Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-20

General or Indirect Analytic Continuation and the Lipschitz Nature of the Radius of Convergence_2

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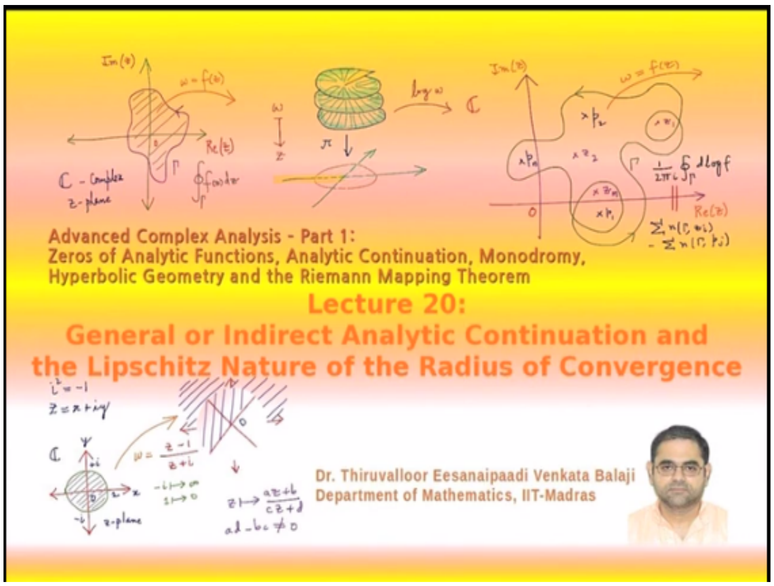

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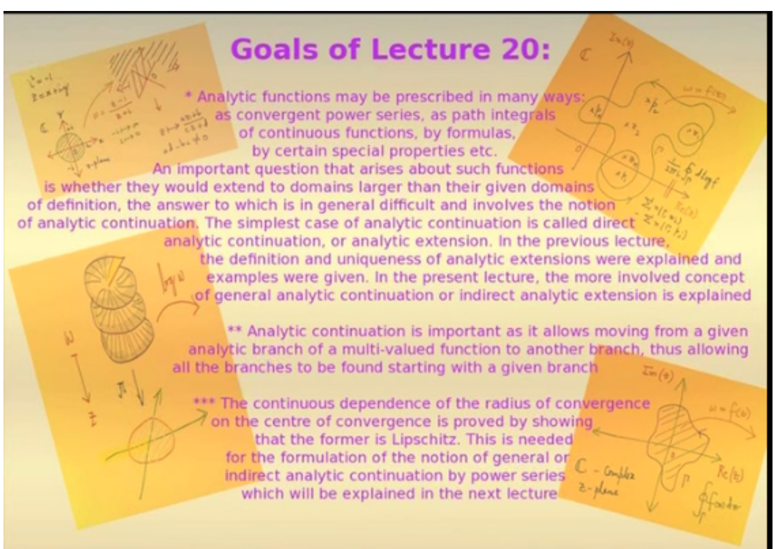

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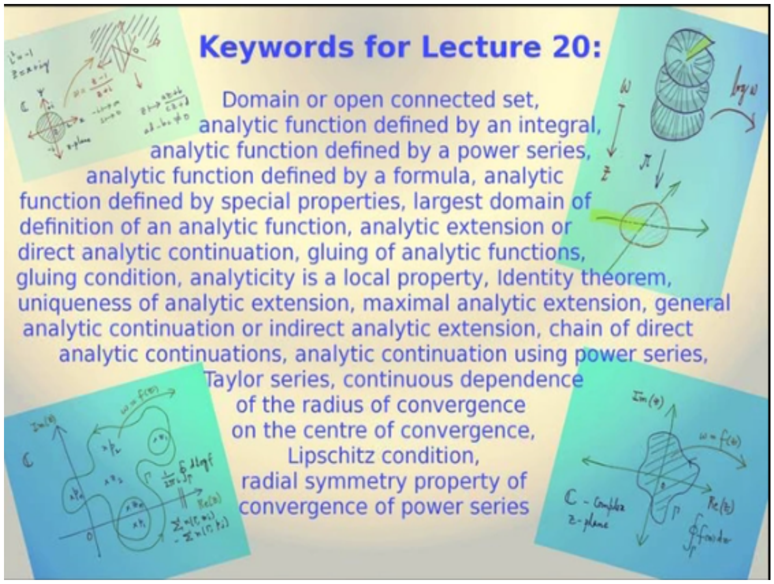

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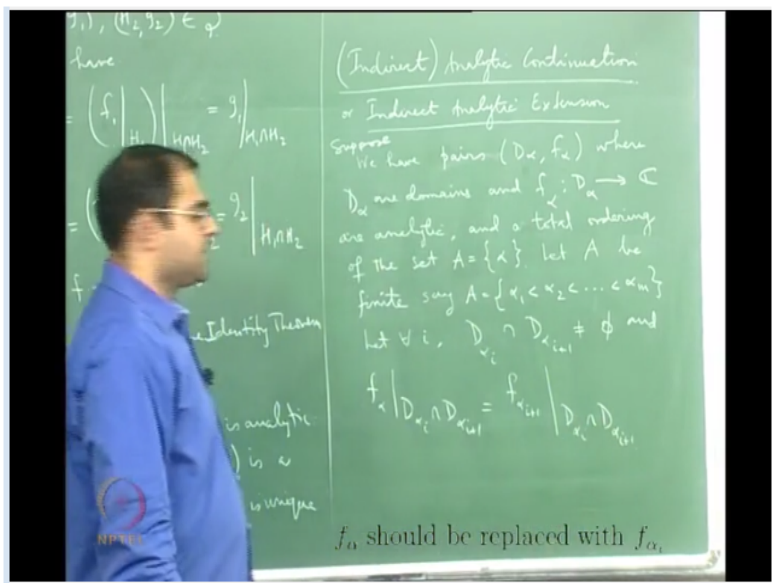

So, here is the definition so, indirect analytic continuation or indirect analytic extension so, it is this topic that is which is which provides lot of variety. And gives you some new things okay so, what is this indirect analytic continuation. So, the idea is a following so, what you do is you have so, in the case of direct analytic continuation what happened is you have to two domains which intersected and you had functions.

And analytic functions on this domains which agreed on the intersection okay now what you do is that you do extend it from 2 to any finite number okay. You extend it from 2 to any finite number but do not insist that all the sets in your collection do not insist that the all intersect. But you only insist that you know if there is an ordering every member intersects with the next.

So, what you do is you do something like this so, you have do we have domains so, we have pairs suppose we have pairs D alpha f alpha where D alpha are domains and f alpha from D alpha to c are analytic. And an ordering a total ordering of the set a is equal to a set of all alpha okay. So, you know you have a collection of pairs D alpha, f alpha indexed by alpha running in an index set capital A alright.

And these pair each of these pairs consist of a domain namely an open connected set and an analytic function on the domain right. And on this set A you have total order a let A finite for simplicity okay say A equal to alpha1, alpha2, alpha m okay. So, A is given like this okay. And I put this strictly less than to say that there is an because of total ordering okay there is since there is a total order .

In fact if it is finite then total order is always there alright but the point is why I want the ordering is a following. So, the point I want is let for every i D sub alpha i intersects with D sub alpha i+1 and f alpha press into D sub alpha I intersection D sub alpha i+1 is equal to f alpha i+1 restricted to D alpha I intersection D alpha i+1 okay. So, suppose I have this situation right.

In fact I could have taken the set as 1, 2, 3 etc... m and then I could have simply called it as D1, D2 etc... D1 I could have simply called these pairs as D1,f1 D2,f2 etc. Dm, fm it should have made it is easier to write down. But the reason why I am doing this, this is in practice this parameterisation could be even continuous not even discrete. And I am going to think of it as coming out of path alright.

So, but I am looking at only here I am only you looking at the finite case so, the situation is like this. So, the situation is so, the diagram for this will look like this okay. So, you know the diagram will look like this.

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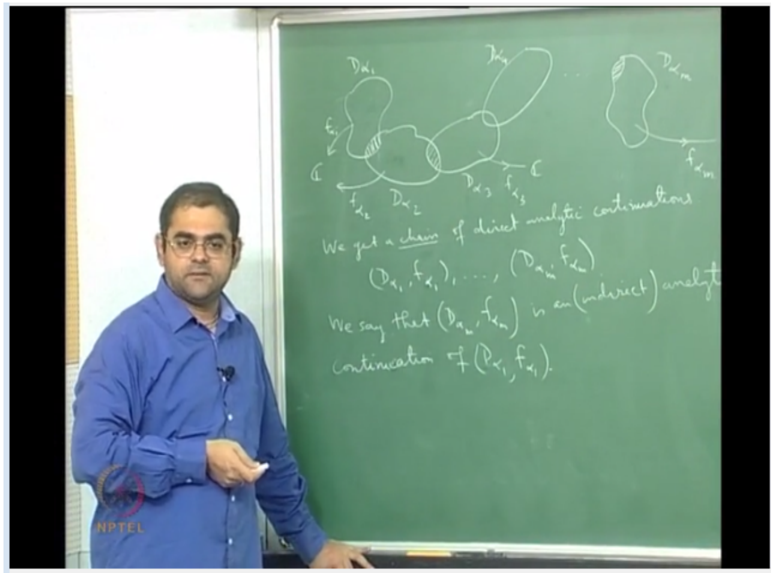

So, we have so, here is D1 so, this is D alpha 1 then I have this is D alpha 2 and they have a non-trivial intersection here. Then well D alpha 3 may or may not intersect D alpha one but has to intersect D alpha 2. So, it will be something like this so, here is a intersection and then you have something here which is D alpha 4 and so on. And it goes on like this until I end up with D alpha m which has some intersection of the previous remember okay.

And what is happening is that I have function f alpha 1 which is analytic on D alpha 1 with values in c. And I have this function f alpha 2 which is a analytic and D alpha 2 which is analytic on D alpha 2 and it is complex valuehood function. And on this intersection f alpha 1 and f alpha 2 the so, here it should be alpha I and they should be alpha i+1 right.

And of course I will have to take i from 1 to m-1 okay for this to make sense. So, **so** I have this situation so, you can see what is happening D. In other words what you are having is a chain of direct analytic continuations okay we get a chain of direct analytic continuations. So, D alpha 1 f alpha 1 D alpha m, f alpha m. This is a chain of direct analytic continuations namely this direct.

The pair D alpha to f alpha to is a direct analytic continuation of D alpha 1 and f alpha 1 okay. Then the pair D alpha 3, f alpha 3 is a direct analytic continuation of d alpha to f alpha namely f alpha 3 and f alpha 2 will coincide on this intersection. And this goes on alright so, every pair is a direct analytic continuation of the previous one okay. And it is also direct analytic continuation of the next one okay.

So, you have a chain of direct analytic continuations and then what we say, we say that the function that you get at the end f alpha m. We say that this D alpha m f alpha m we say it is an indirect analytic continuation of the first pair okay. And usually the word indirect is something that I am stressing but if you see a general literature the word indirect is not mention.

But I am stressing indirect because in general it will be like this okay of course if there are only two then it is there is no difference between direct and indirect oaky. But the reason for calling it indirect is something very strange what can happen is I can start with a pair okay. I can have a chain okay such that the ending pair has a same domain as the starting pair okay.

But the function I get will be completely defined okay so, this is a strain sink that can happen what can happen is I can start with a pair here. I can have a chain but the last member has the uh domain the same as the first member. But the function is different such a thing can happen and you know what is actually happening in this case what I as actually happening in this case is that these function elements the first one that you started with and the last one that you got which is again a function of the first piece okay.

These two are two branches of an analytic function they are different branches you arrive from one branch to another branch in this way okay. That is the whole point you we saw in the previous lectures that you know if you have a general analytic function. If you take if it is derivative does not vanish at a point then you a neighbourhood of that point it is it has an inverse which is also analytic alright.

That is the inverse function theorem it is 1 to 1 and it has an inverse but on the other hand if the derivative vanishes then it is a critical point. But even for the critical even at in a neighbourhood of the critical point we have seen that you can get branches of the inverse function. They are functional inverses not actually inverses because in a neighbourhood of the critical point it will be a many to one function okay.

It will behave like z going to z power m where m-1 is the order of the critical point and you will get m branches for the inverse function okay. And these m branches will leave on a Riemann surface as a single function but the process of going from one branch to another branch if you do it locally that comes via indirect analytic continuation.

And of course you must understand the whenever you have trying to talk about branches of a function. That function is certainly not it does have a singular point and it has a branch cut and so on and so for okay. So, **so** that is importance of indirect analytic continuation okay, that is the motivation for the study of indirect analytic continuation. But this needs to be formula is to using a proper theory.

So, this is a first step you define what is meant by a chain of direct analytic continuations okay we say that D alpha m, f alpha m is an indirect analytic continuation of the first one D alpha 1, f alpha m okay. And the point with indirect analytic continuation let me again repeat and stress is if you start with a particular function and you do an indirect analytic continuation.

And assume you come back to the same domain you started with you will end up you could end up and if you try hard enough you will always end up with all possible branches of that function. So, you can get all the branches of the function okay by this process okay that is the importance of indirect analytic continuation so, the word indirect is usually omitted in standard literature.

But I am stressing it okay so, let us formulise this one would like to treat this in two possible ways that is the two possible ways of proceeding now. And I do both of them the first one is trying to do it using power series okay so, let me let us think of indirect analytic continuation using power series. So, the idea is the following.

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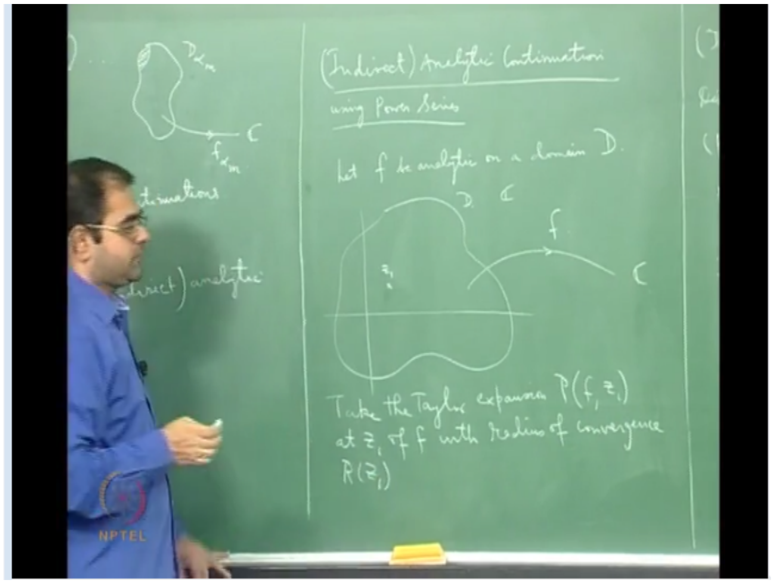

So, let me write that down indirect analytic continuation using power series so, basically what you are doing is this means that you are trying to do indirect analytic continuation where your pairs D alpha f alpha they consist of a domain and a power series which is convergent and represents an analytic function on the domain okay. So, you are trying to get a chain of direct analytic continuations of functions.

You are defined by power series let f be analytic on a domain D okay let f be analytic an a domain D right and so, I will let me draw picture so, here is my complex plane and but here is my domain D. And here is mu function f okay and I can do the following thing if you give me a point z1 in the domain okay. Then the function f is analytic at z1 certainly because it is a analytic on the whole domain.

Since it is analytic at z1 I can write out it is power series centred at z okay. So, the power series centred at z1 as you know will be power series which has certain radius of convergence okay. And that radius of convergence the function the power series will represent the function f within that disc of convergence which is the disc centred at z1 with radius equal to radius of convergence.

And corresponding to this power series expansion so, take the Taylor expansion p f z1 at z1 of f with radius of convergence R of z1 okay.

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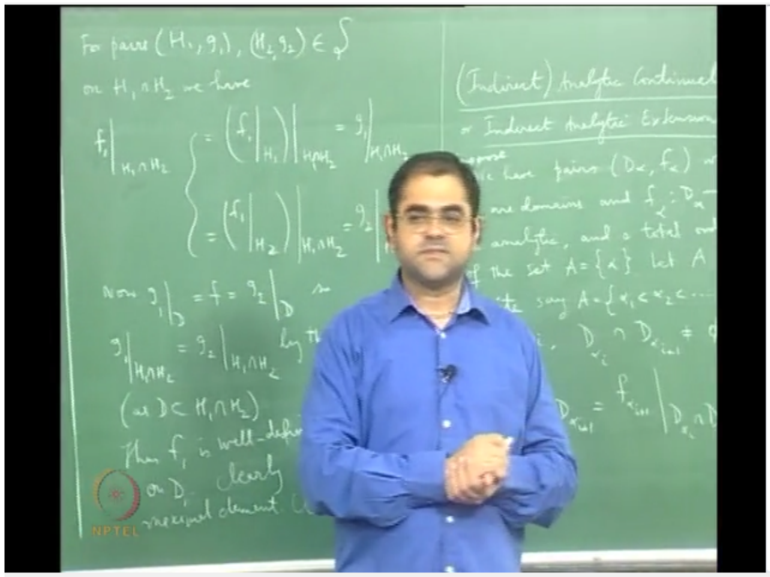

So, you know what this Taylor expansion is of course p f z1 is of z is just you know zigma n equal to 0 to infinity z-z1 to the power of n/factorial n into f n of z1 where f n is the nth derivative of f. This is a Taylor expansion the Taylor expansion and a point Taylor expansion centred at z1 is given by this formula okay. This it is just expanding f in terms of a power series centred at z1 which means at your expanding f in terms of powers of z-z1 positive integral powers of z-z1 okay.

There is a Taylor series of it okay and of course the Taylor quotients are given by the nth derivative, the nth Taylor quotient is just the nth derivative evaluated at the at that point divided by factorial f right. And R of z1 is the radius of convergence of this power series it is a radius convergence okay of course you know I want to look at the case I want to eliminate a silly case.

The silly case is when the radius of convergence is infinite okay if the radius of convergence is infinite it means that extension an entire function okay. Because if the radius of convergence infinite means the disc of convergence is whole complex plane it is a disc centred at z1infinite radius. So, the whole complex planes contain and if you take the that analytic function that will be a maximal extension of this where D, f.

And the maximal extension is a entire so, actually this f is restriction of an entire function and there is nothing special about it okay trying to extend an analytic function is serious only for analytic function which has singularities why do you extend an analytic function. You try to extend an analytic function just to discover where it could have singularities where it could have branch points and so on.

But is the function is entire there is nothing to check it is it extends to unique analytic function of the whole plane and the story is over. We are not interested in that okay so, I am going to only consider the situation where the radius of convergence is finite okay. Assume f does not extend to entire function so, the radius of convergence is always finite for every z1. So, you have that means you are considering a function which has only finite radius of convergence okay.

So, **so** the picture is like this see if you give me z1 then there is some finite disc surrounding z1 with radius equal to radius of convergence or z1 okay. Now the first key point that is very important to this formulation of indirect analytic continuation with respect to power series using power series. He is the fa that you know if I now vary this z1 okay by vary the z1 after all z1 could have been any point in D.

If I vary this z1 then of course these this discs of convergence will vary and the radius the corresponding radii of convergence will also vary. But the key point is this variation in the radius of convergence is continuous okay.

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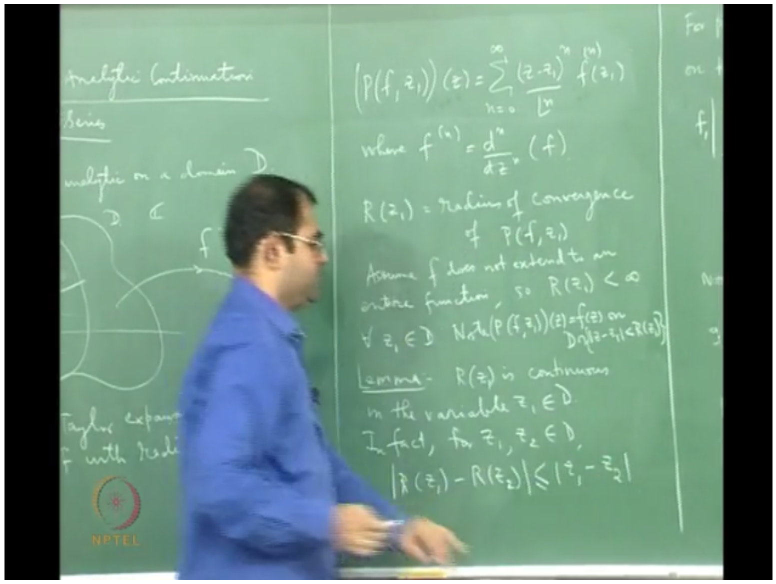

So, **so** here is a fact so here is a lemma R z R z1 is continuous in z1 belonging to D okay mind you I could have use the variable point of z. But if I use the variable point of z I have to put a different variable for the variable of this power series okay mind you that this power series will coincide with f of z on D intersection this radius of convergence. I mean D intersection this disc of convergence.

So, you must that understand that so let me write that note p of fz1 of z is equal to f of z on D intersection mod z-z1 strictly less than Rz1. This is of course going to be true the this if you take the analytic function define by this power series. And you and look at the intersection of this disc of convergence with the original domain you started with there it will give back the function f.

Because after all you know that the Taylor series of a function converges to a function to that function at every point in the disc of at every if the domain of the function okay where it is analytic okay. So, this Taylor series this fn pf z1 as an analytic function is just f when you take the intersection of the disc of convergence and the original domain right.

In other words I am just saying that this is a direct analytic continuation of f it is a direct analytic continuation of f okay but the point is this Rz1 is a continuation function of z1. And so, okay I have this scripts a little to crampt is continuous in the variable z1 belonging to D. In fact for z1,z2 in D modulus of Rz1-Rz2 is strictly lesser than is less than or equal to mod z1-z2.

In fact this is the equation, the equation is R of z1-Rz2 is less than or equal to mod z1-z2 okay. The movement you have something like this you can see very clearly that R is a continuous function. Because given an epsilon positive okay if I fix z1 and make z2 variable and if I give epsilon given an epsilon positive how will I make mod Rz1-rz2 less than epsilon.

I just have to make mod z1-z2 less than Rz2 so, I will have to choose delta equal to epsilon in the l epsilon delta definition of continuity of R okay. So, I can simply choose given epsilon I can simply choose delta equal to epsilon. So, this in equality tells you that trivially the function R is continuous according to the epsilon delta definition of continuity okay. And how does this come about this simply comes about by properties of the uh disc of convergence and radius of convergence.

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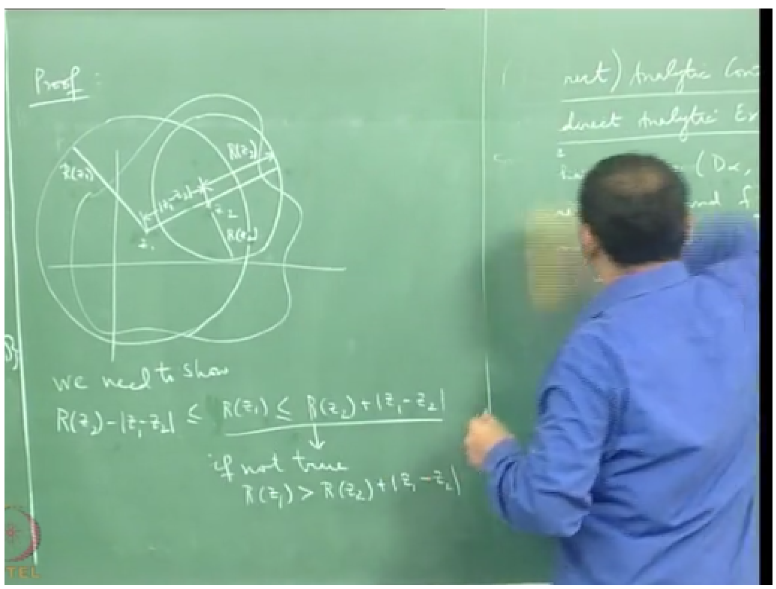

See so, let me quickly indicate the proof of that proof so, you know the so, let me draw a diagram so, I have situation like this. So, I have this D so, here is my z1 and say I have z2. And I have this disc of convergence at z1 with radius of convergence Rz1. So, you know have some disc like this okay and then z2 I have another disc and well this length is Rz1 that length is Rz2 okay.

And well you know if I draw this, this radius this radial line for the this centre at z1 then you know that this is Rz1 this is Rz2 and this distance is mod z1-z2 okay. And we need to show what you need to show to show this is an equality you will have to show that R modulus of Rz1 that is Rz1 lies between Rz2+modulus of z1-z2. And rz2 –of mod z1-z2 this is what will you have to show right this is what you have to show alright.

And I will explain why this is true it is enough to proof this because we are prove this by interchanging the roles of z1 and z2 by symmetry you will also get the other in equality. So, you will have to just understand why this is true okay and why is this true because if you contradict it you will get a contradiction. So, if this is not true if not true what you will get is you will get Rz1 is greater than Rz2+ modulus of z1-z2 this what you get right.

And I claim that this , this is a contradiction this will give a contradiction this is a properties of the so, called radial symmetry of the radial symmetry property of convergence. So, I will just have to use the fact that you know if a power series as a finite radius of convergence. Then on the circle of convergence certainly there is at least 1 point where it will where the corresponding function will have a singularity okay, see the proof of this claim is suppose at every point on the circle of convergence.

Suppose you can extend directly analytically the function to an analytic function then what you are saying is that the analytic function itself extends to a disc which contains a circle of convergence okay and that is the contradiction to the property of radius of convergence, property of radius of convergence is that inside the disc of convergence the power series will converge on the boundary you cannot say anything.

But outside the boundary it has to diverge that is a property of radius of convergence, the radius of convergence is a smallest such number such that I mean it is a number with the property is a unique number with a property that inside the circle of convergence the function I mean the power series converges outside it diverges you cannot say anything on the circle of convergence.

But from this definition it follows that if at every point on the circle of convergence you can directly analytically extend the function. Then this function is analytic on the whole it will be analytic on a bigger set than the disc of convergence that is not allowed on a bigger open set or on a bigger open disc than the disc of convergence which will include the circle of convergence and that is not allowed.

Because outside the circle of convergence the power series is suppose to diverge that is the property of the radius of the function. So, if this is the case then what will happen is that this circle centred at z1 radius R z1 will certainly contain the circle centred at z2 with radius rz right this is what you remember, you see this distance is mod z1-z2 okay, this distance is mod z1-z2 .

So, if I draw it like this, this is mod z1-z2 and this remaining distance is Rz2 okay, so this distance is mod z1-z2 this remaining distance is Rz2 their sum Rz2+mod z1-z2 okay. so, this sum is Rz2+mod z1-z2 and if Rz1 is greater than that okay then it means that this bigger disc the disc centred at z1 radius Rz1 contains this smaller disc okay, but then it means that the analytic function the power series centred at z2.

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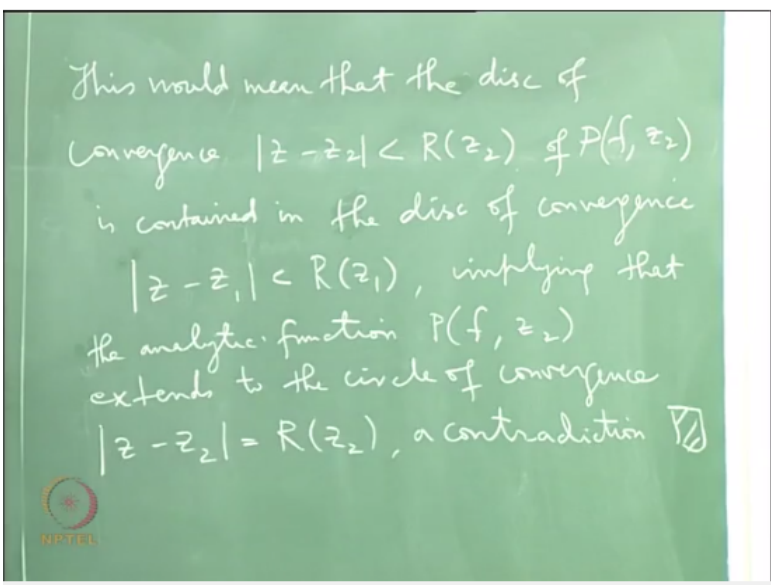

That analytic function extends beyond the circle of convergence which it should not okay do you understand that. So, let me write that down ad that will give you the proof. This will means that will be the disc of convergence mod z-z2 less than Rz2 is contained in of power series of f centred at z2 is contained in the disc of convergence mod z-z1 lesser than Rz1 imply that the analytic function Pfz2 extends to the circle of convergence mod z-z2=Rz1 a contradiction and that finishes here okay that finishes okay.

So, if you contradict this unique quality what will happen is that one of the discs lies inside the other okay and then it will tell you that the power series in the smaller disc is extendable to an analytic function even on the boundary of the smaller disc. But there is not suppose to happen if an analytic function if you take the analytic function defined by a power series and if it does finite radius of convergence.

Then on the circle of convergence there has to be at least one singularity okay that is because the circle of convergence is suppose to be defined uniquely as the circle inside which the power series will converge always and outside which the power series will always diverge okay. So, the fact is that property tells you that you get this re quality and if you interchange z1and z2 in this inequality by symmetry you get this inequality.

And both put together you get this inequality and this inequality tells you that the radius of convergence at each point is a continuous function on the domain okay and this lemma is the crucial starting point to define and to treat analytic continuation using power series okay, so I armed with well this lemma I can make definitions while do that in the next lecture right.