is that the function itself convert they are the same ok.

That is the requirement, so this is the continuity requirement, so what so what so if you have something like this then we will say we then say that fa fb is the result is the result of analytic continuation of fa non gamma on the path, so you are saying starting function here the starting function here is the analytic function here ft ok, when you start here it is fa.

At that any fb so you are saying fa the analytic function FB is analytically continue along the path to give the anti function fb ok. So this is another definition when you say when one function is analytic continuation of the other ok along a path and what I was trying to explain the previous lecture was that these two definitions are one and the same, these 2 definitions are no different ok.

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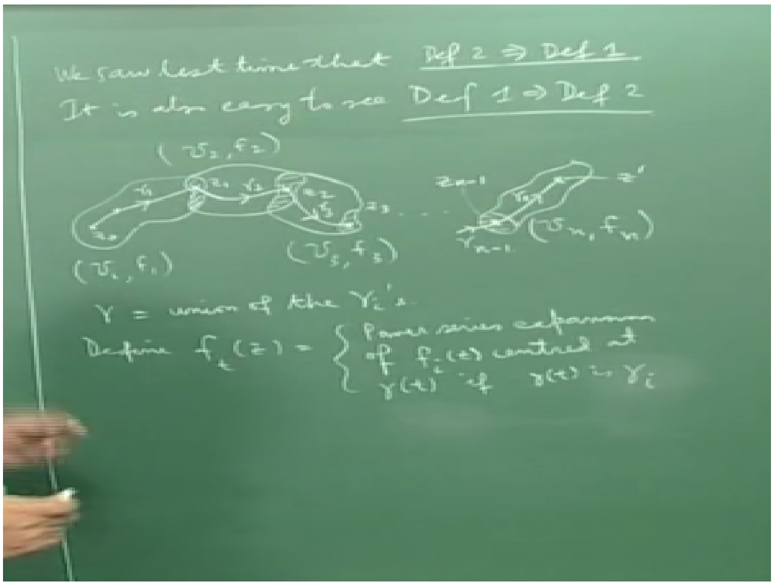

So what we say we saw we last time we saw last time last time that definition 2 implies definition may be a prove that if you give me an analytic continuation along a path then you can then the final function the function at Fb is an indirect analytic continuation of fa in the sense of the first definition namely it is obtained by a finite chain of successive direct analytic continuation.

This is what I prove last time ok and therefore in other word I am saying that this definition is this implies this definition ok and the converse is also true it is also easy to see that definition one implies definition 2 ok, it is also easy to see to see definition 1 implies definition 2, it is so easy to do it see because you know let me draw the diagram here, so here is my so you know here is my U1 this F1 and have something like this U2 F2, this is U2 F3 and so on.

And at end I have some Un Fn ok U2F2 is a direct analytic continuation U1F1 U3F3 direct analytic continuation U3F2 and so on and the final one U1F1 is a direct analytic continuation of Un-1 Fn-1 and then we say that according to the first version Fn is an indirect analytic continuation of f1 ok. Now I want to say that fn is also an direct analytic continuation of F1 along a path in the sense of definition 2.

How do you prove that is very recently what you do is recently started with the point z0 in U1 ok then you choose the point here Z1 in U1 intersection U2 ok and since Z0 and Z1 belong to the same set U1 which is connected and open mind you and open connected set is path connected okay. So the path connected say in a topologically is always connected set.

But you see what happens the connected need not be path connected, but if the connected set is open set then it will be path connected. So so this is a open connected set which is path connected and therefore I can join z0 and Z1 by path ok and I can call the path as gamma 1 ok and then I can choose a point here are calling Z2 and then again Z1 and Z2 belong to U2 which is again a domain is connected with path connected.

So I can join Z1 and Z2 by path gamma 2 ok and then I can go on like this. So you know I choose a point Z3 here which is this is an intersection of U3 and U4 and you know I connect z2 to z3 because both of them belong to U3 which is path connected by path gamma 3 and I go on like this, I end up with point so I end up with a point Zn here.

So this is a point Zn and it comes it is join to zn-1 which is intersection of Un-1 and Un-2 by a path gamma n ok and yeah this has be zn-1 like this ok. So each gamma I lies in Ui so and it connects z i-zi ok. So gamman-1 has to lie Un-1 and it has to connect zn—zn-2 ok. This also will be and then finally I chose this point if you want let me call z prime ok and I join it by another part is gamma n.

So this is a path gamma n and finally what I will do is I put gamma equal to union of all the of the gamma x you take union of all these paths ok, you take union of all this paths and of course a path alright, which is concatenation of path U we have 2 paths if you join them again you get a path alright. So you put them all together a single path right and then what you do is define ftf z to be equal to so you know this the definition.

The definition is you define it to be power series expansion of fi of z centre at gamma of t if gamma if gamma t belongs to to U1 ok. So I mean by that I mean gamma tf if gamma t is a portion of the path the gamma t is is gama mi ok. So what I am doing is I am doing a following thing, see for every point from z0 to z1 ok on this path and simply taking the power series of f1 centre at that point this make sense ok.

Then from z1 to x2 for every point along this path gamma 2 the corresponding power series is a power series expansion of the function f2 is defined on analytic centre at that point. So in this way I get this ft ok and it very clear that I have given you a given you a power series 5st of course the D is going to very in such a way that t for initial value of t it is going to be z0 and final value of t is going to z prime alright.

And what going to happen is to check that this is a this is a analytic continuation along with path I have to only check that look that for any t for t prime close to t the ft prime and ft f is the same function, but that is obvious because you know if you take any if you take anything t if you take any point it is going point gamma t it is going to be on some gamma if t ok, on that gamma ift in a small neighbourhood of t it is going to just represent a F5.

This is all fts are just power series expansion of the same function fi, so the condition that this condition for that t prime close to t ft=ft prime is automatically z alright because let me again repeat if you take any particular value of t and look at particular value of gamma t is going to be going to be one of the gamma is going to a point on gamma is and is going to be on gamma I it is going to be point gamma.

Then all the fts there all the ft prime nearby of t prime nearby t they are all just expansions of the function fi because fi is a function on Ui inside which you have taken the portion of the path gamma i ok. So you get this condition that for t prime close to TFT is equal to FTP automatically get this condition. Therefore it is clear that fn is analytic continuation of F1 along this along the path gamma ok.

So so let me write that down clearly ft=ft prime=fi for t with gamma of t belonging to gamma fi gamma i ok. So hence fn is the analytic continuation of f1 along gamma ok. So what this is telling you the definition one is also the same is definition 2. So this so definition one is the same as definition 2 alright. So either you see as analytic indirect analytic continuation as a chain of success finite chain of successive direct analytic continuation see it like or you see analytic continuation along a path using parameters by power series both of them ok.

So this is one thing that that was aim of the previous lecture and now, now I need to make a few I need to make a few statements, so the first statement is of course that so looking at it from the from the point of view of the analytic continuation along a path there are several observations ok. So the first observation is that if you think of analytic continuation along a path then the radii of convergence are the

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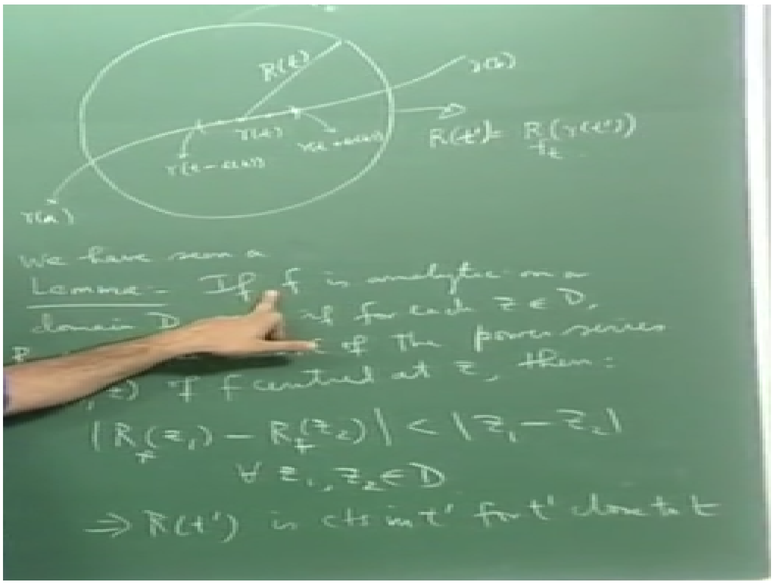

This function r of t is a continues function of t and this an of t is also continues, so that is the first one. So so here is a lemma if given a given an analytic continuation ft of z=sigman=0-infinity an of t z-gamma t power of n to the power of n modz-gamma t an an of t and r of t are continuous functions of a. So the first thing is that this is more or less intuitively clear ok.

This is more or less intuitively clear, but then in a little bit just be quite and little use a lemma that we prove and result that we proved in an earlier lecture ok may be first two or three lectures ago ok and we recall it. So what is the proof the proof is you see so you know so the diagram so let me again draw diagram so gamma is from a b to c this is your path.

Where a b is an close interval a b is a close interval on the real line ok of course by gamma you also think of the geometric part that it raises and also the function gamma ok and it is abusive language I am using both at the same time but this gamma, but when I am thinking of a path this gamma is fix, this function is fix the which means that I am dealing not just with geometric part.

But I am dealing with a fixed parametrization of that, that is what I want to tell you ok, so the path can be parameterized in many ways, so when I say I am look at path gamma and I mean I am fixing my not only my geometric path that is the path raise as we moves but I am also fixing the function gamma, that something I remember right. Now you see you know you also have this function a, so you know an of an will be a function again from ab to company

And r will also be a function from a b to C, an is going to give you the coefficient of for every t an of t is going to be the coefficient of it is going to be the enough n+1 coefficient of FT and this n+1 because I am starting with N=0 ok and r for every t r of t is going to be the radius of convergence of the power series f that is not a right. So an and r are both functions on this close interval a b on which gamma is defined ok.

And the claim is that a n and rf continues to this ok continues complex plane so of course I have to say I have to say more this is this does not actually not complex value it take values on the real line and positive real values ok. So it is a here in fact I can put I can use in fact change it to r or greater than or equal to r greater than 0 which is subset of c because I am giving you for each t I am always assume that the radius of convergence is positive.

So it is an analytic each ft is a convergence power series ok and it is not by conversion process I mean something that has positive radius of convergence ok it is not 0 it is always positive and of course I teach if you have an analytic function at point then you expand it in a power series about that point you will get a positive radius of convergence because if analytic at a point.

Then it is analytic in disc surrounding that point and that disk is certainly contain in the disc of convergence of the power series for that expansion of that function cantered at that point and therefore the the radius of convergence therefore certainly positive, so the rt are always positive. Now want to tell you is that see the check the type to check that you know a function is continuous.

If you have to check it local ok, check the function and it is enough to check the condition local, so you check the an and rn continuous enough to check them locally ok. Now you look at this, so you look at this so let me write that right down to check an and r are continuous it is enough to check they are continues locally ok because continues is a local path ok right.

And now here is where I am going to use this fact, I am going to use this fact in this definition of analytic continuation along a path, it says that you know for nearby points for all t prime near t ft and ft point should represent the same function ok, that is the key that is the key hypothesis, that tells you to check that they are continuous looking. So you know so start with start it in a t in a b ok.

Then there exist an epsilon of t of t such that so this is something that we repeatedly used in the previous lecture where we prove that definition 2 implies definition one. So for every t there is a exist of positive epsilon t such that for every t prime in t-epsilon t t+epsilon t intersection a b we have ft for prime we have ft prime=f ok. So this is just writing in terms of epsilons this continuity conditions ok.

So the continuity condition is that you know if t prime is close to t then the power series ft prime and power series ft represents the same function the same analytics and just putting in more precise language the condition that t prime is close and I am just saying that t prime belongs to an epsilon t neighbourhood of t inside it ok and for every t prime and epsilon t neighbourhood of t inside a b.

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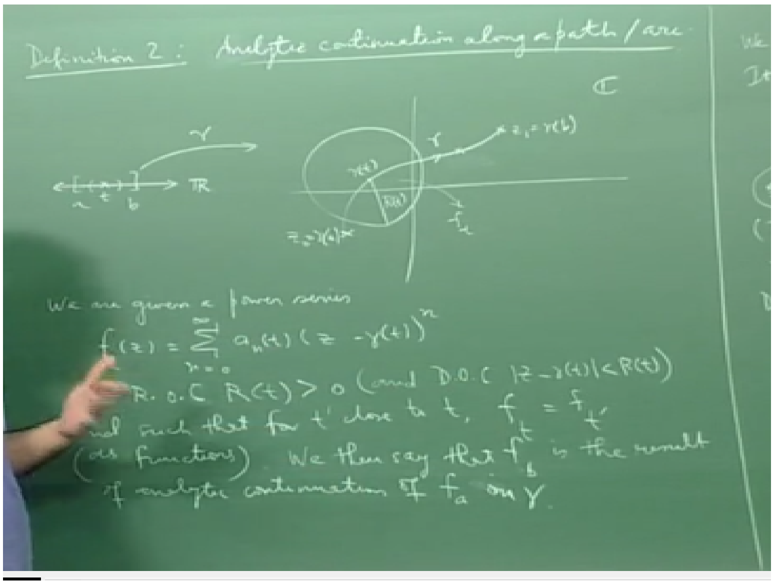

I am saying that ft prime is left is this translation this condition, so the condition that the power series the power series very continuously ok. Now you see now I want to so you know I will draw a diagram so here is my t here is my point gamma t, this is my path is gamma of a this is gamma t and then here I have I have this disc of convergence radius of convergence Rt ok.

And what is going to happen is and I am look at all t prime close to t so which means I am looking at you know a portion of the arc like this ok I am looking portion of so when t prime is lying in this neighbourhood so I get a neighbourhood like this ok. So this point will be assuming that gamma t is an interior point otherwise it could I will get only one side of it.

So it is in gamma t is an interior point of this of this part that lies between gamma and a and gamma b I will get this school interval so this point will correspond to gamma of t-epsilon t and this point will correspond to gamma of t+epsilon t ok and well for all t prime inside this open arc ft prime ft of z ok. So you see on on this on this whole disc the function ft lives ft is define on whole disc.

Because it is after I have taken ft is the power series centre at gamma t radius and within the disc of convergence you know the power series represent an analytic function, it is infinitely differentiable and r is derivatives we have the same disk of convergence ok. So if Ft is analytic function that leaves here and you know this f the other power series ft prime close to ft they are the same .

They all represent the same function ft because of this choice of epsilon t ok. Now watch you see you know we have seen lemma here is another lemma so this lemma is using another lemma which I prove the couple of lectures ago and the lemma is if f is analytic on a domain d and if for each Z denominator b rz is radius of convergence of the power series power series t FZ of f centre z.

Then r if modulus of rZ1-rz2 is less than modZ1-z2 for all z1 and z2 may essentially the statement of the radius of convergence is a continuous function of z, ok I prove this I prove this lemma a couple of lectures ago. And now what this lemma will if I apply this lemma then you see that then you know if you then if you see them for t prime the correspond to point here ok.

If I apply this lemma you can see that radius of convergence Rt will be the continues function of t because r is a continues function of z and on gamma z is continues f of t. So Rt which is composition of continues functions is cutting ok. So what I want to say is that here Rt is actually r of gamma of t rt prime is r of gamma of t prime ok, it is r of z with respect to the function ft ok. See throughout all that for all t prime line in this range ok.

All the functions all the power series represents the same function namely ft ok and therefore if you look at the radius of convergence they are the radius of convergence of FT at the various points ok. And therefore rt prime for t prime in that in this in this small neighbourhood of t ok and you know t prime is a continuous function of gamma I am sorry gamma is a continuous function of t prime.

And r sub f is a continues function of variable inside. So Rt prime is continues function of r t. So **so** this implies that Rt prime, so here you know if you want some notation I can put here if you want to make in sphere I can just saying Rz I call it rft this is radius of convergence of the power series expansion of f at z ok. If I put that f to make it clear.

So in this case I just get subset f here alright that might help, so what I am looking at these Rft Rt prime is just rft it is the release of convergence of power series expansion of FT at the point geometric because ft is same as ft prime ok. And and you know if you give me a point and you give me an analytic function the power series representation the power series expansion around that point is unique.

It is an it is an identify theorem for power series ok, the power series expansion of analytic function about a point is unique ok therefore there is why you get ok and so rt prime is a composition of gamma and rft ok and it is composition of continues functions so rt prime is continues in t prime for ft prime close to it. So what I I manage to show I manage to show that Rt prime.

R is locally continues function of t ok for every t you should take the prime sufficiently close to t ok. I have proved that R is the continues function of t prime ok a function which si globally defined functions is locally continuous is global economy because continues is a local. So I will just verified it locally and how I am able to verified locally because locally the point is all the power series they are all representing the same function.

That is what I mean, so that is what I mean and so long as if you have fixed analytic function then you know this lemma tells you that the radius if you expand it as power series at different points then the radius of convergence are continues functions of the points, that is what this lemma tells you ok. So this tells you that Rt prime is a continues function of t prime for t prime close t and this means that Rt is locally continues which means Rt.

Because function is locally continues that is all ok. So this establish the fact that r is function now I have to show that an is the continues function that is easy that also will be easy why is that so can write in line, it simply again the fact that in that for all t prime in that neighbourhood of epsilon neighbourhood t inside a b all the ft prime all the function given by the by the power series ft Prime is they are same as ft.

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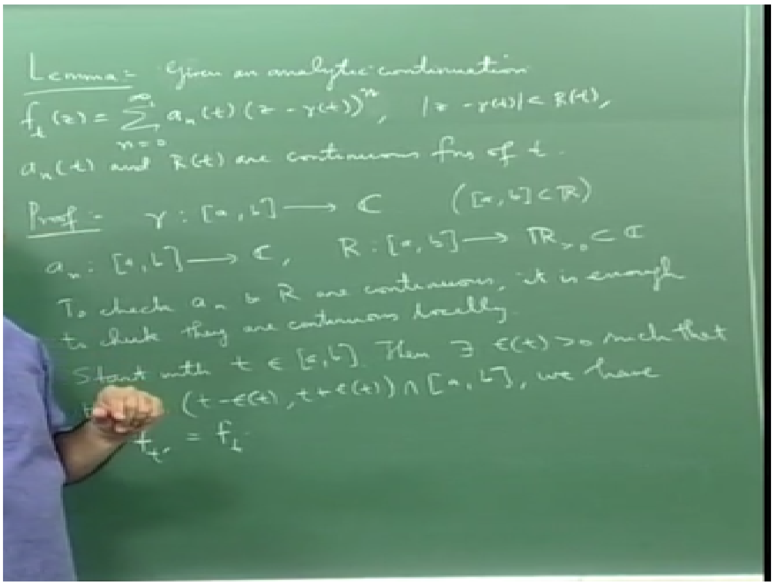

So this is the same analytic function, so you see an of t prime is what, is a of t prime is t this is the n+1 coefficient ok in the power series expansion of FT prime in the power series ft prime ok. What is that if you take a power series of an analytic function what is n+1 coefficient, it is just the n+1 derivative evaluated at the centre divided by factorial n+1 ok. So this wil be just this is really just so it is an t prime is n+1 coefficient of ft prime.

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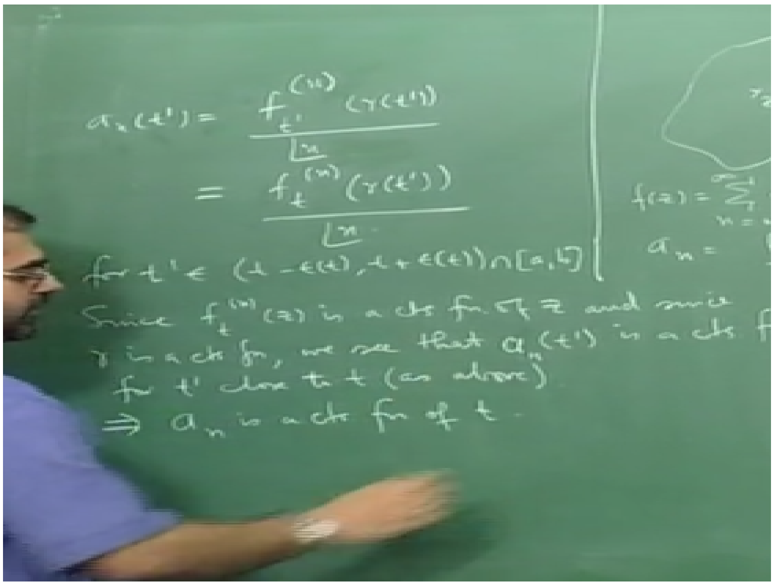

So it has to be given by n+1 derivative of ft prime centre at gamma t prime/factorial n+1 right, you know if you have domain d and if you have point z0 and if you have analytic function f on that then you know if you write if you try Taylor expansion of f at z0 you will get f of Z=sigma n=0 to infinity an z-z= to the power of n and what is the an. This an is nth derivative of f ah probably this yes this should be n not in +1.

So I will get nth derivative of t at the centre factor n, you know this just Taylor formula ok, so I am just applying it after all an t prime is the is the corresponding to ft prime ok corresponding to ft prime or ft prime the corresponding an is an t and therefore an t an t prime has to be given by the nth derivative of ft prime evaluated at the centre which is gamma t prime/factorial n.

That is what I am writing but then you know but you see the point I am using is t prime cose to t ft prima is same as ft you see all the for all the t prime close to t the ft prime is all one in the same ft. So this is equal to ft n at gamma t prime/factorial for t prime in that neighbourhood may be t-epsilon t t+epsilon intersection neighborhood as because in that all t prime in that neighbourhood ft.

And ft prime represent the same analytic function ok ft and fy prime representing same analytic function then the nth derivative of ft should be the same as n derivative of t because we have seen the same function, the derivative depends only on the function ok. So I get this, but now you see you see you know but now you are done why because you see if you have a analytic function then you know all is derivative also analytics ok.

So each nth derivative of ft is also analytic, so it is continues and gamma is a continues function of t prime therefore ft n of gamma of t prime is a composition of continues function, so it is the analytic function of t prime. Therefore you get an t prime is a continues function of t prime, for all t prime close to ok. Since that is it, so you have prove that an is locally the continues function of t and therefore globally continues function of t ok.

So since ft of n z are continues function of z and since of course gamma is a continues function you see that the ft prime or an t prime is a continues function of t prime for t prime close to t as above, close to t means t prime lie in epsilon t neighbourhood of t since I view, so this implies that it tells that it tells that therefore an is locally continues and therefore continues function.

So that finishes proof of this lemma ok, so this lemma tells us something that we told you already clear when you define analytic continuation along a path is given by a family of power series along the path ok with centres along the path and you want that family power series locally to represent the same function along the path. This condition actually that is what lemma explain.

This tells you that this condition ensure that the coefficients a n also ensures that the radius of convergence of each ft in the R that is also continues function ok and which we intuitively feel should be cleared ok and then in the next what I am going to do I am going to tell you that I am going to tell because of this very important fact. The important fact is if you give me a fixed parameter of gamma of path ok and you give me an analytic continuation like this. Then everything is completely fixed by the initial function ok.

Ft for every t greater than or equal to every t greater than a including b is completely determined by fa. So I am saying in other words if you give me a path they start if you give me a path of the parameterization start with analytic function at the starting point, then if you do any analytic continuation on the path the each of those functions that you get in between b, they all unique ok. So the everything depends only on the starting function and that we usually use this continuity right. So I will explain that in next lecture.