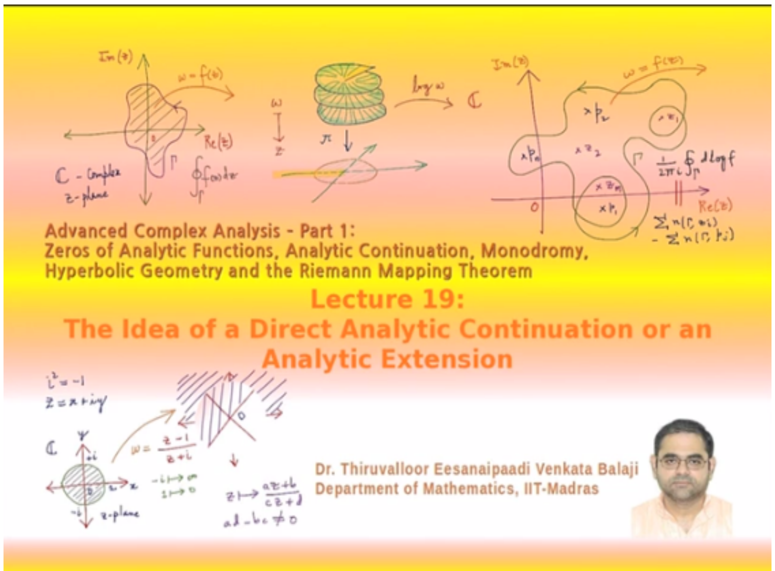
Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

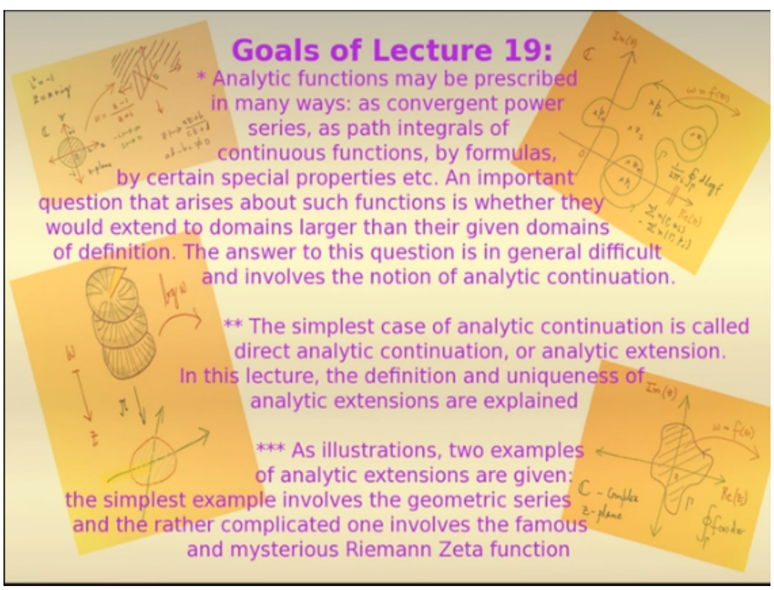
Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology-Madras

Lecture-19
The Idea of a Direct Analytic Continuation or an Analytic Extension

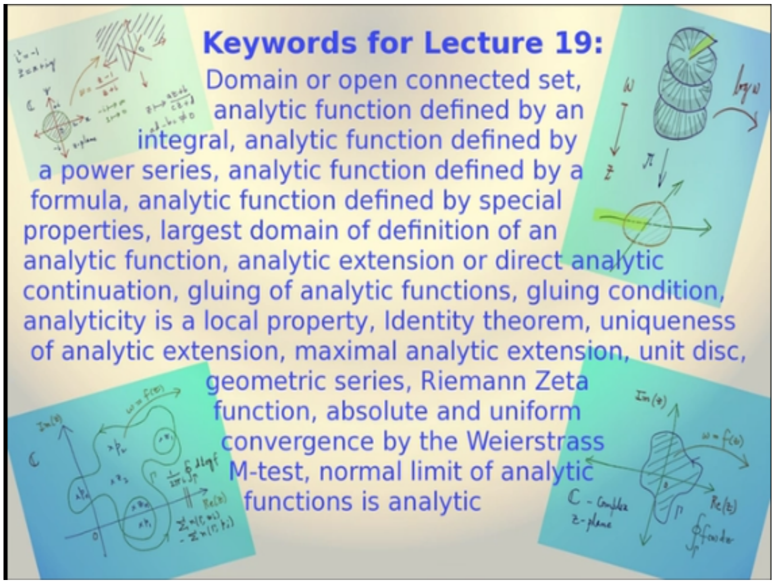
(Refer Slide Time: 00:05)



(Refer Slide Time: 00:10)

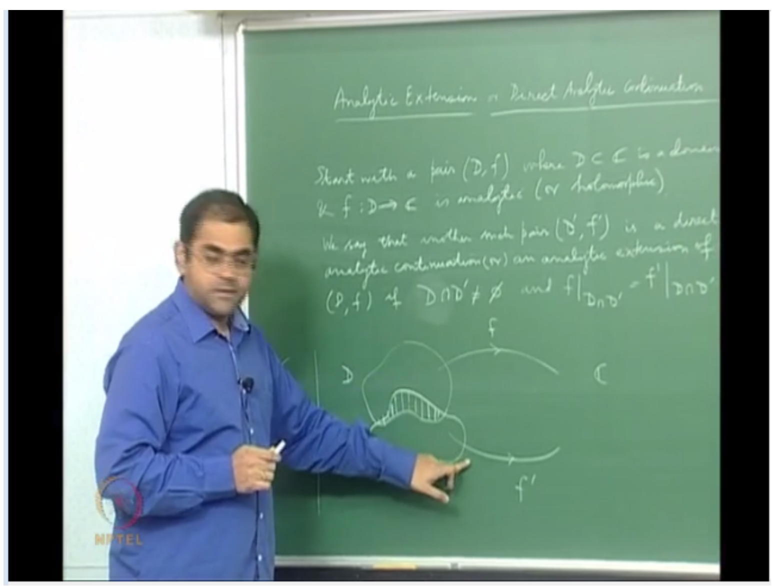


(Refer Slide Time: 00:17)



Okay, so what we need to discuss now is analytic continuation okay, so the first thing that I am going to talk about is what is called as analytic extension okay, so let me explain this analytic extension.

(Refer Slide Time: 00:46)



So, what is this notion of analytic extension see the idea is that to give an analytic function okay there are many ways alright of course I am considering an analytic function on a domain which is an open connected set of the complex plane alright and to an analytic function can be given by a formula or it can be given by a power series alright or it can even be given as the integral of a function okay there are so many ways alright.

Now but the problem is that if an analytic function is given in a certain way say it is given by a power series okay centred at a point of course you know that the power series will converge in the disc of convergence okay. But outside the disc of convergence what happens you do not know alright similarly you may I may give an analytic function by a formula or by some properties in a domain I do not know whether that formula will hold outside the domain okay or whether it will define a proper define an analytic function outside the domain okay.

So, the question of trying to see how far you can find an analytic function how far means on a largest possible open set on which you can defines an analytic function largest possible open connected set okay that is the question that we first need to understand okay. so, you know I will give an example so you see so let me say this so the analytic extension the other word I would like to use is direct analytic continuation.

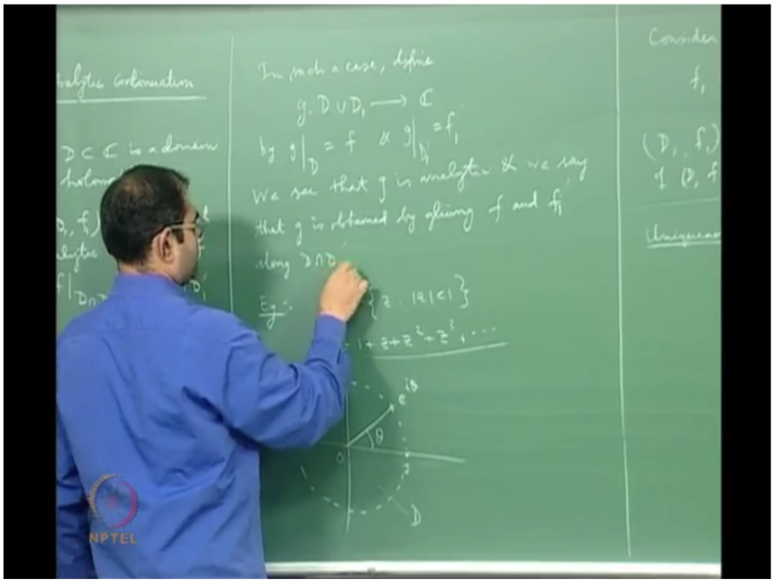
This is the other word I would like to use and what is this, so you see start with a pair D, f where D in C is a domain which means it is an open connected set of course non empty and f from f is a function that is 4defined on D it is a valid function and it is analytic on D okay analytic or holomorphic okay start with a pair like this okay we say that another such pair D prime, f prime okay is a direct analytic continuation or analytic extension of the original pair D , f.

If f restricted to if D and D prime intersect and f restricted to D intersection D prime is equal to f prime restricted to D intersection D okay. So, the roughly one picture that you can think of but of course is not the best picture. Because I am in this picture I am only considering bounded domains which is simply connected so, you know this may be domain D on which you have function f with values in complex plane.

And well and this side is of course is a complex plane and you may have another domain D prime and I could have an analytic function f prime defined D prime. And the condition is that where D and D prime do meet and in the intersection which is also an open set f restricted to D intersection D prime is a same as f prime restricted D intersection D prime.

We say that the pair D prime,f prime is a direct analytic continuation of the pair d,f okay. And of course you can see that it is also the same as saying that D,f is an direct analytic and continuation of D prime, f prime and why what is was special about this. This the thing that special about this is that these two functions glue together to give an analytic function on the unit okay.

(Refer Slide Time: 07:18)



So, what you can do is you can define g so, **so** in this case define g from the union to see by g restricted to D is equal to f. And g restricted to D prime is equal to f prime you define it like this. This definition makes sense because if you go to D intersection D prime g restricted to D intersection D prime will be f is equal to D intersection D prime and g restricted to D intersection D prime will also be equal to f prime restricted to D intersection D prime.

But they are one and the same because of this condition okay so, this is s the glue in condition it tells you that the function f and the function f prime. They give one and the same analytic function on the intersection okay and the intersection is non-empty right. And what we say that g is obtained by gluing f and f prime okay. We say we see that g is analytic and we say that g is obtained by gluing f and f prime along d intersection D prime okay.

So, you have just glue the functions together to give f2 got 2 open sets 2 functions defined respectively on those two open sets. And what you have done is your put those functions together to get a function of the union of the 2 open sets and it make sense because they coincide on the intersection alright and of course analyticity is not an issue. Because analyticity is locally defined this function g is on this set if you take a point in the union.

The point has to either lie on in this set or in this set and if it lies in this set then g is the same as a f and f is analytic. So, g is analytic if the point is on this set then g is f prime so, again it is analytic so, g is analytic. Because analyticity is a local property okay so, this g that you define this analytic right now the point is that so, **so** this is a very simple concept alright.

But the point is that the way in which f is defined may be very different from the way in which f prime is defined okay. So, in other words I am saying whenever you want to specify a pair D,f or D prime,f prime we are trained specify an analytic function an open set okay. But that can be given in many different ways you can give for example analytic function can be given as a power series.

Because you know power series is a analytic function okay it can be given by a formula alright involving say some standard functions like polynomials the standard trigonometric inverse trigonometric exponential logarithmic functions okay or it can be also given by an integral of an analytic some other analytic functions alright with a variable being the upper end point of the path of the integration which starts from fixed point of the domain provided the integral is well defined okay.

So, there different ways of defining the analytic function now the problem is that from the way in which analytic function is defined. It is not at all sometimes is not very easy at all to guess that it really extends to a region beyond which it is defined okay. So, I will give you an example so, the point is that you know the first question is if you give me a pair can you find a an analytic extension a direct analytic continuation for that.

In a domain which is which contains points different from the original domain oaky so, the problem is suppose I start with a pair D,f namely analytic function on domain D. My question is can you find an can I extend it to an analytic function on a bigger set. So, in this case you know what has happened is both D,f and D prime, f prime have been extended to g on D union D prime okay.

So, question is therefore the idea is therefore to try to see if at all there is a largest open set on which you can extend it okay that is the first question. The second question is on the largest open set the new function that you get can you describe it in some way okay that is the question. So, to just to tell you the kind of things the immediate things it happen so, I will give an example so, you know the first example is rather very interesting .

So, take D to be a set of all z such that mod z less than 1 is unit disc okay. And take f of z to be 1+z+z square+z cube the geometric series okay you take the function and take the geometric series. Now you know that f of z this is a power series centred at z okay so, I draw a diagram my situation is like this. So, here is unit disc so, I have unit disc so, this is my D and here this is given by power series.

You know the radius of conversation of power series is 1 alright and therefore inside the disc of convergence okay. The disc of convergence in this case is unit disc okay the region inside this unit circle okay there you know the power inside the disc of convergence the power series always represent analytic function. And what is the analytic function that it represents it is that analytic function for which if we write out the Taylor series expansion at the centre of the disc.

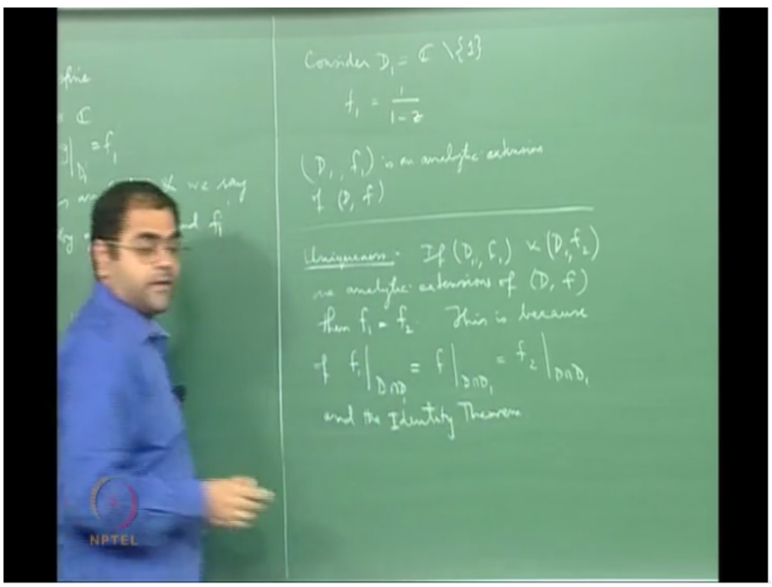
You will get back the power series so, this is an analytic function okay for this analytic function if you again try to write the Taylor expansion at 0. You will get back this series so, I am just saying that the if you start with a power series okay centred at a point. Then inside the disc of convergence around about that point the power series represent an analytic function for that analytic function if you write the Taylor series about that point.

You will again get back the power series because the power series this is the statement that you often see in a first question complex analysis which says the power series of an analytic functions okay. And the proof is the essentially the fact that the power series if you take a discs which is if you take a disc close disc which is contained inside the disc of the convergence.

The power series convergence uniformly and absolutely and in fact therefore what will happen is that it can be differentiated infinitely many atoms and the every time you differentiated the new power series will also have same radius of convergence the fact that you can differentiated tells you the reason analytic that is why power series are analytic okay.

So, **so** here is an example so, here is my power series and now the point is so, here is my pair D,f alright. Now you know you can easily guess that there is a you know there is a way to extent this okay there is a way to extent this. Because you know this if you consider.

(Refer Slide Time: 16:31)



Now you consider D prime to be the plane-the point1 and we consider f prime I think f prime is a very bad notations. So, let me use D1 f1 is well 1/1-z okay you take this now this function f1 is certainly analytic on though this is a reciprocal of a polynomial. And you know reciprocal of a polynomial is of course an analytic function everywhere it is an entire function.

And the reciprocal of an analytic function and also continue to be analytic so long as denominator does not vanish. And the denominator vanishes only z equal to 1 alright so, this so for as so long as z is not equal to 1 this is an analytic function. So, here is the another pair and the fact is that these two this pair is direct analytic extension of this. Because, you know for 1/1-z if you take the function, and try to write out the power series at the origin.

You will exactly geometric series which is which you know as you write 1/1-z as 1-z to the -1 and then use binomial theorem and expand it you get the geometric series 1-z to the -1 is 1+z+ square that is what I have written. So, you can see that D1,f1 is an analytic extension are direct analytic continuation of D,f okay. And the point is that it is rather what is really mysterious is the following.

You see if you consider the functional form as given by a power series in this form the function does not leave at any point on the boundary okay. Because you know if you take any point on the boundary it will be of the form e power itheta. It will be a complex number of modulus 1 so, it will be the of the form e power itheta for theta real okay where of course theta is this angle if you want alright.

And if I plug-in e power itheta into this series then the nth term is e power intheta it is modulus is 1. So, if I evaluate this power series at any boundary point of the circle the nth term always has mod 1 therefore the nth term does not go to 0. And if you know for an numerical series if the nth term does not go to 0 it cannot converge at necessary condition for a power series to converge is necessary condition for a numerical series to converge is that the nth term should go to 0 okay.

So, this tells you that this power series does not leave even on the boundary it leaves strictly inside okay. It is life is only inside even on the boundary it does not make sense okay but notice that this is equal to this which leaves everywhere except the point 1 okay 1/1-z of course gives everywhere except for z equal to 1. Because z equal to 1 is a pole of order 1 it is a simple pole it is a 0 of order 1 of the denominator the function right.

So, modulus story is that if I give you a an analytic function in a certain way the way I have defined the analytic function may restrict it from extending you know the form I given it to you beyond the region that I have given. So, I cannot except this power series to extent even to the any point on the unit circle okay. But that does not mean that the analytic function does not extent.

The analytic function as a function is actually 1/1-z it extents the problem is with trying to only look at it as a power series. It is a the power series has limited life only on the interior of the unit disc but it does so, what you must understand is so, what is that is happening. I have this power series that certainly gives me an analytic function. But just because the power series has a life only sense make sense only inside the unit disc does not mean that the analytic function.

It represents lives only there the analytic function which actually it represents may live on all much bigger open set. And the whole theory of analytic extension analytic continuation is try to find whether given a function on a domain which may be given by a power series or whatever it is. You have to find out whether it really extends okay and it is a pretty involve kind of problem alright.

I give you another example so, here is which is very similar to this and that is the example of zeta function. So, you know incidentally before I do that I want to tell you that you know if I try to make an analytic extension of a pair. The analytic extension that I will get if I take an analytic extension of a given pair okay. Then that extension is unique and that is because of identity okay see so, here is a uniqueness.

Uniqueness is well if D prime, so, even here I think it was bad notation to you have used f prime. Because it would confuse you with the derivative of f so, maybe I will better late than never I will fix D1 f1. And change everything to I will change all this super primes to sub primes so, let me do this. So, of course when I wrote f prime I did not mean the derivative of f okay. So, let me change that sorry for the bad notation okay.

So, let the point I want to make is if D if D1,f1 and D2 and D1, f2 are analytic extensions of D,f. Then f1 is equal to f2 okay this is just a consequence of the identity theorem because what you will get is you see why is that true because you know if you take f1 and restrict it to D1 intersection D or there is D intersection D1. I will get f restricted to D intersection D1.

And that is suppose also be equal to f2 restricted to D intersection D1 okay this equality is because f1 is a analytic extension of f. And this equality is because f2 as analytic extension of a f so, what will happen is you have two analytic functions f1 and f2 defined and the same domain D1 and on a smaller open set D intersection D1 which is non-empty on a non-empty smaller open set the they are equal.

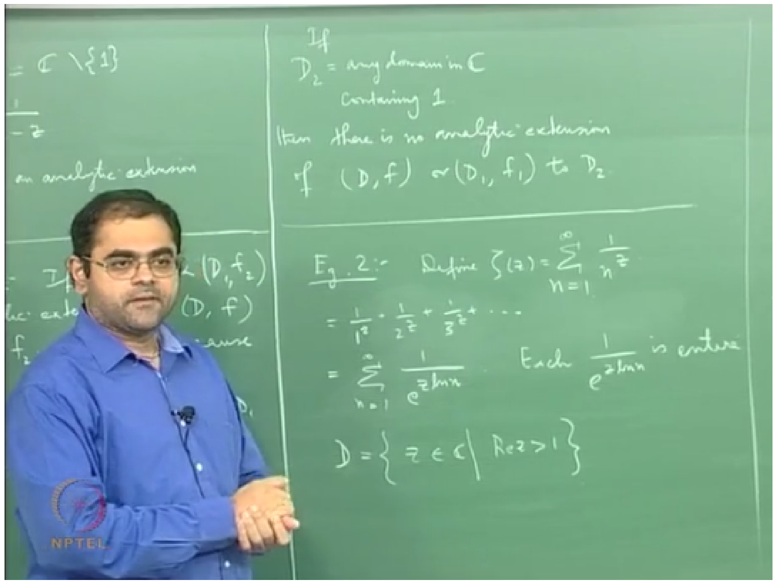
Therefore they are the identity theorem will tell you that they are throw out equal so, **so** let me recall what the identity theorem says. The identity theorem says if you have two analytic functions defined on a domain and suppose they are values are equal even on a convergence sequence of points okay. And with the limit of the convergence sequence in the domain of analyticity okay.

Then they are always equal they are value at every point is equal so, to check the two analytic functions are equivalent domain all you have to do is to find just one sequence of convergence sequence of points in the domain which converges to a point in the domain. And verify that for each point of the sequence the two analytic functions in the same value okay. So, and of course if you give me a non-empty open set I can find.

So, many convergence sequences there okay because it always contained a disc and I can always take an convergence sequence of points going to the centre of the disc if you want okay along the radius. So, the identity theorem will tell you that f1 and f2 are one and the same. So, they direct analytic continuation to a given domain is has to be unique okay. So, this is so this is because of this and the identity theorem okay.

So, the analytic direct analytic extension is unique alright now. Now what I want to it understand is that of course there has situations where if you give me a domain and you give me domain this intersects it analytic extenture not extensions are not possible. So, for example you know if I if you in this example suppose I continue with this example.

(Refer Slide Time: 27:27)



Suppose I take D2 to be D2 is equal to any domain any domain in c containing the point1 okay. Then there is no direct there is no analytic extension of this function to the point1 okay there is no analytic extension of this function the point1. Because of the simple reason that the point1 is a singularity see the maximum analytic extension of this extension is the function 1/1-z.

That is the maximum analytic extension that is the largest possible open set on which this the function representing this power series. The analytic function that represents this power series can be extended okay so, though the function is 1/1-z the largest set on which is analytic is leaves out the point1. Because the point1 is a pole it is not a removable similarity if a point is a removable similarity.

Then you can correct it by defining the value of the function at that point to the limit that you get as you go to that point. But this is not a removable similarity it is a pole it cannot be corrected okay so, if you take any domain which contains that pole there is certainly no analytic extension of this function to that okay. So, then there is no analytic extension of D,f or D1,f1 to D2 there is no analytic extension okay.

So, of course this fact I made about uniqueness is true for any analytic extension and here I am writtening back to this example alright. So, there are so, the idea I start you cannot except a analytic extension across a singularity across singularities right. Now so, the other example that I want to talk about is the Riemann zeta function okay.

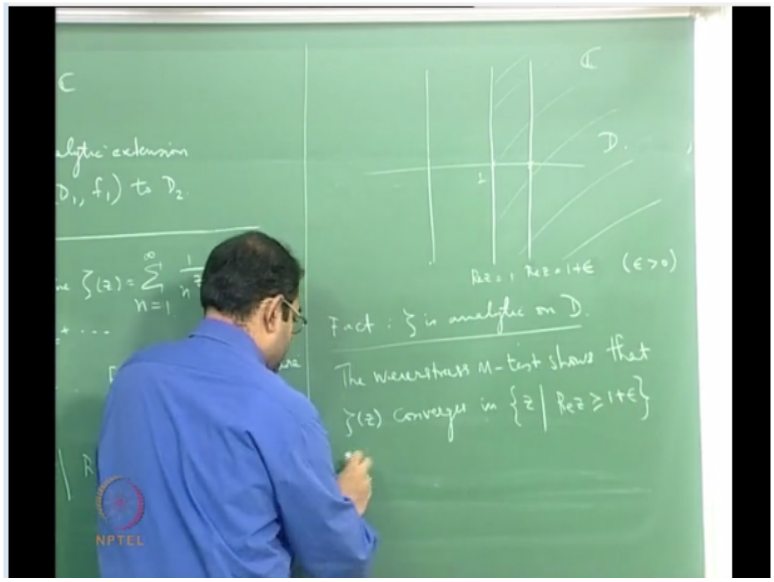
So, that is the function which is so, you know you define**,** so let me give this example D so, this is another example let me call this is example2. So, this is example1 so, let us go to example2 define zeta of z to be let me take it from a 1 to infinity okay. So I will start with n equal to 1 to infinity and so, this is just if you write it. It will be 1/1 power z+1/2 power z+1/3 power z and so on right.

So, I will define it like this okay and by that I mean of course zigma n equal to 1 to infinity as I just explain it I s1/e power z lan n okay. And the point you must understand is that each of these functions lan n is a constant okay lan n is a real logarithm of n that is a constant. And z into lan n is of course an analytic function because it is a polynomial of degree 1 okay. It is just the analytic function z multiplied by a constant and e power that is also an analytic functions.

So, each of these functions is an entire function each so each function 1/e power z lan n is entire each of these is an entire function okay. And mind you since I have put an exponential the denominator can never vanish. So, I can put it in the denominator okay so, whenever I put something in the denominator I always be worried about whether it vanishes.

But what I have put in the denominator is an exponential and you know exponential never vanishes. So, is always defined and is always analytic so, these are all entire functions so, if you take the sum is again an entire function okay. But where so, there is a there is an issue there so, what you do is you, you take the so, here is my so, let me write D let me defined D to be the right half plane okay to the right of the point to the right of the vertical line passing through z equal to 1 okay. So, you define it like this so, you take so, here is a diagram.

(Refer Slide Time: 32:59)



So, here is the complex plane okay and here is the point 1 and I draw this line this is the point real part of z equal to 1 okay. This is that line in real part of z is x of this, this is the line x equal to 1 line parallel to the y axis this is the imaginary axis. And I am taking this region so, this is my D it is a right half plane okay. So, D is a set of all z in c complex numbers such that real part of z is greater than 1.

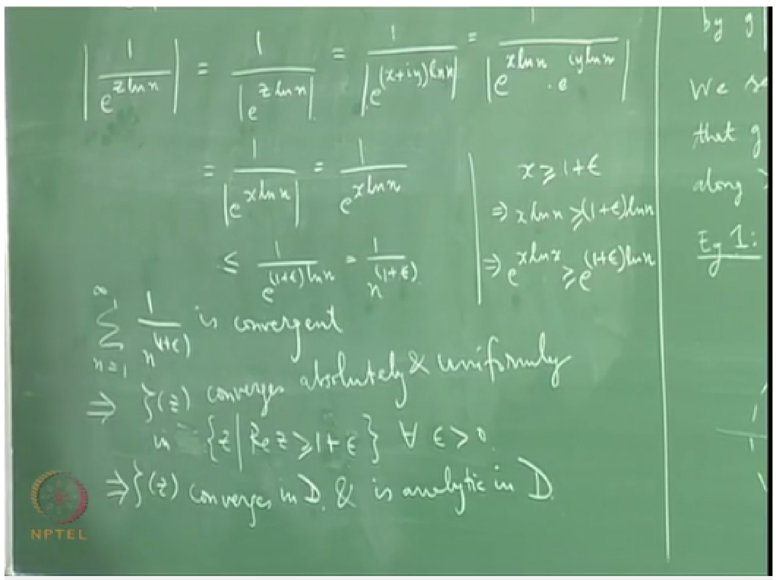
Imaginary part can be anything okay so, this is a right half plane it is a right half plane that I sto the right of the line. The vertical line passing through z equal to 1 which is given by the equation into real part of z equal to 1 okay now the fact is that zeta is actually an analytic function on D. So, fact is zeta is analytic on D okay function zeta is analytic on D, so in fact probably you have seen this in a first quotient complex analysis.

First of all these of this is a series okay it is a functional series with the mth function given by 1 by e lan n okay, first of all a functional series need not converge. So, first of all for this to make sense as a function it has to converge okay and the fact is it will converge okay and then the next fact is that this convergent function once it converges it defines a function zeta it is called the Riemann zeta function okay.

And the fact is that this Riemann zeta function is actually analytic okay, so the so what is the , so let me briefly recall how you prove such a thing. So, what you do is well you take a line to the right of this even by real part of z=1+epsilon where 1+epsilon where epsilon is positive and you can take epsilon as small as you want but you take a vertical line to the right of this line okay.

And what you do is you use the weierstress m test to show that this function converges uniformly and absolutely on the right half plane starting from this line. And including that line so, the weierstress m-test shows that is zeta converges in the set of all z such that real part of z greater than or equal to 1+epsilon both uniformly and absolutely that is the weierstrass m-test okay. And so, you know one can easily check that it is pretty easy to check that.

(Refer Slide Time: 37:03)



So, let me here in this closed right half plane given by real part of z greater than to 1+epsilon what you have is is z is x+iy and x is greater than equal to 1+epsilon okay. So, what will get is it is an estimation so, if you calculate ee power if you take the uh nth term of that functional series it is 1/e power z lan n okay. You see so, it is equal to 1/x+iy lan n which is that is right 1/.

Because it is multiplicative so, it is x to the lan x lan n D power iy lan n and as you writely pointed out the mod of this is 1. So, end up with 1/ e to the x lan n it is what I get that is good and now I think you more or less than because x is greater than equal to 1+epsilon x lan and is also greater than in to 1+expsilon lan n. And e that we will also be greater than that and the reciprocal will go the other directions.

So, you see x greater than or equal to 1+epsilon we will tell you that epsilon n is greater than or equal to 1+epsilon lan n. And that lost together e to the x lan n is greater than or equal to e to the 1+epsilon lan n that is because lan n is nis greater than or equal to 1. So, lan n is positive that is why this in equality holes and then the exponential function for dl is an increasing function okay.

So, this holes so, this will be the same as e to the x lan n because is the real number okay. And that is going to be less than or equal to 1/e to the 1+epsilon lan n and that is 1/n to the 1+epsilon this what you get okay. So, if you take the nth term of the series then in modulus it is dominated by this numerical term okay. And if you take the corresponding numerical series for this, what you will get is.

You will zigma n equal to 1 to infinity 1/ n to the power 1+epsilon okay if you take this, this is convergent. This is a fact that you would have learnt in any first course in analysis if you take zigma 1/n power alpha okay that will always be convergent for any alpha greater than 1 alright. So, this is convergent so, the moral of a story is your hole if you, so you know if I start with the this functional series for the zeta function.

If I take the absolute series that means I take the series that you get by taking absolute term absolute values for each term. Then the absolute series is dominated by this numerical series which is convergent okay and this is exactly the situation of the weierstrass m-test which says that whenever a functional series is dominated by which is absolutely dominated by a numerical series which is convergent.

Then the original functional series converge is uniformly and absolutely this is the of course it is since the absolute series itself so, first of all there is a dominated convergence theorem it says that if a series is dominated by another series. And the dominating series converges then the original series converges so, what this will tell you is that if I take zigma if I take the absolute series for the series correspond to the zeta function.

That will converge and you know it is also fact that absolute convergence implies convergence okay. Therefore that will also tell you that the original function functional series for the zeta will converge. So, the moral of a story is the original series that was used to define the zeta function that converges both absolutely and uniformly. And that happens for every epsilon greater than 0 and therefore it if I make epsilon small enough.

I can cover every point in the right half plane right to the right of the point to the right of the line x equal to 1 by taking epsilon small enough I can cover every point. Therefore these gives you the fact that the zeta function the series for the zeta function actually converges for every point lying in that right half plane to the right of the point to the right of the line particular line passing through z equal to 1 okay.

So, **so** this will implies that zeta z converges absolutely and uniformly on in the set of all in D which is the set of all z such that real part of z greater than or equal to 1+epsilon. This is not D 1+epsilon for every epsilon greater than 0 okay now from this there are 2 facts it will you can get the first fact is that zeta of z converges in D there is a first fact. Because any point of D I can make it lie in a region like this by taking epsilon small enough okay.

So, zeta converges in D okay the second fact is not only it has it converge I now claim it is analytic. I am claiming that this zeta is analytic why is that so that is show because of a fact that I had stated in several lectures ago namely whenever your sequence of analytic functions if it converges normally okay. When you have, whenever you have a series of analytic functions, or if you have a sequence of analytic functions**.**

If the sequence converges normally then the limit function is also analytic okay this is the fact that I stated in an earlier lectures and the proof was essentially that the limit function normal convergence will mean that it is convergent on converge is uniformly on compact subsets. And this uniform convergence will ensure that the limit function is continues and then what will happen is that analyticity will follow from Morares theorem.

And you use also the fact that because of uniform convergence you can interchange limit an integral okay. So, you can go back to that earlier lecture where I true in detail that you know if you have a normal limit okay. If you have a normal limit of a analytic functions in the limit function is also analytic. So, what is happening in this case is that you can consider this function zeta as this series.

But what is the series a series is just limit of partial sums and the partial sums are analytic. Because they are sums of analytic functions in fact they are the partial sums are entire functions oaky. So, you the partial sums of this series are entire functions they are analytic functions and the series converge is uniformly on any such closed right half plane. Therefore on the on a interior of such a closed right half plane.

It will represent an analytic function because the only condition that is required is you must have normal convergence. You must have convergence which is uniform on compact subsets but you here you have uniform convergence on the whole closed half plane that is the right half plane including the boundary which is a line to the right half z equal to 1. So, that theorem will tell you that zeta is actually analytic that is how you get analyticity okay.

So, **so** and is analytic in D okay so, this is a zeta function this is a famous zeta function and so, here is my pair with consist of the famous zeta function. And this domain which is the open right half plane okay which consists of all points the right of this line okay. And the question is can this be analytically continue so, this question is very similar to the simple first example that we saw.

The simple first example also consisted of a function which is defined by a series on a domain. In this case series was a power series and the domain is was the unit disc and one could see by introspection by inspection that well one could see that it is just 1/1-z. And therefore one was able to guess that it could be extended to the whole complex plane except the points z equal to 1 which is a pole okay.

Now they are amazing facts is a following the amazing fact is the zeta function also admits an analytic extension to the whole plane except the point z equal to 1 exactly like the geometric series okay. When you try to extend the geometric series the whole plane it admits an extension.

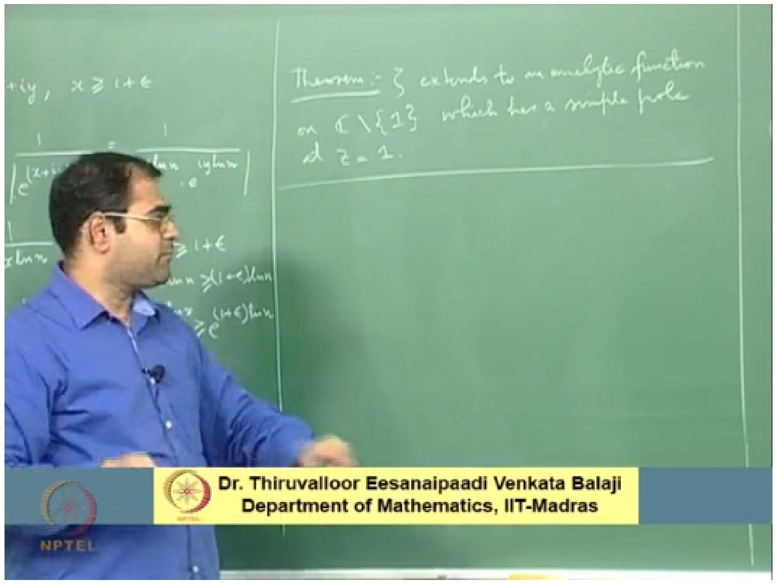
And the extension has a trouble only at z equal to 1 and a z equal to 1 what kind of trouble does it have it has a pole of water 1 okay.

It has a simple pole of it is a simple pole that is it equal to 1 amazingly you enough zeta also extends to the whole plane except the point z equal to 1. And z equal to 1 it has exactly a simple pole okay so, this is what happens. But to prove this is not trivial it needs theory of analytic continuation analytic extensions okay. So, the point I want to make is that trying to find whether an function has an analytic extension is not an easy problem okay.

But it happens that in the cases zeta function it can be extended to the whole plane-this point what is the problem with this point at this point you know what is going to happening the series becomes the harmonic series oaky. And the harmonic series is it does not converge okay so, that is the only trouble some point okay is hot to believe that it can extent all other points on this line except the point z equal to 1 where all the other points on the line also same to be very difficult points.

But the truth is it exceeds everywhere the only point where it cannot be extended is z equal to 1 where it becomes the harmonic series and that for the extended zeta function the zeta function which is the full extension of this function. The whole plane-the point z equal to 1 that point z equal to 1 is only a simple pole okay. So, this is the beautiful fact okay and we will have to prove it which I will try to prove it okay.

(Refer Slide Time: 49:38)



So, I just want to summarise by saying that the problem of finding an analytic extension can be very simple or it can be pretty complicated okay. Here I am zeta extends to an analytic function that is zeta admits an analytic extension on c-1 which has a simple pole at z equal to 1 okay. So, this is a so, this needs proof we will try to prove this at some point so, this is the story about trying to extend an analytic function.

Now I want to tell you that it is purely of a matter of abstract mathematics to ensure that you if you start with f pair consisting of a analytic function on a domain okay. A pair consisting of a domain and an analytic function on it namely the pair of the form D, f then there exist a maximal extension of that okay. So, we continue with this discussion the next lecture.