Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-22
Analytic Continuation Along Paths via Power Series Part B

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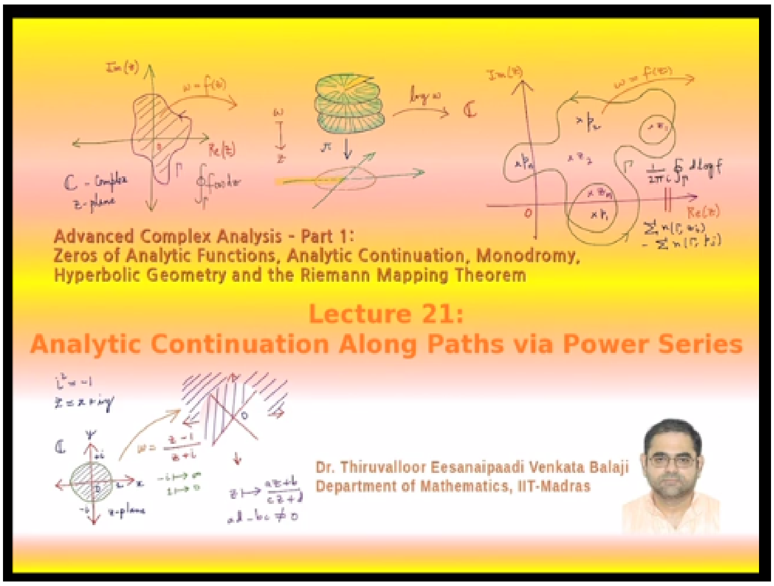

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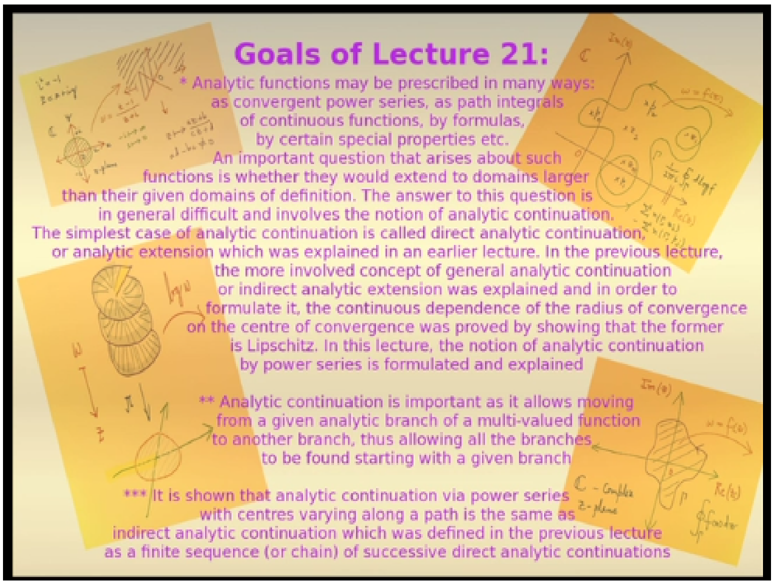

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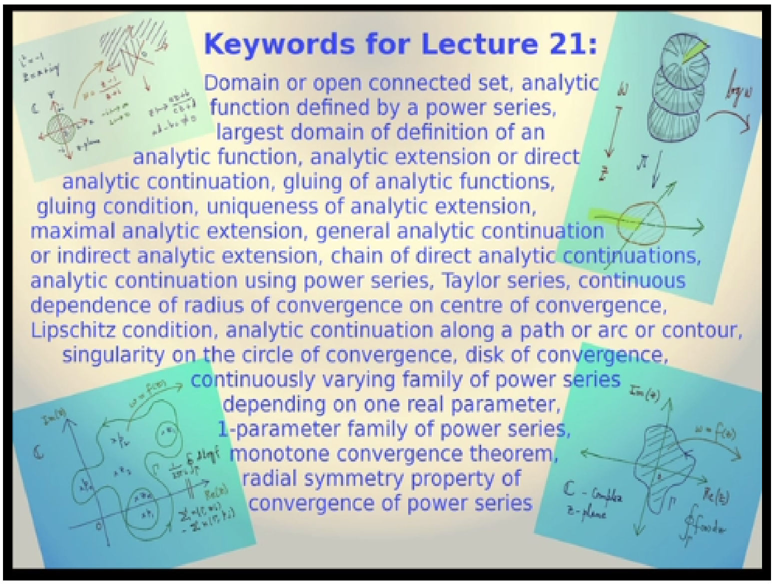

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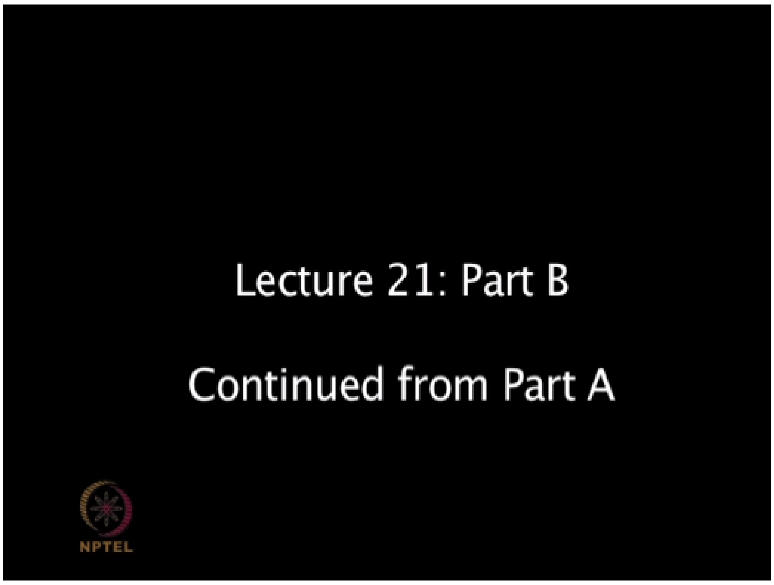

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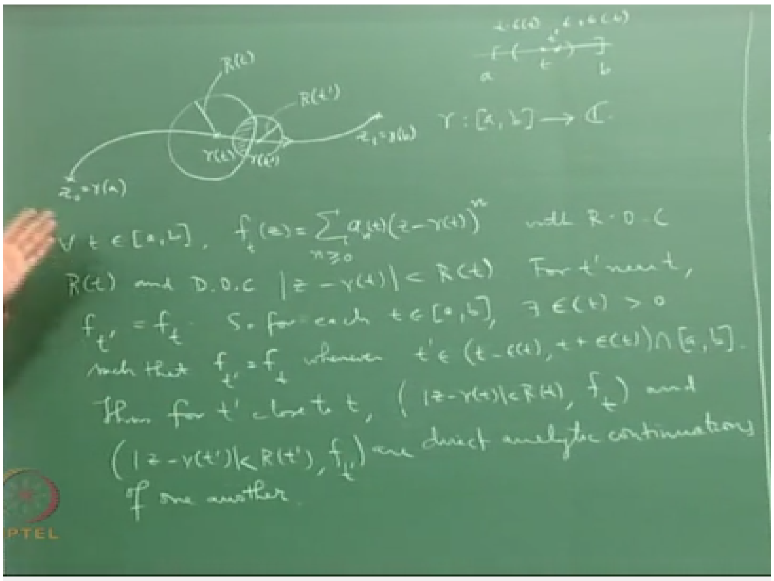

I have this situation where I have a path varying at so this path which is given by contains function gamma from the closed interval a b on the real line to complex line and of course this point is z0 which is gamma a and this terminal point of the path is Z1 is gamma b and what we defined as indirect analytic continuation in terms of power series is an expression.

It is a expression involving power series with continuously vary with respect to the parametric t as t varies on the on this close interval on the real line ok . So for every t in a b we have gamma, we have the function f t z which is given by this power series convergence power series centre at gamma t so it is an expansion in integral powers of z-gamma the

And with coefficient a n which are again function of t, n is less than 0 and of course with varies of convergence R(t) and risk of convergence mod z-gamma t is less than R(t). So this is a so this is a one parameter family of power series ok, the parameter is t right, and we want this we want as t moves which means you are thinking of the point gamma of t moving on this path.

So t is moving on this interval ok from a to b gamma of t traces this path ok and we want the power series to move to be b continuous, in the sense that we want successively the power series in neighbourhood being direct analytic continuation and then that is the condition. So the condition is a. for it t t prime near t F t prime is equal to ft. This is the condition we have ok.

So so in other words so for each T e (a, b) there exist an epsilon there exist an epsilon t less than 0, such that f t prime is same as ft whenever t prime is in is epsilon neighbourhood of t, so t prime belongs to t-epsilon t t+epsilon t intersection+a, b. This is the condition that we put to say that so if you have gamma t here corresponding to a point t.

So I draw front of the real line here and this is a and this is b and I have t I have a neighbourhood and open interval centre at t given by t-epsilon t t+epsilon t as that for every t prime in that neighbourhood the power series at t prime and power series at t are in the same analytic function ok. So of course when I say the power series are equal I mean that the analytic functions represent are equal in t.

Of course in the intersection of the it will be in a neighbourhood of it will be in the neighbourhood of it will be in this neighbourhood ok. So what is happening is here is gamma t and then I have gamma t prime and you know there is a gamma t has a power series with certain radius of convergence and what I want is that if I draw the power series centre at gamma t prime.

What I want is that in this intersection I want the power series at gamma t and gamma t prime represents same analytic function, that is what I want ok. So so at gamma t this is the this si the disc of convergence with radius equal to radius of convergence equity R(t) at gamma t prime and have another disc of convergence with radius of convergence R(t) prime ok.

And then I in this disc of convergence I have ft which is analytic function of Z with this power series expansion it is a Taylor series expansion of Ft and here I have in this disc I have the power series set t prime ok and what I want is that in this intersection of these two discs I want ft and ft prime to be the same alright. So what is essentially means and I want this to happen for all t prime in a all t prime are simply close to it.

So in other words if you take any such t prime then what happening is that ft prime and ft are direct analytic continuation of each other, in fact their one in the same function of the intersection. So they are direct analytic continuation of each other. So thus for t prime close to each close to t ft prime , so if I take the pair with corresponds to the disc of convergence for ft prime.

I mean for ft and for ft prime, they these 2 pairs are direct analytic continuation of one another. So if I take mod z-gamma t less than R(t). This is the domain disc of convergence of ft and the other one is z-gamma t prime lesser than R(t) prime and also t prime are direct analytic continuations of one another. This what I have and the claim is that this definition of t.

So the claim is that this definition of thinking of a one parameter family of power series ok, such that close by power series are the same that means they have the same analytic function this is a claim this gives a an equal definition of an indirect analytic continuation. So the claim is that the power series at b is an indirect analytic continuation of power series at a.

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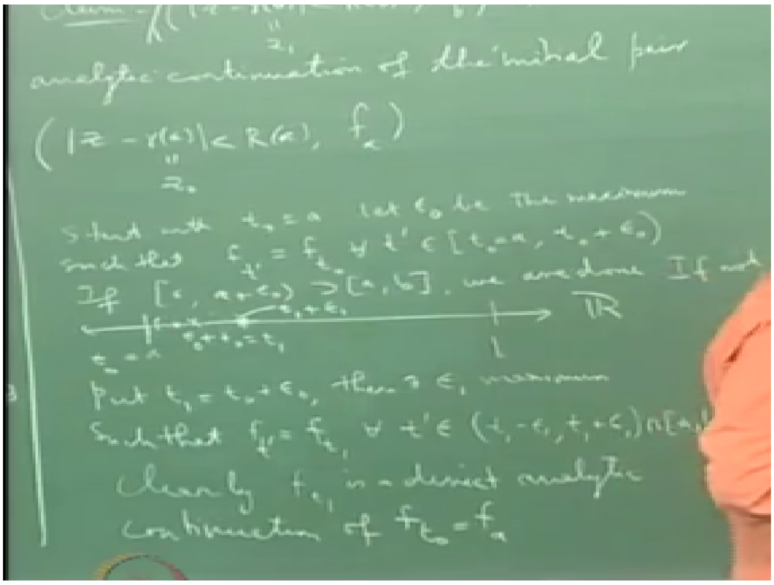

So the claim is that the the power series claim is modz-gamma b gamma a other gamma b lesser than R(b) which is b is gamma b is z, so this is z1, power series at b which is gamma this is the power series of gamma b which is at power series at this point z1 is an indirect analytic continuation analytic continuation of the initial the initial pair.

This is the initial the initial pair this is the final pair this is the final pair of initial pair which is at the starting point gamma a is z0 ok. SO this is the this si the claim and the true is the claim I will have to show that you know our definition of indirect analytic continuation the original definition of indirect analytic continuation is z there are finitely meaning you know direct with the chain of finitely many direct analytic continuation.

That is our definition of indirect analytic continuation, that is what I have to, so what I will have to show is that these is a plain from this by chain of direct analytic continuation ok, now see this what we have to prove it more or less in due to obvious but then it means the poof, ok it is because it gives the centre part of it, so so the point is that while the definition insist that nearby power series are equal ok which means that we are saying as t varies a small neighbourhood .

You are getting the same function ok, it is clear that nearby power that the function taht we will get nearby or one in the same so they are direct analytic continuations. But if but it does not imply that this and the starting the power series starting point for power series at ending point the analytic functions that they represent or indirect analytic continuation of each other.

That is not imply okay, their definition only says nearby power series you should take 2 nearby points then the corresponding power series are direct analytic continuations ok, but it does not tell you that for example anything is an indirect analytic continuation of something before ok. So that is the power series at any point is an indirect analytic continuation of power series at a point before ok.

That means we need to so how does one tool is well one essentially uses some topology, so the first thing is what you do is start with t0=a let epsilon 0 epsilon 0 b say the maximum such that ft prime=ft0 for all t prime in t0=a to t0 +1 epsilon 0 correct. So choose ah we choose the maximum epsilon 0 see the definition says that you give me any point then there is a will there is a small neighbourhood ah of the parametric corresponding to the point.

Where the power series are all equal alright. So I start with t=t0 then I will get epsilon neighbourhood effect, t0 where all the power series are equal ok, then what I do is so you know in this way so my inductively I do the following thing, so I have this is a, this is b on the real the real line ok and a is t0 and what I is now I have I have this epsilon 0 +t0.

And for all t in this interval for all t prime in this interval ft=ft prime, I have that ok. Now what are you doing start with epsilon+t0 I will call that I will call that as t1 ok, you should call that as t1 then there it is an epsilon 1 maximum such that ft ft t1=ft prime=ft1 for all t prime in well it is epsilon 1 neighbourhood about t1 alright. So t1, t1-epsilon 1 t1+epsilon 1 intersection a, b ok.

This is again adjust this condition that the locally the power series for sufficiently close for parameter are the same and this applying that condition ok and of course I if if you know if this interval already contains I am done, ok if the interval a, t0+ a, a epsilon, epsilon 0 contains it contains a, b we are done, because in that case you are saying that you are just saying that the we are just saying that the power series of b is same as the power series at a, alright .

And I mean what you are saying is that the power series but the function defines the power series in b mainly fb is the same as a function presenting the power series at a right, that is what we are saying and we have done and which means that this function is actually the same as this function and there is a whole neighbourhood there is a whole open set which contains this whole part where fa*fb define same function ok.

So it means that is actually an extension is a direct extension of fb he is a direct analytic extension of fa ok and of course is a direct analytic extension is also an indirect analytic extension ok, it is the chain consisting of just one pair ah I mean just couple of pairs ok. So so if we stop here, but if it is not ok we continue ok. so what we do is if not if nto we do the following things.

We put t1 is this and then you again try to look for a neighbourhood around in the surrounding t1 where ft1 where ft prime is the analytic function represent by ft prime is same as the analytic function of ft ok and clearly ft1 ft1 is a direct analytic continuation of ft0, this is fa, this is because is because for t prime close to t1 and t prime as of in t1.

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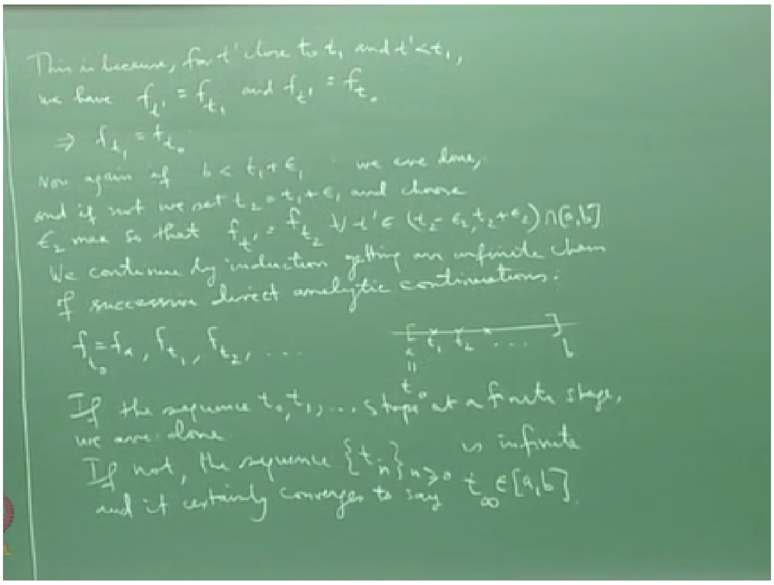

We have ft prime=ft1 and ft prime is also equal to ft0 ok because ft prime=ft0 the movement t prime goes to that is have in t1 alright that is how t0 that is how epsilon 0 goes to that is how t1 of source right. So you have both which implies that ft1-ft0 ok, so so in this way what does happen is that you know I have I have this point I have chosen this point t1 as a ft1 is direct analytic continuation of ft0 ok.

And you know it now I continue this process, I now continuous process until I reached the other end until my parameter t reaches b alright, so you will have to show the parameter t if I do this process I have to show that t should b at some point. Now again if well this ends so well b is less than or equal to if b is strictly less than t1+epsilon 1=t2 we are done ok. We will be done.

If not and if not we said we said t2=t1+epsilon1 and choose epsilon2 maximum, so that ft prime=ft2 for all t prime in t2-spsilon2 t2+epsilon2 intersection a, b, ok. So you so you continue this process we continue by induction by induction getting an infinite chain of direct successive direct analytic continuations. So what you get is well I have first I want to write the domains.

So I have ft0 which is fa and then ft1, ft2 and so on, and the point is that and this corresponds to choosing points on the on the on the parameter interval. So this is t0 and I get t1, then I get t2 and it goes on like this, ok. And every ftJ is a direct analytic continuation of ftj-1 ok and go on like this. Now the point I want to say is that my claim is that these ts ok. There are 2 possibilities to the sequence of ts.

One is I might the sequence of ts could just stop ok ok, with a sequence of ts stops the final stage that I am done ok, because then I would have been the should be a finite chain of direct analytic continuation starting from fa and ending at fb which will tell you that fb is an direct analytic continuation of fa ok. See the only problem is that I need a finite chain for for in fb, so that fb is an indirect analytic continuation of fa.

I need only a the only thing I need is a is a finite chain ok, it is a finite chain that is bothering alright, so if the sequence t0 t1 stops at the finite state we have done we have done because you got a finite chain of successive direct analytic continuation this starts with fa and ends with fb. And that will tell you that fb is an indirect analytic continuation of fa ok. So this case can be ruled out ok.

This case is now ok, what is the other case that the other case is that the sequence is an infinite sequence ok, and if it is infinite sequence there are 2 possibilities ok. it is an infinite sequence which is a which is a increasing sequence ok. So it has to its bounded by the mountain convergence it has to converge ok. So if it so there are again 2 possibilities if it converges to b also I am done ok.

The only problematic situation is when it converges to point before b and in that case you get that cannot happen because we' will get contradiction ok. So if not the sequence tit n ended with 0 is infinite and certainly converges to and certainly converges to say the infinity in a, b ok. So you are having a so I am using a monotone convergence theorem here.

I have a sequence of real numbers which is bounded above and which is increasing then it has to converge ok. And the whatever the limit is I am calling that limited t infinity and it has to it has to belong to this interval closed interval a, b because it is closed, a close set it will always contain the limits of sequences ok. So this t infinity is in a b and if this t infinity is b then I am done ok.

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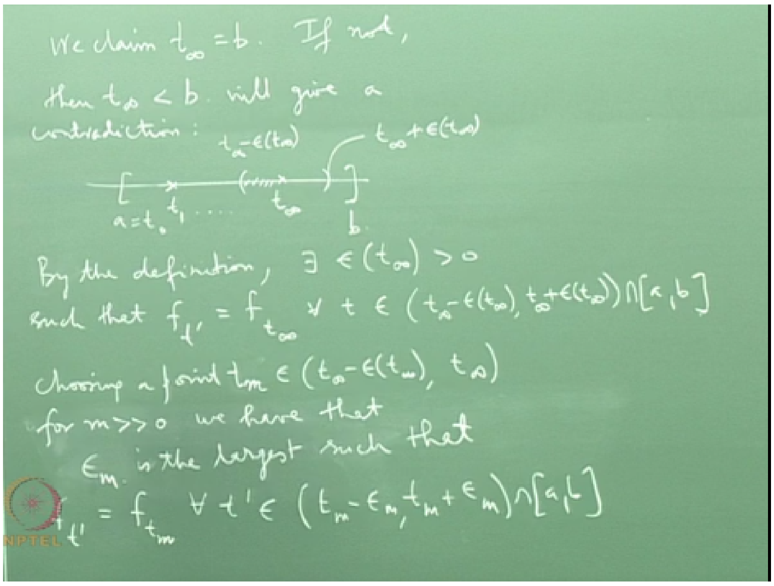

We claim we claim t infinity=b ok, we claim t infinity is the point b alright, if not then t infinity is the point strictly less than b ok and that will give us a contradiction because you know see the situation is so you have a this is t0 and we have t1 and so on, suppose it convergence to t infinity and this t infinity is strictly less than b, will give you a contradiction as follows.

What will happen is well you see the condition this condition that the power series ft you know it locally should match the power series in nearby points also applies at t infinity ok. So at at by by by the definition there exist epsilon of t infinity and there equal to 0, such that ft prime=ft infinity for all t prime in an epsilon t infinity neighbourhood of t infinity.

So it is such write this t infinity-epsilon t infinity, t infinity+epsilon t infinity intersect it I will get this, ok and now this will give me a contradiction because you see if you choosae a if you choose a so you know I I get this neighbourhood like this and this left hand point is t infinity-epsilon t infinity right end point is t infinity+epsilon of t infinity of course I am worried about right end point it could go beyond b also ok.

But what is important for me is the points before t, so if you choose a point if you choose a point t prime there when you will be in trouble you choosing a point for m sufficient to lot because of tral every neighbourhood of t limit point contains point at as close as you want of the sequence. So I can find a tm there, alright and then but what is the contradiction the contradiction is we have that ft prime=ftm or all t prime in tm-epsilon mtm+epsilon m intersection a, b.

Now this is how you are choosing see what you have understood is htat tm+epsilon which is mind you this is t, this is tm+1, this tm+epsilon m is tm+1 aright and this tm+1 is to the left of t infinity ok this tm+1 is to the left of the so I have this, so I have t infinity here I have on this on this side I have t infinity-epsilon of t infinity, ok and I have on the other hand I have chosen tm here ok. I have chosen tm here.

And then I have then there is a there is a neighbourhood you write tm-epsilon,m tm+epsilon m alright, and tm+epsilon m is is to the left of t infinity okay. So this is tm+1 this is tm-+epsilon this is tm+1 right, but the point is I get contradiction here ok ah hi will get a contradiction here in the sense that if I take any point here at a point here for the t there if I call this is this t as t prime.

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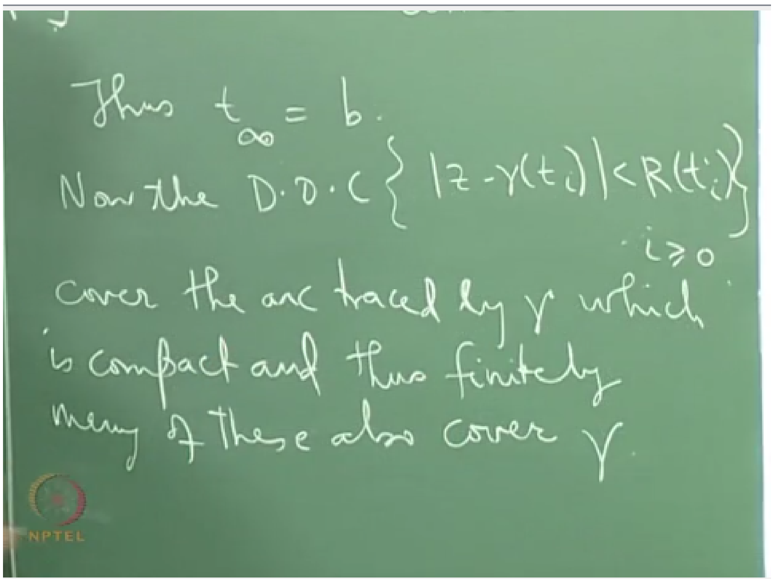

I have that ft prime=ftm ok I have that, on the other hand I will also have ft prime is also equal to f t infinity ok and that means that you know this will implies that this t infinity has to be strictly less than epsilon the distance of t infinity from tm has to be strictly less than epsilon m which is not correct ok.

So this implies distance of t infinity t from tm b lesser than epsilon 1 which is which is not correct. This is what we get, see please try to understand I have this left side of the neighbourhood of t infinity with length epsilon of t infinity said that for every t prime that the power series at ft prime is the same as the power series at ft infinity alright, in that neighbourhood I choose for m sufficiently large tm ok.

And for that tm I have an epsilon m which is the maximum interval around tm, said that ft prime=ftm alright, but I have found a point outside that interval ok, so you know I have so you know I have points I have this point t infinity so the outside that the interval and such that the analytic function representing ft infinity is the same as analytic function representing ftm.

That is the contradiction to the definition of epsilon m, ok and that contradiction tells me that this case may not happen, so thus thus we end up with thus t infinity is actually=b so I will just have to say that now since I know that t infinity is b then I use the compactness of the arc to get you know finitely many points in the sequence with corresponding discs of convergence.

You know covering the covering the arc alright and then it will follow that fb is in analytic continuation of fa, correct so now the discs the disc of converges modz-gammati less than R(ti) ah i greater than z0 cover the arc place by gamma gamma which si compact and thus finitely the meaning of this discs also cover they also cover it on.

This and this rows then fb sequence and indirect analytic continuation and ft continuation, so let us stop.