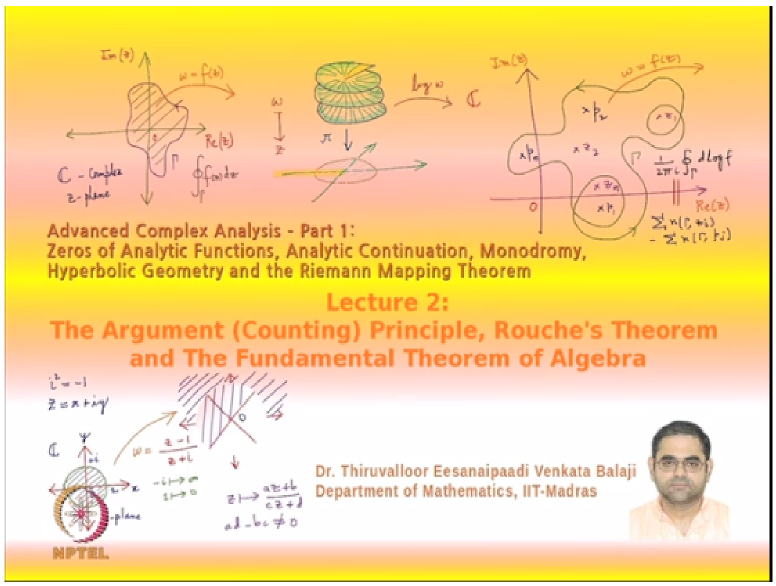
Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

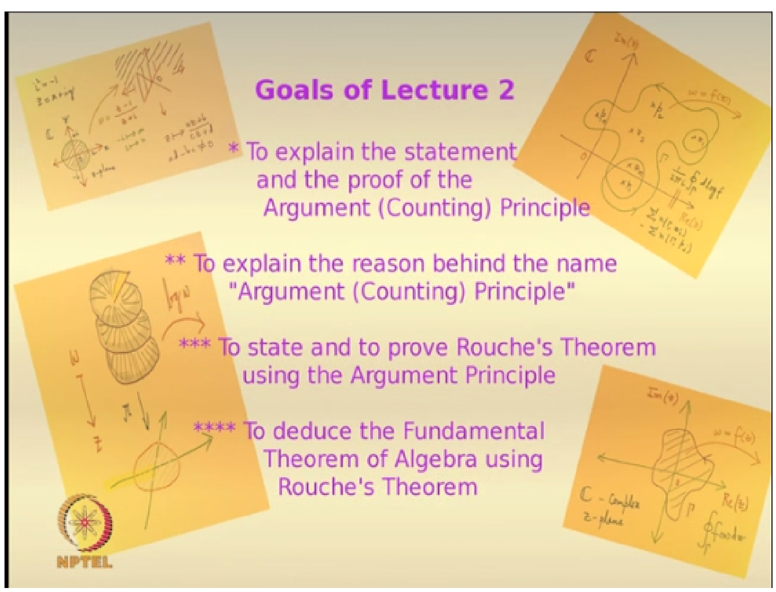
Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology-Madras

Lecture-02
The Argument (Counting) Principle, Rouche's Theorem and The Fundamental Theorem of Algebra

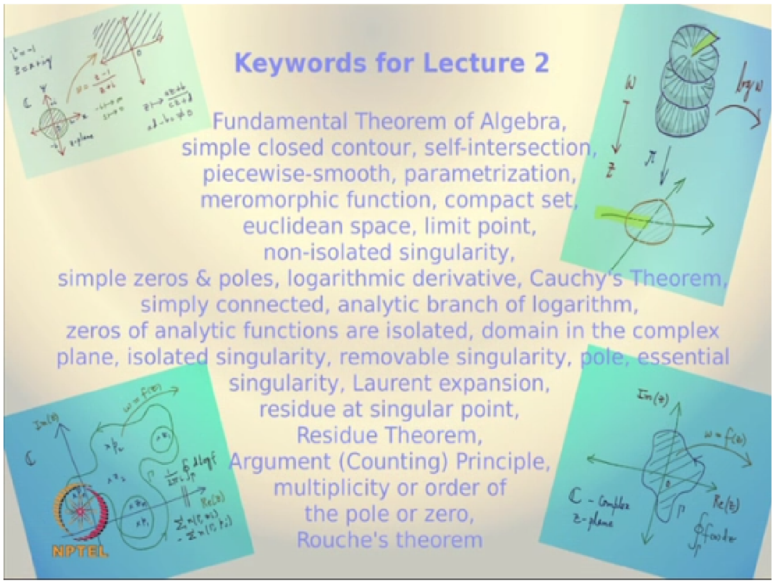
(Refer Slide Time: 00:05)



(Refer Slide Time: 00:10)

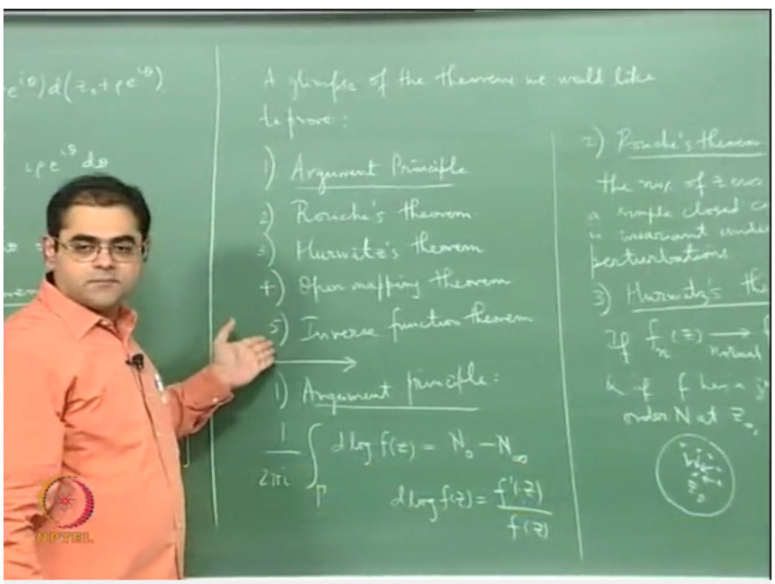


(Refer Slide Time: 00:19



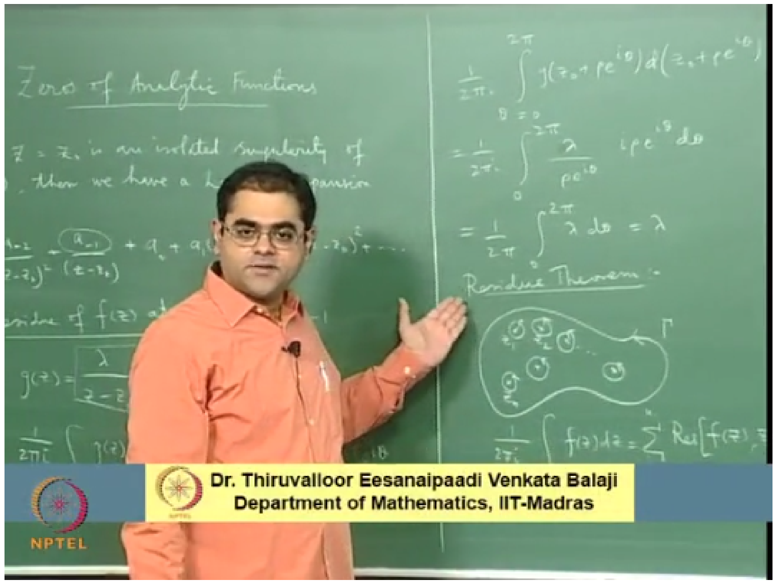
So, let us continue with our discussion, so you know basically we are starting zeros of analytic functions.

(Refer Slide Time: 00:35)



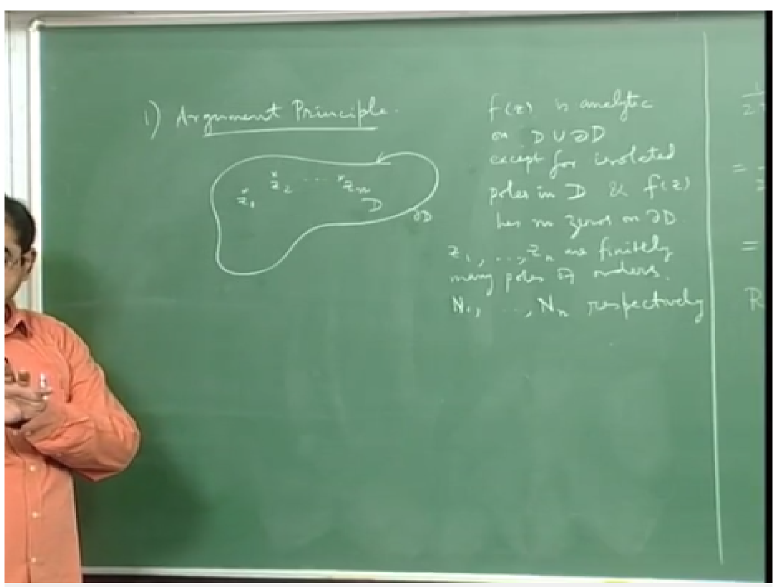
And our aim is to begin with the argument principle.

(Refer Slide Time: 00:40)



This is essentially correlative residue theorem and then try to prove some of these important theorems with the Rochia’s theorem, Hurvitz's theorem, open napping theorem and inverse function theorem. So, let me begin with the argument principle so let me start here. Of course you can look at a proper proof of the argument principle in any standard book on complex analysis.

(Refer Slide Time: 01:04)



But I keen try to tell you how you get it, so basically we are looking at so you are basically having a you know a contour the simple closed contour here, so simple of course means that it is it does not cross itself there are no self intersections and the and when I say contour it is piece wise smooth okay, so the when you write a parameterization of this contour.

Then the considered as a function of the parameter it is continuous and the and this differentiable with respect to the parameter and the derivative of the respect to the parameter is also continuous okay and this should happen piece wise alright. So, that is what the simple closed contour is and we have looking at a function f of z which is .

So, you know I will call this domain as the interior I call does D and I will call the contour is tous of D, so the partial tou or dell depending on what you use to this tou D is always the boundary of D and that is a boundary contour and f of z is assumed to be is analytic in on tou on the union D union tou D except for isolated poles in D okay .

And of course that has no f of z has no zeros on the boundary and f of z has no zeros on the boundary okay, so this is as same this is as assumption, so which means you see there are **are** so **so** there are poles the isolated poles they are isolated somewhere it is and the fact that there are isolated poles already means that there are only finitely many of them okay .

And so there are points z1 z2 etc., zm which are z1 through znr finitely many poles of orders well if you want n1 and so on so let me use capital N1 okay N's of n okay respectively okay and of course we must understand that the fact that there are only finitely many poles is a constant as (() (04:45) because you see you already assumed that there are only isolated poles.

And you know functional shares only isolated poles is called meromorphic function that is the language that we use the meromorphic function is a function which the only similarity it has are isolated poles okay in the reason where it is define. So, of course it is not defined at a at the post one places where it has similarities they are suppose to be isolated poles okay such a function is called a meromorphic function.

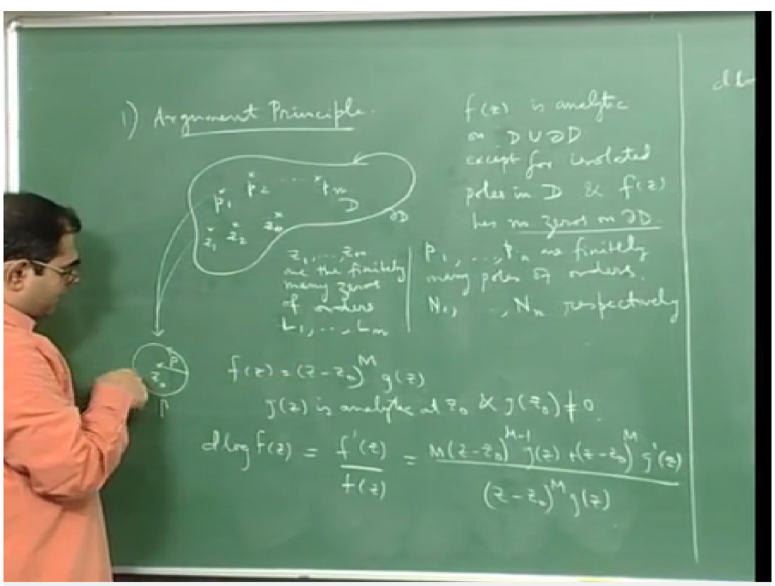
So, basically f of is a f is a meromorphic function but on the boundary it is analytic and it is never 0 on the boundary okay and see there are that it has only isolated poles in this region in this domain D follows from the fact that it follows from the little bit of the topology see because if there are this region D along with the boundary becomes the compact set okay D along with the boundary becomes a compact set.

And you know since u are in Euclidian space in a compact set if you have infinite sub set they will always be accumulation part okay. So, if the number of poles is is in if the number of isolated poles is infinite okay then if you take the sub set of poles that becomes the infinite sub set and that been inside a compact set is certainly have a limit point okay.

And that limit point certainly will also be a pole okay it is certainly not be a point of analyticity and then you end up with I mean you will get a contradiction to the fact that all this that limit point will also be a singularity and you will get a contradiction to a fact that all the singularities are isolated because in every neighborhood of that point you will have similarity.

So, it start isolated you will get a non isolated similarity okay you assume that there are only isolated similarities and their force. Then because of compactness there go finitely in poles and as so **so** these are poles and maybe is a good idea to use .

(Refer Slide Time: 07:09)



Let me use p1 to pn to be poles okay, so let me use p1 to p1 pn and then of course similarly there are only finitely mean zeros okay. The reason is because I have already told you the zeros of an analytic function are also isolated okay, the zeros of an analytic function are isolated this is a I mean this is a result of that you should have come and cross in a first quotient complex analysis.

And what is the reason there the reason is actually identity theorem if you want okay see if you have a 0 that is not isolated okay, it means that in every neighborhood of the 0 you can find another 0. So, you can find a sequence of zeros which go to the 0 okay and therefore what happens is that your function is 0 on a set which has an accumulation part and identity theorem with then tell you the function has to be identically 0 okay.

If a power and that again also is true if you look at a power series you have power series and if it vanishes at a point and it vanishes at a point in neighborhood of that point that means you have a sequence of points where it vanishes and finally tends to a point the sequence tends to a point where also it vanishes then the power series has to be completed the 0 series all the quotients will vanish okay.

So, the only way in which you can have a non-isolated 0 is that is 0 everywhere okay for an analytic function. In other words if you have zeros will have to be isolated, so the set of zeros will also form a will also be isolated and again the same compact argument will tell you that there are only finitely when zeros and so okay. So, all those by as say z1, z2 etc., z sum m okay, so z1 through zm or the finitely many zeros or of orders l okay, L1 etc., Lm okay .

And of course there are more zeros on the boundary on the boundaries the functions are analytic okay there are no similarities on the boundary. But I am further assuming that there are no zeros on the boundary okay all the zeros are only inside there are only finitely one okay. And of course when you take a 0 or pole you have to count it with multiplicity okay, you have to worry about the order of 0 or the order of the pole alright.

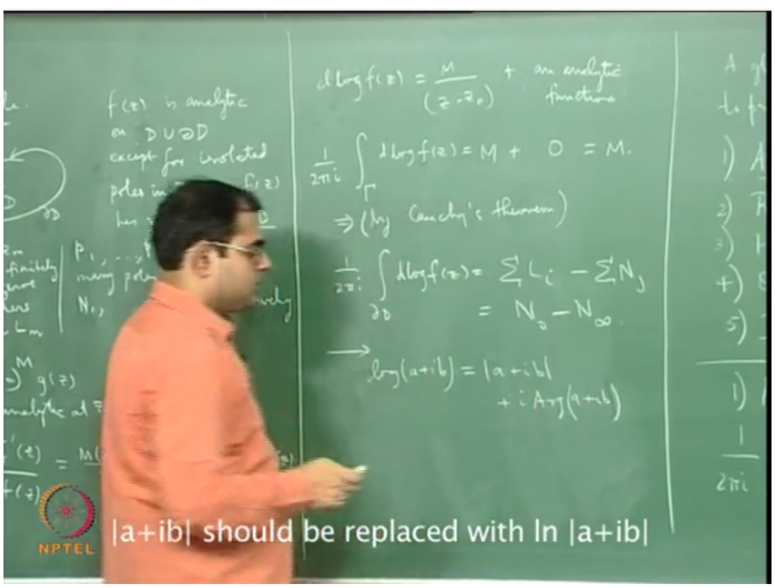
And well in the point is you know if you are looking at a let us assume that you are looking at a simple 0 or a pole I mean you have to looking at a single 0 or a pole okay. So, suppose I have a point let me call just call it as z0 okay and I surround it because it is isolated from the other zeros or poles suppose I surround it by very small disc whose border is a whose boundary is a circle okay.

And again I can a parameter is a circle as I mean if I take the radius to be rho small enough radius then there is no other 0 or pole here okay. And then the f of z can be written as z-z0 to the power of m let me give something m should 1 into g of z okay where g of z is analytic at z0 and g of z0 is not 0. Of course if m is either positive or negative okay, m is positive if z0 is a 0 okay.

And m is negative if z0 is a pole alright if m is positive then m is the order of the 0 at z0, if m is negative then –m is a order of the pole at z0 okay. And now you know if you calculate d log fz okay what you will get is will get I mean this is essentially by definition is f dash of z by f of z this is the logarithmic derivative alright. And if you calculate it what you will get is well if you differentiate this with respect to z.

You use product rule what is get is m into z-z0 to the m-1 times g+z-z0 to the m g dash of z divided by f dash by pie which is just z-z0 to the power of m g z that is what you get it okay. And if you expand it out then what you will get is as follows you will get well.

(Refer Slide Time: 12:28)



And that will turn out to be, so D log Fz will be so you know when I say this I am worried about f only in this small neighborhood of z0 when I were chosen the radius rho very small and where there are no other zeros suppose okay zo is the only 0 output okay. And there is no 0 or pole even on the boundary okay, so I am taking such a small neighborhood, such a small disc .

And that is possible because all the zeros impose they are all isolated okay they can be separated from each other by small discs okay . So, D log fz will turn out it to be well if I calculate this I will get well I will get n/z-z0+ you know here I am going to get an analytic function, see what you must understand is see g the see the function g cannot see the function g cannot vanish anywhere here.

Because is the function g vanish at the certain point that will also be 0 for z okay but I have assumed f has more than zeros outputs. So, in principle g does not vanish at the g does not vanish and at z0 and it is analytic. So, this piece, so the second term is g dash by g okay and g dash by g g0 vanishing at z0 okay is an analytic function okay it has you know if of if analytic function does not vanish at a point.

Then one by that analytic function is also analytic at in a small neighborhood of the point okay, so if you want to get shrink further the neighborhood if you really want okay. So, the fact is that the second term will be g dash by g that is a logarithmic derivative of g and that is analytic okay. And now you know if I integrate 1 by 2 pie i integrate over gamma okay yeah integral over this gamma of D log fz.

If I do it you know I am going to integrate this and you know if I integrate this part that is the integral we have already calculated you will get M which is the residue, so you get M+if you integrate this part you will get 0 because it is Cauchy's theorem says that if you integrate an analytic function you are going to get 0 over close call simply close call, so the net affect is that you get M.

So, what you see is that if you do if you take a small enough circle surrounding a 0 or a pole and you compute the integral 1by 2 pie i of the logarithmic derivative you end up getting exactly the order of the 0 logarithm and now all you have to do is simply surround each of this zeros and each of the poles by set small discs okay and then use the Cauchy's theorem in the region that is gotten by taking away from the these discs where the function f is completely analytic.

And the Cauchy's theorem the integrate over the boundary is sum of the integrals over each of these small discs but the integrals over each of the over small disc cum gives you the number of gives you the order of the pole of the 0 and therefore you get the residue theorem namely you get you so implies by Cauchy's theorem that 1 by 2 pie i integral over the boundary D lof fz actually you give you the number of you will get.

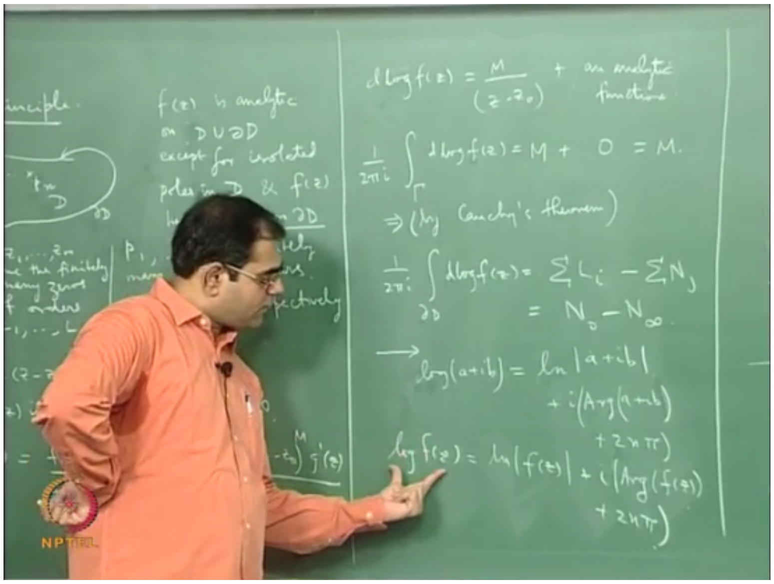
If there are zeros you will get all the Li's, so you will get sigma Li-and if it is a poles you will get the –Li's okay see mind you if what you must understand is if this is the pole then M is negative okay. And when I write N is an order of **of** the pole P1 then this N1 will be this N will be –N1 okay. So, what you get is sigma Li-sigma M Nj which is precisely number of zeros-number of poles, N's of zeros is number of zeros N's of a hideous number of poles okay.

So, this is basically the argument principle okay it is an application of residue theorem+cauchy's theorem. So, of course the question is why z called the argument principle okay, so the answer to that is that this quantity here is actually the chains in the argument of f of z as z varies over it is boundary okay. So, why the reason why is called the argument principle is the following you now me you write log a+ib where a and b are real numbers okay, i is of course always square root of -1, a square root of -1.

You always naively write it as mod of a+ib+ i times argument of a+ib okay, this is how you define a logarithm+ but of course this is not the you know this is a multiple value thing. In fact you can add to this argument you normally add you know +2 n pie okay and various values Ni suppose to give you the various logarithms the only requirement is that a+ib should not be the 0 complex number should not be 0 okay.

And the problem is the when the 0 the argument of 0 is not defined okay, so mod is 0 is of course the argument is not defined. So, it should not be 0, so you of course you get a logarithm for every non 0 complex number and all these logarithms they all differ by integral multiples of 2N pie I mean integral multiples of 2 pie i okay. And in the same 9 way you can write well you can write log fz as you know log mod fz I think I forgotten there is a log here which I forgotten okay yeah I forgotten this yeah forgotten that.

(Refer Slide Time: 19:10)

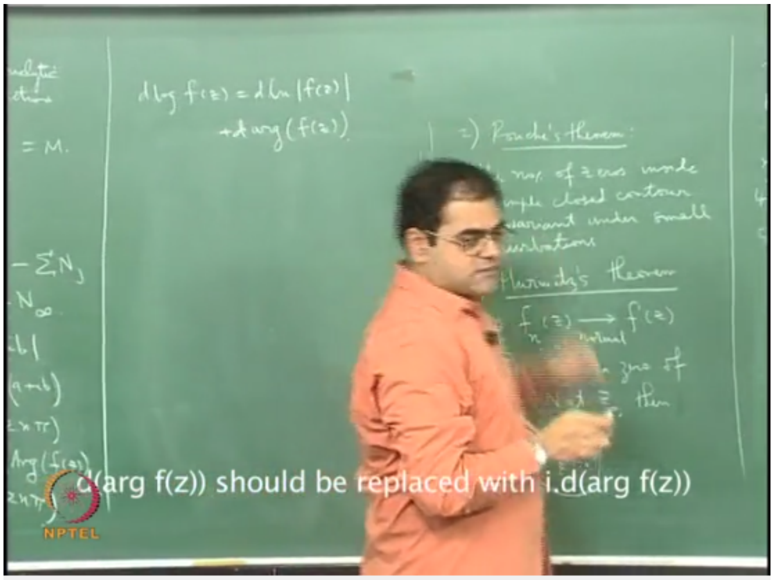


So, this is Ln which is log natural log to the basic and this of course is also log to the basic for that matter only thing is this is the complex logarithm that is the real logarithm to pole through basic okay. So, if you write log fz you write you will get a again naively you will get lan mod f z+i times argument of fz to zn pie you can write something like this okay, but then the there is a this is okay if you fix a particular value of z okay.

But then the problem is that you know if you try to write this uniformly for all values of z in a domain then you might end up in trouble in the sense that you see the problem is that this log fz need to define an analytic function okay. You what you will get is will have to do some kind of slitting of the domain to you have throw away some parts to the domain to get what is called a branch of this logarithm okay.

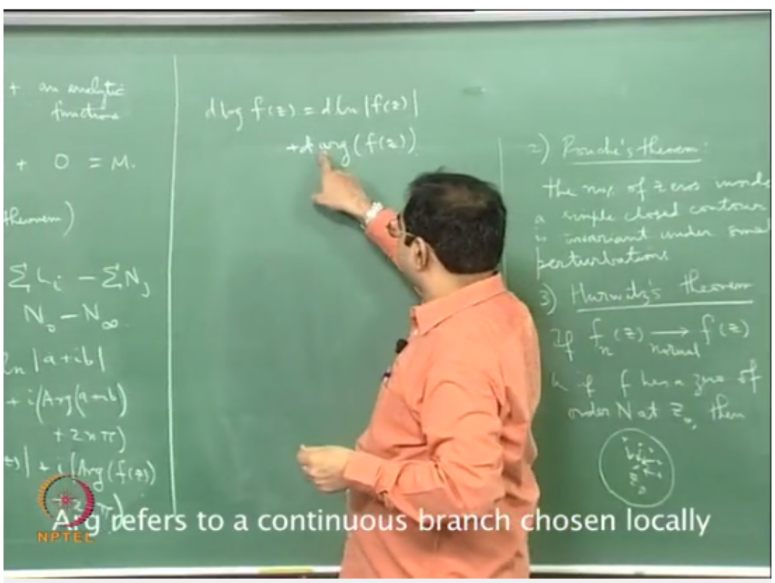
And the only case when you will be able to get a I mean 1 case where you can always get a logarithm will be if your domain is simply connected and the function never vanishes on a domain you can always find the logarithm okay . But the point is that you know.

(Refer Slide Time: 20:46)



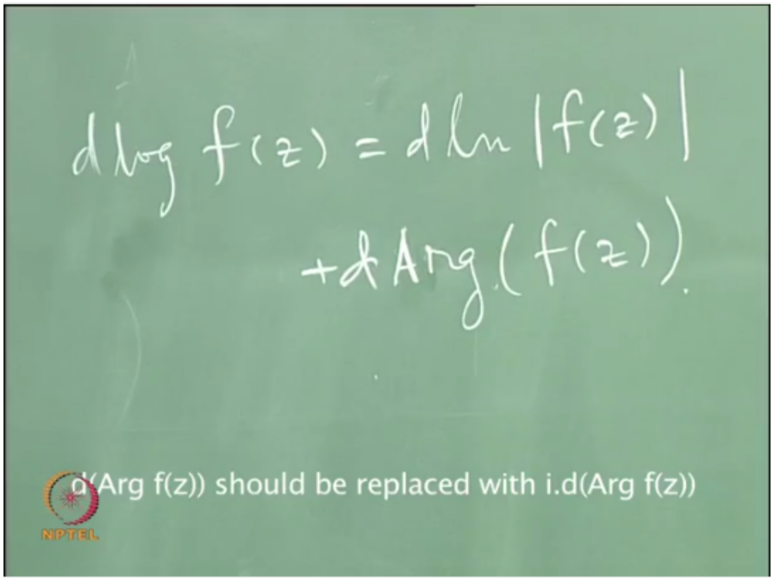
So, you know you can naively write d log f z is d lan mod fz+d you should argument alright you can write it like this the where d is suppose to be the difference as you change z along an arc okay you can change z along an arc and you can make sense of the difference in in these values

(Refer Slide Time: 21:20)



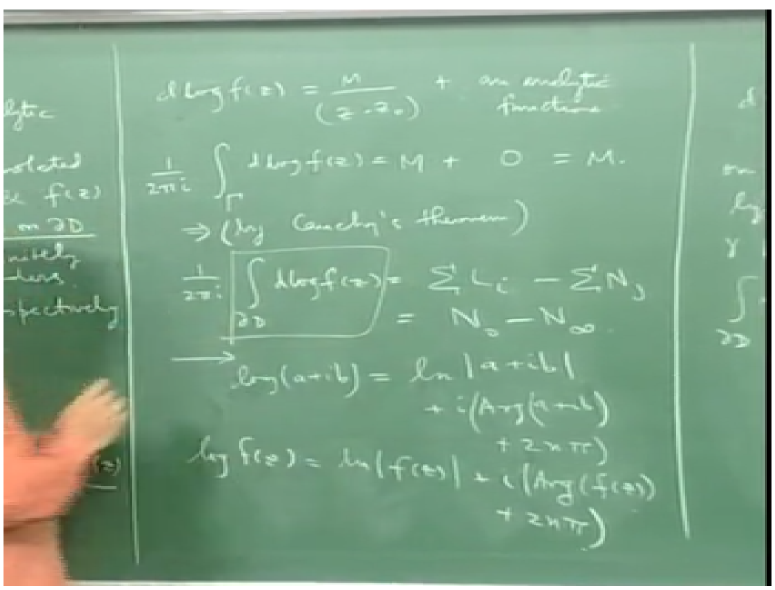
And of course here you choose a particular branch of the logarithm. So, what I do that I rather put a capital A and it should be able to choose a particular branch of that logarithm. So, in particular what I am saying is that you can do this on a nice arc, for example you can do this on this boundary arc you can always do this on the boundary arc you can write that.

(Refer Slide Time: 21:49)



And why you can do that is because on the boundary z can be written as a smooth function is a smooth function of a real parameter okay. So, then you can so you know on tou D if it is which is parameterized by gamma of t alright where t is a t varies on the interval on the real line okay .

(Refer Slide Time: 22:31)

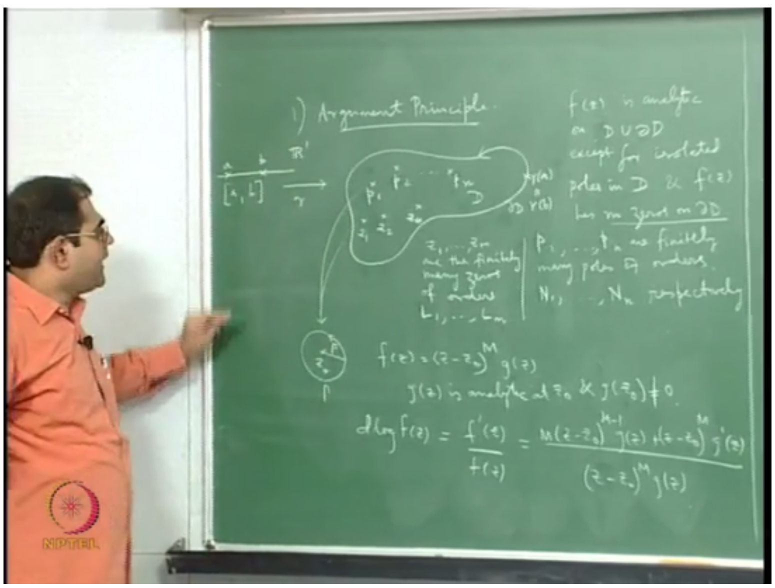


Gamma piece wise smooth which means that you know gamma is a continuous it is differentiable with respect to t continuous with respect to t, differentiate with respect to t and the derivative with respect to t is also continuous piece wise that is what piece wise smooth piece okay. So, gamma is piece wise smooth with respect to t okay what you can do is you can write integral over tou D D log fz as integral from a to b.

If you want integral over gamma of d log f of gamma of t and that will be integral over gamma of d lan mod f of gamma of t+integral over gamma d argument of f of gamma of t okay. There of course here you can choose a particular single valued branch of the logarithm and actually what will happen is , so you know what I am just trying to compute what this integral is what is the logarithmic integral over a single closed contour like this.

If you compute it what will happen is see this part will be 0 okay, the this part of the integral will vanish okay and this part of the integral will give you the difference in the argument of the function from the starting point to the ending point. Of course you know when we do this integration over the contour when you parameterize it.

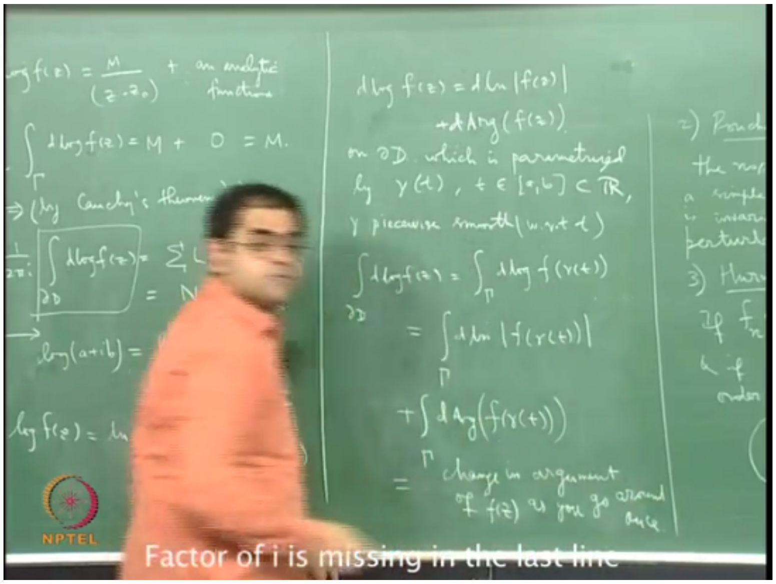
(Refer Slide Time: 24:54)



Then you know you choose some point which you take as gamma of a and that is also equal to gamma of b, gamma is a parameter is if you know path and the parameterization is a map. So, you think of parameterization as a map a, b is interval on the local line I mean on the on r1 and gamma is a function. And of course gamma of t has a real part has an imaginary part.

And for different values of t you are going to get different parts you are going to get different complex numbers as t changes from a to b gamma traces this path and the starting is equal to the ending point. So, gamma of a is same as the gamma of b okay and the fact is that if you go around once like this the integral will vanish okay. And the second integral will essentially give you the change in the argument of f of z as you move across the gamma.

(Refer Slide Time: 26:01)



So, what you will get is see this will turn out to be just f of argument changed in the argument of f, change in the argument of f of z as you go around once along the boundary tou d which is now parameterized by gamma okay this is a change in all that. So, the fact is that if you calculate this you know this logarithmic integral here actually getting the change in the argument okay, you are getting the change in the argument of the of the function alright.

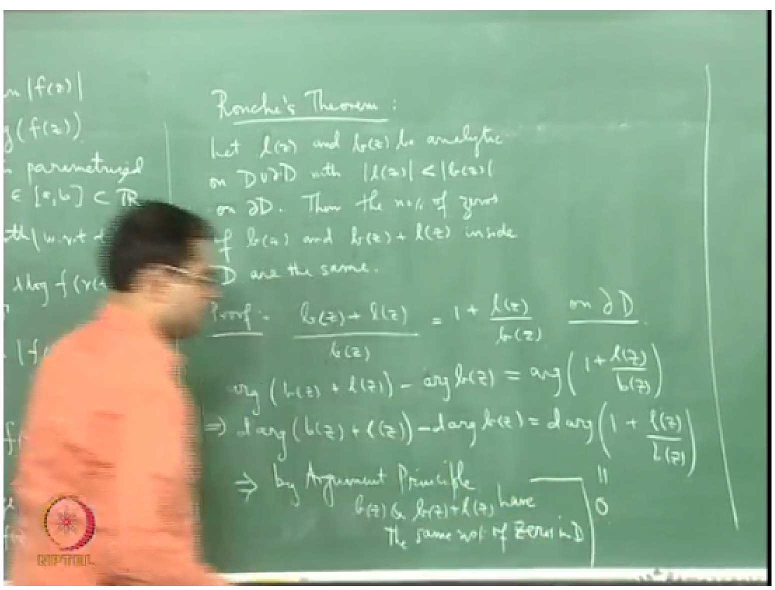
And mind you in all these calculations I had taken Arg to be a fixed determination a fixed branch of the logarithm. If I taken a different branch of the logarithm you know any 2 different branches of the logarithm will differ by a constant multiple of 2 pie i but since you are taking the difference this value will not depend on which branch of logarithm you took okay, instead of taking Arg which is 1 branch.

Suppose I have taken r prime which is some other branch okay then Arg and r prime will differ by say is up to m pie. But when you take the difference the 2 M pie will cancel out, so this quantity which is a change in the argument that is not going to change okay. So, I mean roughly what you must think is that what you must try to understand is that I when you calculate this logarithmic integral over the boundary you are actually getting the change in the argument of the function okay.

And the change in the argument of the function could be 0 or it could be something it all depends on what the function is and it depends very much on the zeros and poles of the function inside that is what the residue theorem says, what the residue theorem actually says is that it says that you see the change in argument of the function is 2 pie times the difference between the number of zeros and number of poles counted with multiplicity.

So, this is the another way of looking at the at a argument principle and this is what lends the argument principle it is name okay. So, when you calculate the logarithmic integral I mean you calculate the integral of the logarithmic derivative or the closed curve what you actually get is a change in argument. And the argument principle says that this change in the argument is 2 pie times the number of zeros-the number of poles inside okay, that is why it is called the argument principle.

(Refer Slide Time: 29:12)



Now the advantage of that is that we can now through Rosche's theorem, so here the Rouche's theorem so f of z and so let me use L of z and b of z the analytic functions on D union tou D where you think of this kind of a diagram okay. So, D is simple closed contour which is why and I mean tou D is a simple closed contour which is piece wise smooth and D is the interior.

And you take this whole region along with the boundary and it is analytic there okay there 2 the both functions are analytic there and with mod L of z strictly less than mod b of z in on he boundary okay. So, you know the small ended the small l and small b are suppose to you must think of smaller as little and small b is big and I am just saying that the bigger one is really bigger than the little one okay in modulus on the boundary okay.

Then what Rouche's theorem says is that then the number of zeros of b of z and b of z+l of z inside D or the same okay, this is Rouche's theorem. So, you just see what it says, it says see I have a function b of z which you think of is a bigger function it is bigger than l of z is a smaller function and what you mean by bigger in modulus is strictly greater than the modulus of l of z on the boundary okay.

Then by adding to bz this little function lz you are not going to change the number of zeros itself you are not going to change the number of zeros itself. And so this addition of this little this lz to that bz is not a very small analytic permutation okay, you can think of it is a small analytic permutation and Rouche's theorem is says that if you the analytic function bz and analytically perturbed it.

The resulting analytic function is not going to have different number of zeros than the original analytic function oaky. So, this number of zeros the count of the number of zeros is (()) (32.06) function. So, let me explain when let me first explain what is the idea of the proof, the idea of the proof is here are 2 functions okay, you want to show that we have the same of the zeros inside your domain alright, it is surrounded by this simple closed curve.

Now of course mind you there are no poles okay, there are no poles here, the functions are completely analytic and of course mind you the function b is has no 0 on the boundary, b has no 0 on the boundary. Because you see on the boundary mod bz is strictly greater than modulus and modulus is certainly greater than or equal to 0, so then you tell you that mod bz strictly greater than 0 on the boundary.

So, b has no zeros on the boundary mind you, b has no zeros on the boundary alright, now I want to show that b of z and b of z+l of z have same number of zeros that is what I want to show how do I show it, how do I use the argument principle. The argument principle says is the number of zeros times 2 pie is the integral over logarithmic derivative and that is also equal to the change in the argument of the function.

So, if I want to show that these 2 have the same number of zeros inside all I have to show is that the change in the argument for both is the same. Because it is change in the argument that counts the number of zeros, so all I have to show is that the change in the argument of this and the change in the argument of that over this boundary is the same okay. So, in other words I have to show that the DArg of the bz and the D Arg of change in the argument of the bz+lz they are the same as you traverse as z traverses along the boundary curve okay.

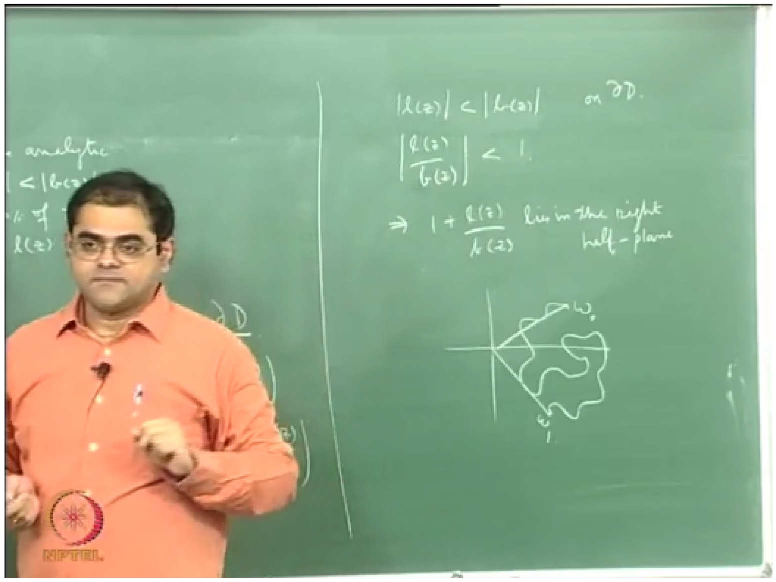
So, that leads you to look at this function if you look at bz+lz by bz okay you will see that of course you know this is 1+l of z by bz okay. And , so you know what this will tell you is that argument of you know the argument of a quotient is the differential arguments. So, you get argument of bz+lz-argument of bz is argument of 1+lz by bz okay, this is what you will get.

And therefore what this will tell you is that you know , so now I want it look at this quantity here. So, **so** this equation is tell you that the change in the argument of bz+lz-a change in a argument of dz=the change in the argument of this quantity and mind you I can divide by the because b does not have any zeros okay. And it does not have any zeros on the boundary, so mind you all this is I am writing this down only on the boundary.

Because I have to compute the argument change that z base on the boundary b may have zeros inside that is not the point, the point is to be done on the boundary. So, I should write this on tou D it does not make senses you take z inside because the z could be 0 of b and then I cannot divide by b of z okay. So, this happening on the boundary where D does not vanish okay.

So, this will tell you that d Arg the change in argument of bz+lz-the change in argument of dz=change in the argument of 1+l of z by d of z this is what it says but the fact is that this is you see**.**

(Refer Slide Time: 36:25)



The reason is you see what you should understand is mod lz is strictly less than mod bz on the boundary okay. So, mod lz by dz strictly less than 1 okay and therefore if you look at this quantity will lie only on the right half plane 1+ **1+**lz by dz lies in the right half plane right I think that is I mean so you see this is a complex number lz by series bz is the complex number that lies in the unit disc alright.

And to that you are adding 1 that means you are translating it to the right by 1 unit okay and therefore it has to go to right of the y axis, the imaginary axis right. So, this lies in the right half plane, so you know basically, so you know so this means see for now the point is that if you look at a argument of this you see the if you have even if you have a variable point let me call it as omega which is say move okay.

Suppose it moves from omega0 to let us say omega1 okay, then you see the if omega1 and omega0 are the same and you are looking at the right half plane, if it is on the same half plane then the change in the argument will be 0 okay. So, **so** essentially what it means is you see if you move from here to here the change in argument will be literally you know this angle and this angle and it will be the this total angle alright.

But if you no matter how haphazardly you go okay but if you come back to the same point okay then your change in argument will be 0 alright. So, what this tells you is that this is 0, the change in the argument is 0 and that will tell you that the change in the argument of b+l is the same is the change in the argument of D as you go along the boundary. But then the argument principle will therefore tell you.

That the number of zeros of b+l is the same as the number of zeros of b inside the region bounded by the boundary, that is the argument principle. Because so argument principle actually tells you that the number of zeros is controlled by the change in the argument. So, to show the true functions have the same number of zeros all you have to show the change in the argument are the same for both functions okay. So, this implies at by argument principle see the whole point is you know why I am saying 1 half plane.

It is because you know it should not happen that the curve should not go around and come back, if it goes around and come back and you pick up the change in argument. For example you know if it went around across the origin and came back then it picks up see the moment it because the argument is measured with respect to the origin right by joining the point to the origin.

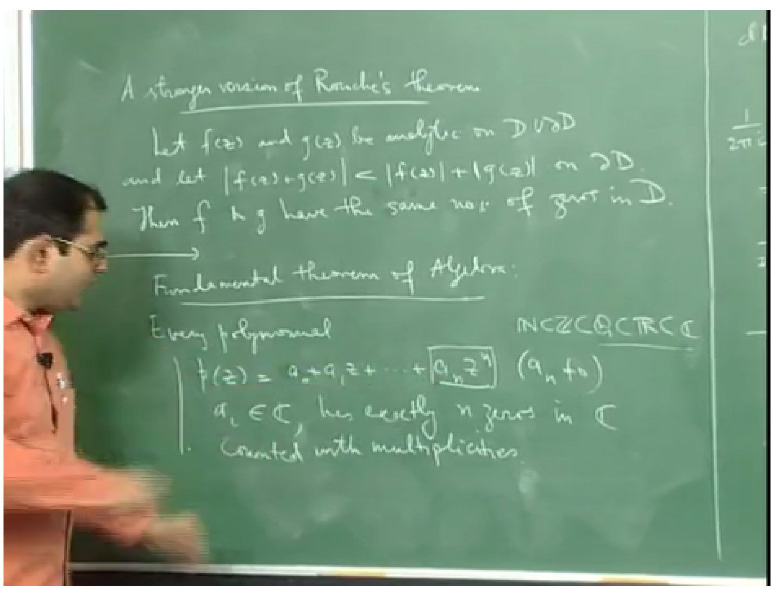
So, if you go around the origin certainly you are going to the argument is going to change by some quantity but if you are on the same side of the origin and the same side of the plane the half plane passing through the origin. The argument is **is** going to independently only going to depend on the initial point and the final point no matter how you move so long as you were in the same half plane okay.

So, that is the whole point that is why I need that it lies in the right half plane okay, I mean I know for sure that is not going to wander somewhere here and then you know come out back come back all the way to this because if you did that I will pickup the 2pie and if you does that twice I pickup 4 pie and if you does that in a different direction I will get -2 pie -4 pies things under such these were happen.

Because it is never wanders outside this right half plane and that is this condition that is because of that condition okay. So, the argument principle bz and b z+lz have the same number of zeros in D okay, so this is an application of the argument principle that is Rouche's theorem. So, there is a you know there is a another avatar of the Rouche's theorem which you can maybe try to prove as an exercise. So, let me rub this part of the board.

But then I mean this what you must understand is that this theorem is pretty powerful, the proof is pretty easy okay. Because you have used basically you have used residue theorem and you have used Cauchy's theorem and so you have used a literally all the results that you have done in a first quotient complex analysis alright. So, it is a very powerful theorem and to illustrate how powerful it is, it is very easy to deduce fundamental theorem of algebra from this okay.

(Refer Slide Time: 42:47)



So, but before that let me a stronger version of Rouche's theorem, let f of z and g of z be analytic on D union tou D and let mof of f of z+gz be strictly less than mod f of z+mod gz on the boundary. Then f and g have the same number of zeros in D okay this is another version. Of course you know mod of f+g is always be less than or equal to mod f+mod j.

But on the boundary it would strictly less than, that is the condition that tell you that will tell you that f and g will have the same number of zeros inside. So, this other version of Rouche's theorem is I have written is a stronger version but probably it is actually an equivalent version and you can easily deduce is an exercise to prove that again sincerely using argument principle.

And this can be deduced from that, that is you can as an exercise also deduce this from that and I think you can also do it the other way. Because essentially it is going to depend only on argument principle okay, so what this tells you is that if you want to say that the number of zeros of 2 analytic functions are same all you need to check is that the triangle in equality becomes the strict inequality on the boundary, that is what it says okay.

So, yeah so you can try that then of course let me give an example as to how powerful Rouche's theorem is, so you can get can get fundamental theorem of algebra, at the fundamental theorem of algebra is the theorem that you take a polynomial 1 complex variable with complex coefficients then all the zeros of the polynomial you can find all the zeros and they are going to be complex numbers okay.

That is precisely the fundamental theorem of algebra, so I mean the reason why is called the fundamental theorem of algebra is because see in algebra what you do is that you try to extend number systems. Because you are trying to solve equations, so you know the people who have courses of algebra you know that you know you start with the natural numbers.

And then you **you** extend them to integers okay and then you extend them to rational numbers and then to real numbers and then to complex numbers and the point is that every time you extended because you not able to solve equations okay. So, a from natural numbers with the counting numbers 1, 2, 3, 4 to the integers you extend because for example if you take the equation x+1=1 the solution is x=0 is not here.

So, you have to include 0 and if you take an equation x+2=1 the solution is -1, so you meet our negatives that is how you have come to the integers and then here you do not get solution for an equation such 2x=3 because the solution x is 3 by 2. So, you go to rational numbers you invert integers non 0 integers and you get rational numbers and then these are all fields okay. And then somehow you get by u move the reason for moving for rational numbers to real numbers is actually to fill all the gaps which are irrational numbers.

So, it is more topological it is, real numbers are kind of the topological and the real number is the topological completion of there of the rational numbers. And then which for example in first course in analysis, real analysis you would have seen is constructed by the method of (()) (46:48) where you define real numbers to be equivalence classes of Cauchy's equivalence of rational numbers.

And equivalence be 2 Cauchy's sequence of the rational numbers as consider equivalent if they if you put them together as a even and odd subsequence of a biggest sequence in that sequence continuous to the quotient oaky. And then the point is that going from Q to R does not help because an equation such like such as x square+1=0 which is be roots of the -1 cannot be solved here.

So, you have to go to complex numbers adding I to the real number system which is a complex number system okay and then the question is now if you have an equation if you have polynomial equation over complex numbers the question is are they going to polynomial equations for which you do not have solutions, I mean the question is we have to further extend it to something bigger.

And the fundamental theorem what it was says you do not have to do it what it says is that you know you now the it is says it the complex numbers are algebraically closed which is a fact that you take any polynomial in 1 variable it 1 complex variable with complex quotients then all the zeros are complex numbers, you do not have to go you do not have to extend the number system further.

So, that is the fundamental theorem of algebra, so every polynomial p of z= you know a0+a1z+e tc., anz power n a n not equal to 0, ai complex numbers has exactly n zeros in c counted with multiplicities. So, this is the fundamental theorem of algebra and the way one proves it is you know it is just using Rouche's theorem I **I** will tell you +in words very you can very elegantly express in words.

So, what you can do is you can get rid of this you know or do not get rid of it and you look at this polynomial know as you make mod z bigger and bigger than the modulus of this polynomial will depend on the leading tou okay. So, in other words if you mod z bigger and bigger that means if you make mod z greater than say a large positive real number R okay.

That means you are looking at the exterior of a circle centered with the origin radius capital R for R large you are looking at exterior and this is called an neighborhood of infinity if you want okay. If you think of complex numbers as sitting inside the Riemann's spear with the pot it infinity on the okay, this exterior over circle is a neighborhood of infinity okay.

And what happens is that for when you go in for mod z greater than capital R or sufficiently large, then you see except for the leading term all the other terms they become very small okay in modulus the leading term will dominate all the other things in modulus okay. So, you know so what it tells you you look at only this function you take such a large R okay and look at only this function okay.

And think of the rest of the terms as a perturbation it is a perturbation because the modus of the rest of the terms is very small and compare to the modus of this because this is the leading term and you have taken modz=R for R very large. So, what does Rouche's theorem going to tell you is going to tell you is the number of zeros inside mod z=R okay that is in the disc mod z less than R is going to be for this whole function is going to the same as the number of zeros of this big function which is a a n z pwer n.

But a n z power n has m zeros a n z power n has automatically n zeros at the origin it is z=0, 0 of order n, so rouche's theorem will immediately tell you that this polynomial will have m zeros and the all can be found in inside the disc of sufficiently large radius, so it is a beautiful I mean you get fundamental theorem for algebra just like that going by thinking of the leading term as you know the big function.

And the rest of the terms as a little function and choosing a disc set at the origin with very large radius okay, how large as large so that the modus of the leading term dominates the modulus of the other terms some of the other terms okay. So, this tells you how powerful Rouche's theorem is okay, so now we are stop here and we continue in the next lecture.