Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-21
Analytic Continuation Along Paths via Power Series Part A

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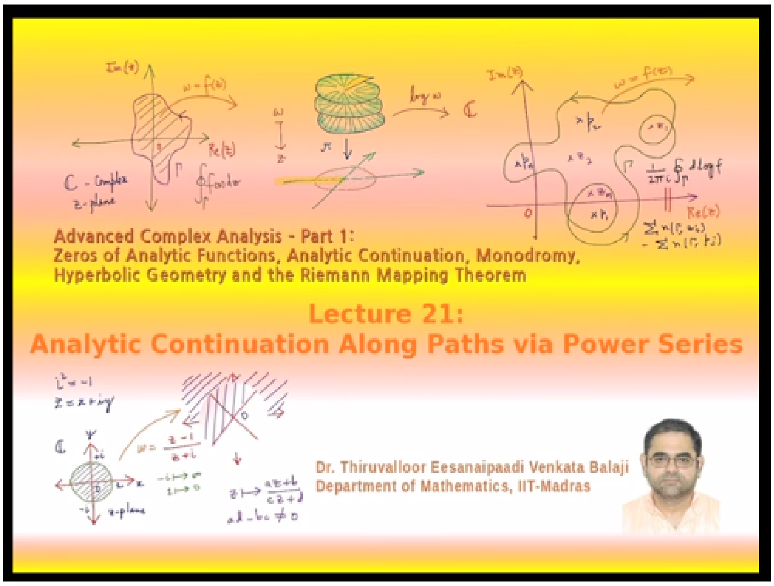

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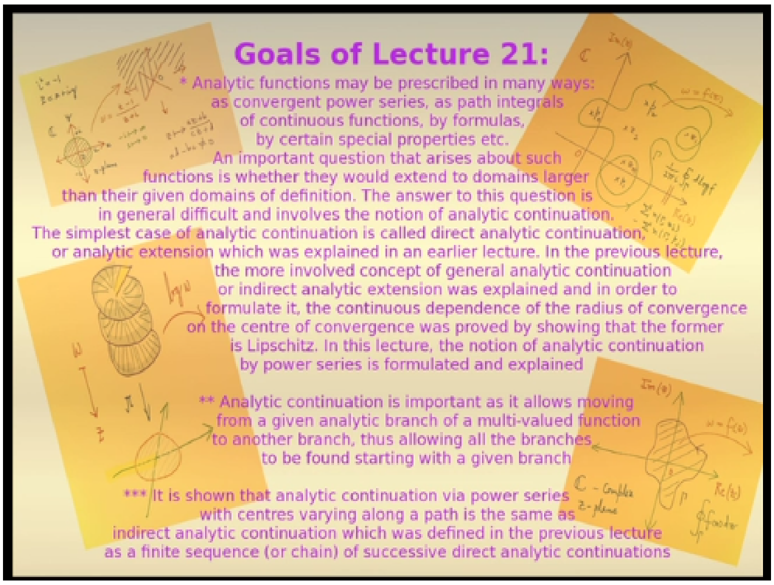

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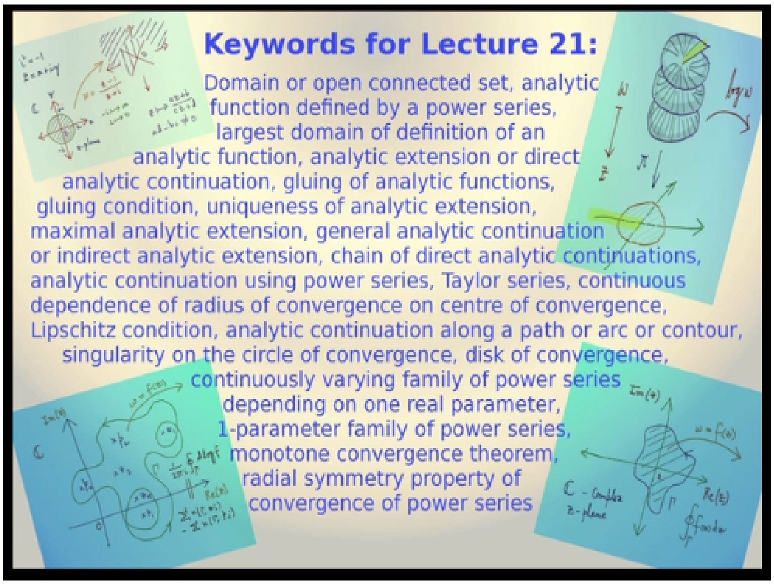

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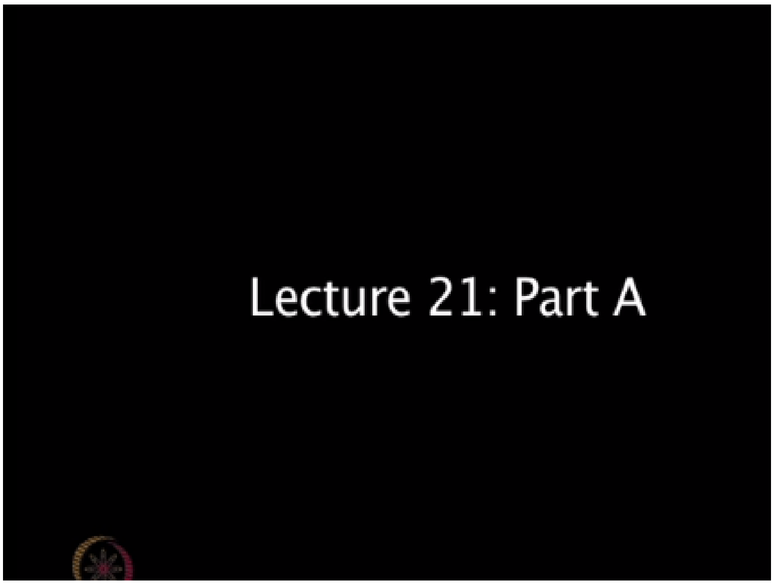

Ok so let us continue with discuss of analytic continuation ok.

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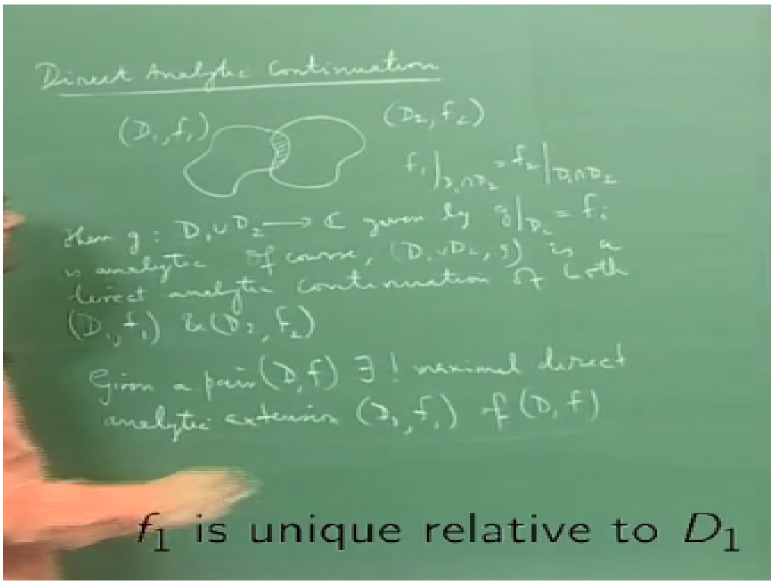

So I told you last time that there are 2 notions of analytic continuation one is so called direct analytic continuation and involves this literally involves growing together 2 analytic functions on 2 domains which will intersect and such that the analytic functions they give the same function on the intersection ok. So direct analytic continuation means that you have diagram like this.

So this is D1 of F1 and we have this D2, F2 and you say D2, F2 is a direct analytic continuation of D1, F1 are conversely if on this intersection which is supposed to be nonD F1 and F2 coincide and so the advantage of this is that you can define these 2 functions grew together to give an analytic function of union ok. So then g D1 union D2 to see given by g restricted to Di=Fi is analytic ok.

Of course the pair D1 union D2, g is set in a direct analytic continuation of both of them, and of course the further direct analytic continuation is unique because of lying between it, which says that if two analytic functions agree on a look at non-empty open subset then the I have to agree on the whole domain ok. So it is the story of director analytic continuation and you of course seen that given a pair D, F there exist a unique maximal extension.

Maximal direct analytic extension D1, F1 of D, F. So we have seen this so we and we saw these are 2 examples of this. So the first example was D1 is the unique disc, and F1 is the power series corresponding to geometry is ok, and all other called that D and F . So you take unit disc which is the disc of convergence of the power series given by geometric series, power series said to be 0, replace the convergence is 1.

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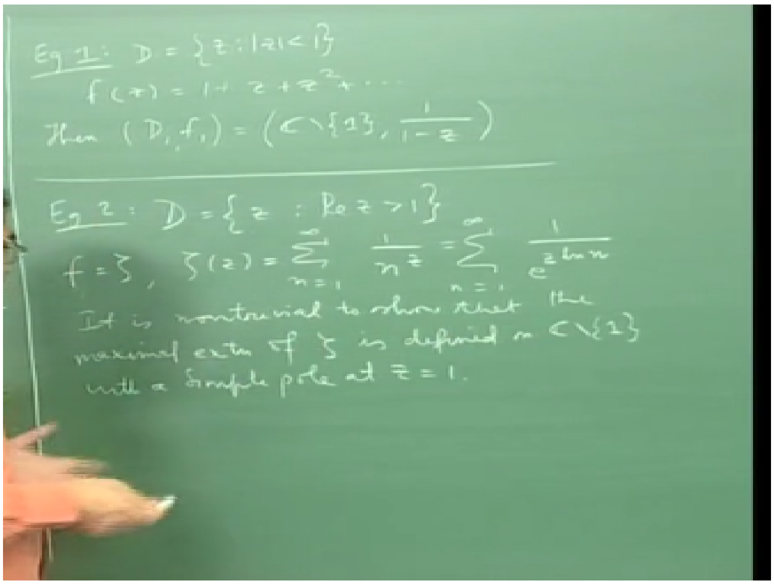

Then the maximal extension D1, F1 is simply the complex plane-0.1 and function which is the maximum extension is 1/1-z is in this. This is pretty easy to see because this power series represents the reference 1/1z units ok, and that is define everywhere in the complex way and analytic accepted the point z=1 where it has a simple pole correct, the this was an easy example ok.

So the other example is that zeta function for which the domain are going to consider as a right half plane, but open right of plane to the right of the line the vertical line passes the z=1. So **so** in this case D is set of z real part of z greater than 1 and the function F id zeta where zeta of z is rieamer zeta function which is sigma n=1 and 1/n power z and this is defined as sigma n=1 infinity 1/3 power z lawn in that lawn n is D usual real log .

So **so** the fact is of course there is some worth in order to taking the z type itself is analytic function ok. So that involves the use of (()) (06:30) which we did last time and of course we also use need to use the fact you need to use the (()) (06:37) to check that zeta that this series converges to a function normally ok in this domain which means is convergences uniform on compact subsets ok.

And then further since we have already seen in earlier lecture that whenever you have sequence analytic functions which conveys is normally to limit function in the limit function is analytic by using that fact (()) (07:12) we can check the zeta is actually analytic is analytic function on this is right half plane. It is with theorem to show that the maximum analytic extents of zeta is defined on just like the geometric series is defined on the whole complex plane-1.

And at what you get up to pole for 1 simply pole ok. so it is nontrivial to that the maximal extension session for zeta is defined on complex plane-0z=1 with a simple pole z=1 ok. So we will see proof of this later these, but in this tells these two examples tell you they give you the the big difference between difficulty ok. Here is an example in which the maximum analytic extension is very easy to understand.

And here is one in which it is very difficult, it is not directly easy to check that you have maximum analytic extension ok I mean any way it is not easy to check what the maximal domain on which the function exchanges z and what the corresponding functions ok one has to do lot of analysis to check that just like in this case the only point where this for this zeta function does not extend is equal to one where of course it becomes a harmonic series and you know.

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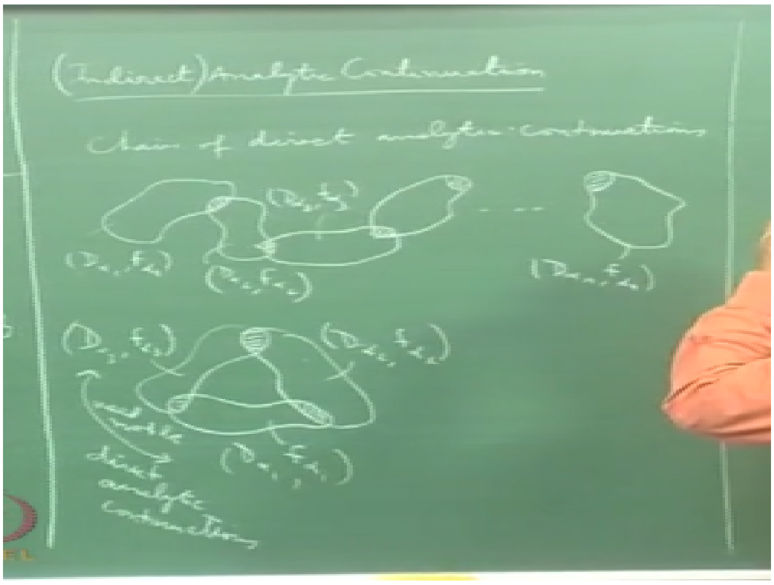

And to that one has to do not finals. So this is the problem direct analytic continuation. Now what we do next is go to this notion of indirect analytic continuation. So now what we do next is both of this motion of indirect analytic continuation and so what is this indirect analytic continuation. So the word indirect is something that I am stressing usually in the literature the word that is used is just analytic continuation and I am stressing the word indirect because I want you to distinguish between these two.

So this indirect analytic continuation something is nothing but a chain of a sequence of direct analytic continuation successively ok. So you know so it is a chain of direct analytic continuations, of course I use of the word analytic continuation I also used the word analytic extension ok in the previous lecture. So that is also often used, so we inception saying direct analytic continuation be of a direct analytic extension.

And indirect analytic continuation is that you can also use the word indirect analytic extension ok. So what is an indirect analytic configuration it is a chain of direct analytic continuation. So you know it is something like this so you have a sequence of domains of course in the diagrams that I am drawing I am I am showing bounded domains ok but they need not be bounded ok.

So you know you have the D1 so you have D alpha 1, F alpha 1, so you know I use so let me write it like that D2 alpha 1 F alpha 1 is the first part which consists of an analytic function of alpha 1 on this domain is the D alpha 1 and that going that has direct analytic continuation to the D alpha 2 F alpha 2 which is define here and then that further has a direct analytic continuation to D alpha 3 F alpha 3.

And so on and finally I end up with D alpha n F of alpha 1 ok. So the point about this is that every every successive there is a direct analytic continuation in the previous pair ok and the point is that every successive pair is directly analytic continuation of previous and the next ok. But there is no relationship between one pair and another pair which is not its predecessor.

And successor ok. so you know D alpha D alpha 1 F alpha 1 if I try to look at D alpha 1 and F alpha 1 and D alpha 3 F alpha 3 there is D alpha 3 F alpha 3 need not be a direct analytic continuation of the D alpha and F alpha because to begin with they need not even intersect as I drawn in the picture ok, of course the fact is that if they intersect and if the intersection is has some intersection common with their second pair.

Then of course they will all blow up to give a single analytic function and D alpha 3 F alpha 3 will become a director analytic continuation of D alpha and F alpha ok. But the problem is that possible these two need not intersect the second problem is even in the intersect the intersection need not have anything in common with their intersections with the intersection of these two and intersection of these two ok.

So you know you could have something like this you could have situation like this ok, so here is D alpha 1 F alpha 1 and hear that is here and that has direct analytic continuation D alpha 2 F alpha 2 and then you can have direct analytic continuation D alpha 3 F alpha and this is a direct analytic continuation of this which means F alpha and F alpha to coincide here.

This is the direct analytic continuation of this that means F alpha 2 and F alpha 2 coincide here and this is the direct analytic continuation well arranged this need not be direct analytic continuation but these 2 need not be direct analytic continuations ok. So such a thing can have. So this is not a function it goes back you probably come back to the you come back to you analytic continuations once.

Then you again further analytic continuations and then end of the function which has which on the which has some region in how many starting domain, but the function may be F alpha 3 and F alpha 1 is not coincide on this intersections ok. So the question is this is a this is a strange thing that happens. So what actually happening is the following what is actually happening is that if you have chain up.

So in this case place you are having a chain alpha where direct analytic continuations, so you have indirect analytic continuations D alpha 3 F alpha 3 is an indirect analytic continuations of D alpha and F alpha 1. Then the question is of course what is how is F alpha 3 related for F alpha that is our question ok. I told you F alpha 3 and F alpha 1 need not agree on situation ok so more generally the question is I have D alpha and f alpha 1.

I start with a particular pack here and then I do an indirect analytic continuations and I finally end up final pack D alpha n F alpha n, my question is how is f alpha n related a performer ok, how is it related. So the fact is the answer is direct is that following the answer to that is well this and this are not totally ended with it, a branches of multi value of analytic function ok.

So like for example this could be a logarithm the branch of logarithm but then this could be some other branch of logarithm ok. So for example in this case you know this could be a branch of logarithm and then when I come back this F alpha 3 could be different branches, this can have if you think of the Riemann surface of log Z is what is happening as you move along the sheets you move from one branch to next branch.

And if you take its image on the comp explain you will see that as you go on one your branch of the starting branch of logarithm after I go around once the odd ones around origin the new branch that you get new function you get with indirect analytic continuations in the old branch, the first one that started with is a new branch of a function.

So the reason why we study indirect analytic continuations is that is allows you to move from one branch of a function to another branch of a function ok. That is the important of studying this. So well I told you that ther is one way of looking at this way of formulating is and that is in fact there are 2 ways of forming indirect analytic continuation one is this as a chain, the other one is to use power series ok, so that is what we were trying to look at last class.

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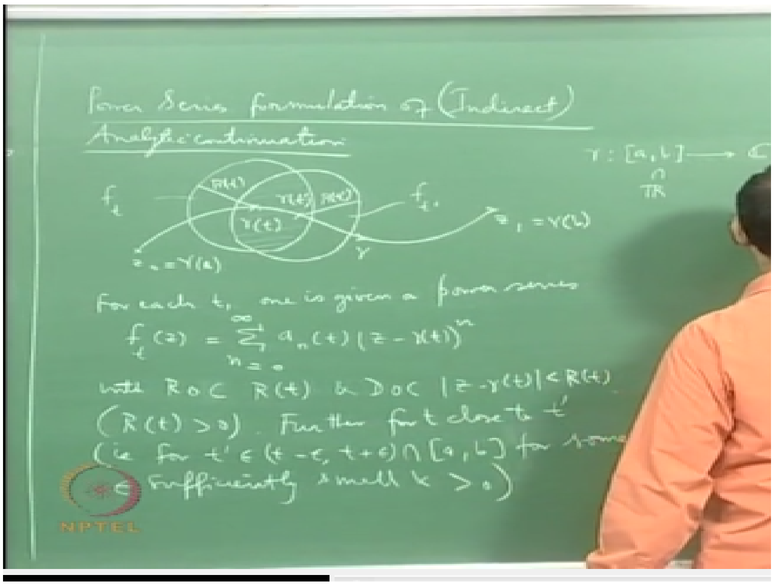

So power series formulation of indirect analytic continuation, so what is power seies formulation it is a it is done like this. So basically the idea is that you know you have start the point z0 you along the path to z1, so this path is gama alright, so gama gamma is a path you have a close interval A B on the real line which if you want you can take it is the unit interval 0,1 ok and you have continuous function from this to become restless ok.

The image of gamma is a path with z0=gamma of a and z1=gamma b this is the starting point to ending point. This is a path alright and the idea is that you know if give me a general point here it is given by gamma of t where t is a point of interval real interval a, b ok and what you want is that at this point you know we want to look at function which is analytic at this point and given by a power series ok.

So what is happening is that you are looking at for each t one is given analytic power series F t of z which is given by a sigma n=o to infinity an of t z-gamma t to the power of n with videos of convergence or radius of convergence R (t) and disc of convergence mod z-gamma t this is less than R (t). So so I have for each t I am given a power series of course so I am giving for each the

I am giving you a power series for each t I am giving a power series and the power series so I have to put F of sub t this is a power series enter a gamma t so it is an expansion in terms of z-gamma t aright with some questions in the questions also depend on t right well if you put t to the t0 that means you are choosing a point in this interval so centre point here and then F ft not z will be a particular power series ok.

And the assumption is the radius of convergence is R (t) is positive ok and the disk of convergence is is given by the disk centre at the centre of the power series and radius equal to radius of convergence of the series ok and of course you know in all this situation you must remember that we are really not interesting the case when radius of convergence is infinite ok.

They are only interest in the case of radius and conversions at this time. Because the radius of convergence infinite it means it is entire ok it means he maximum extension is on the whole complex plane and it the given function is actually a restriction of entire and there is nothing to is nothing complicated happening you just your function is just a restriction of an entire.

So that is not the situation we are interest in that as a very clear to macro environment ok. We are interest in function which have finite discs of convergence namely whose radius of convergence of finite and wires of functions important their important because on the border on the under circular conventions there are similar ok so if a function represented by power series take a power series .

If it has a finite radius of convergence on the set of convergence that is one point this is a single point function. Because no point of singular point of function then I can extend the function to an analytic function to a disc larger which contains a set of convergence and that contract is the very definition one set of ok so why we are interested in functions with you know finite radii of convergence is for this reason.

Because it gives you all the functions which have similarities ok which you can study right. So **so** the situation is that I am so far every point I am give I am giving you a path is ok and we would like this power series we would like the power series to very continuously with respect to b ok that is just like a gamma is a continuous function of t ok which means that this is a continuous path alright.

You would also like F t to continuously with respect to t and one way of getting that is that you know if what we can what you can say is that if you take the t prime was t on the interval you can required that gamma power series F t and Ft prime they are 1 and their present the same and function on the intersection of the tools discuss ok. So which means that you are you t prime close to the

You are assuming that Ft prime is direct analytic continuation of Ft ok. So you know so the additional condition that you are going to put the following if you take suppose this is gamma t prime which is close to gamma t this radius of convergence R t if I take gamma t prime I get another disc here and it will have radius of convergence of R t prime ok and what I am requiring is that for t prime close to p.

If t prime is close to t then gamma d prime close to gamma d is gamma d because gamma is continuous ok and I am requiring that on this disc F t is define and on this disc F t prime is define and what I want is that I want that F t and F t prime are direct analytic continuation of each other in this common region that is F t should be the same as Ft prime in this intersection ok.

So further for t close to t prime which should be thought of with t prime belonging to epsilon neighbourhood Ft in the on the real life that is for t prime belonging to t-epsilon t+ epsilon intersection a, b is what it means ok t prime epsilon neighbourhood of T epsilon for some epsilon sufficiently small and positive ok that is what t prime close to t means or d prime means ok we must.

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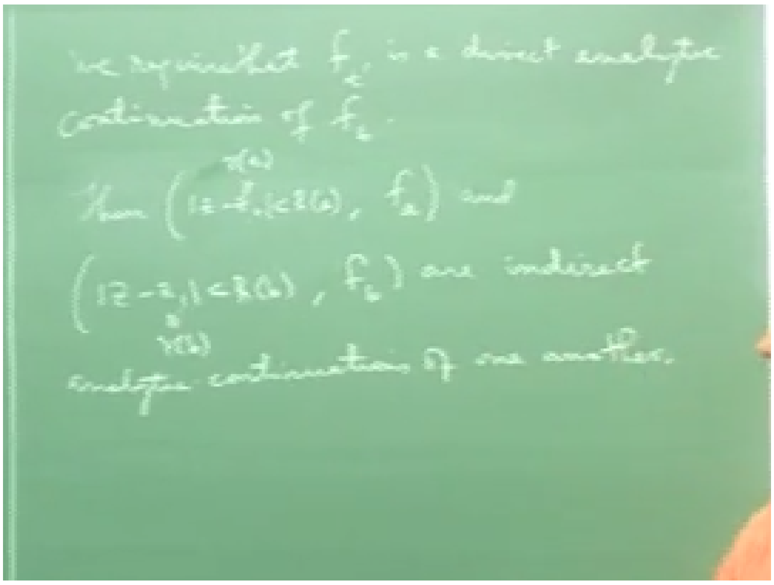

So let me continue here we require that F t Ft prime is a direct analytic continuation of Ft ok so this is the continuation, so this is the condition that you say that you know so what is happening is that as t move from A to B the point gamma t start from z0 and ends with z1 and I as you move close so at every point you are giving me your power series ok.

And if you take nearby points the analytic functions represented by the power series are one in the same on the intersection ok. So what you are giving me is analytic functions which are you are give me a sequence of analytic functions that are moving along this path and in such a way that locally they are their present the same ok and so we say that so that the point I want to make is at formulated like this**.**

Then you know the if you take F sub a which is analytic function at z0 which is gamma a that if you look at that and if you look at sub b which is analytic function and gamma b of gamma b which is z1 I came that they are indirect analytic continuation of each other ok, so then f then this pair which consist of what z-z0+then R (a) z0 is remind you gamma of a, F a of and the initial one and the final one z-z1 less than R(b) z1 is the same as gamma b.

And F b are indirect analytic continuations of one another. They are indirect analytic continuation that is features ok in the sense of the earlier definitions so I am just saying with the new definition and I am just trying to prove that this new definition which involves continuously varying power series that that is the same as origin of definition and prime at the same I mean by saying that I am trying to say that this is more you know stricter formulation of this which involves in analytic expression.

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