Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology-Madras

Lecture-18
The Algebraic nature of the functional inverses of an analytic mapping at a critical point

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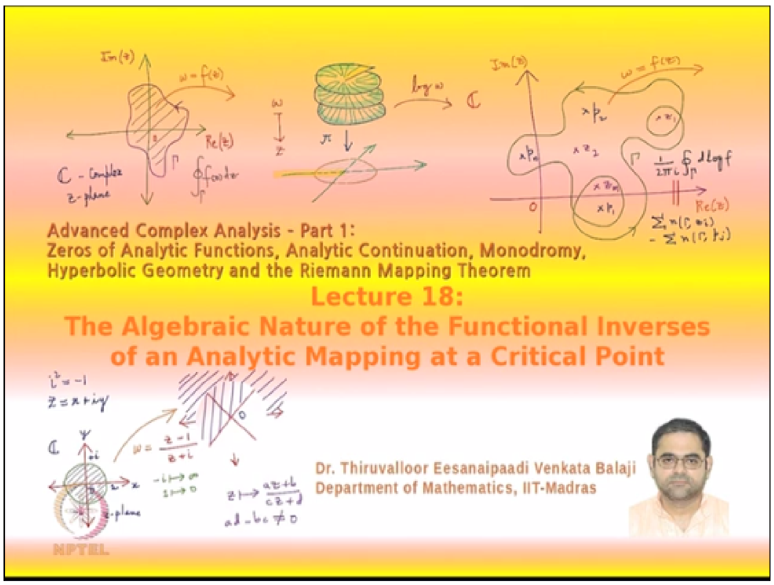

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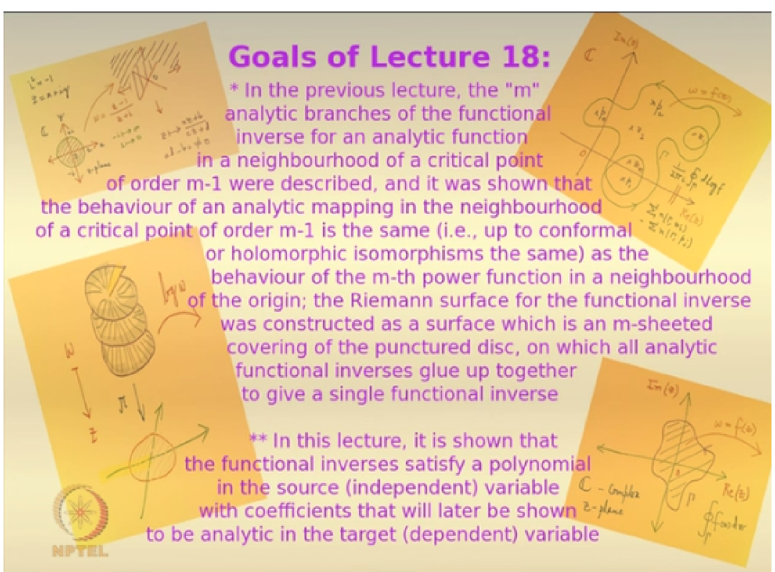

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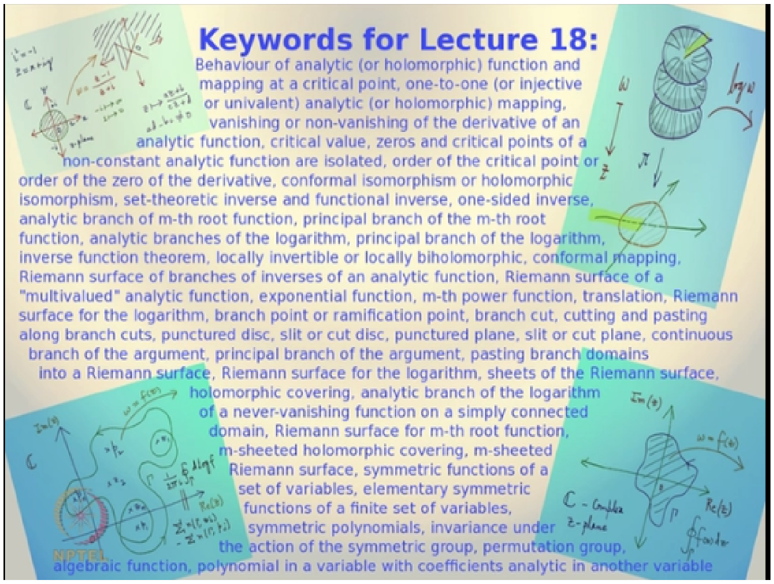

Okay, so what I am going to do now is tell you few more facts about the behavior of a function in a neighborhood of the of a critical point. So, let me recall what we have seen the last lecture.

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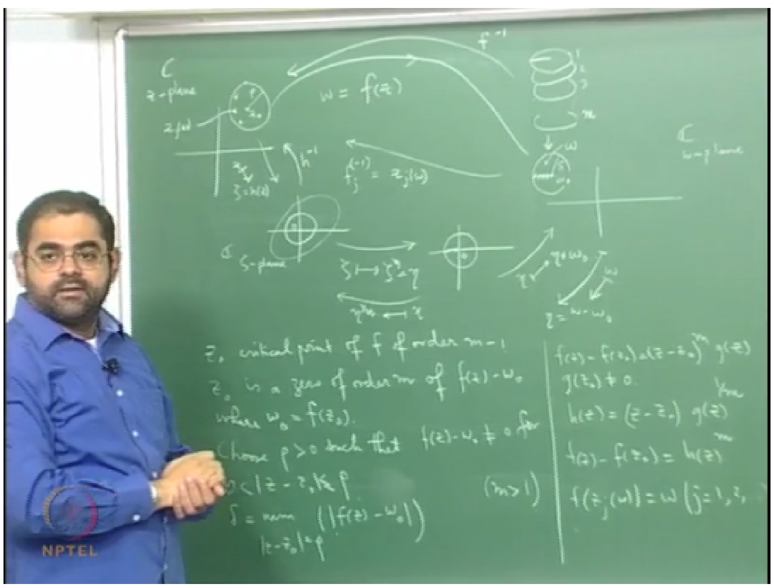

So, you see we have the following situation you have the function w= f of z and with well z0 is a critical point of f of order m-1 and we interpret it as z0 is 0 of order m of f of z-w0 where w0 is f of z0 is a critical value okay and we of course we choose row greater than 0 such that the 0 of f of z-w0 namely z0 is isolated there in a disc centered at z0 radius row okay.

So, f of z –w0 is not equal to 0 for 0 strictly less than mod z-z0 strictly less than or equal to row and choose or let and of course we have delta is the minimum over mod z-z0=row of mod of f of z-w0. And then what we do is that we have this diagram if you recall where we took this disc in the complex plane the z plane centered at z0 radius row alright.

And then we are looking at this map is into so let me give some space above because I want to draw something above. So, this is into the w plane and z0 goes to a point w0 and I am looking at a at this disc at centered at w0 and radius delta where delta is chosen like this okay and the fact is that so the function goes like this and what is happening is that we have .

So, and this whole transformation is broken down in such a way, so that it conformably looks like z going to z power m okay, so how it has broken down is that you have first of all a change of variable I do not what I called it. So, it is z going to zeta which is h of z if , so please let me know if this notation is conflicting okay.

I **I** think it was h of z and then so this goes into the any of the copy of the complex plane which is now the zetta plane and the image of the is this of this disc is a something that looks like a disc okay because h where of course we have set we have made the following definitions we write f of z-f of z0 is z-z0 to the power of m into g of z g of z0 is not equal to 0 okay.

And then we write h of z to be z-z0 times g of z to the power of 1/m okay and mind you g does not vanish on this disc and therefore there is a an analytic branch of a the mth root of g okay effect there are m of them and you choose one of them and then you call that as h alright and then you see that if you write it like this f of z-f of z0 becomes h of z to the power of m okay.

And so what you do is that you first go by h then you z0 goes to the origin because h of z0 is 0 alright. So, it z0 goes to the origin and then image of this disc is going to be well you know something that is looks like a disc conformably, so you know I am drawing something like this but I am taking a small enough disc there and well and from here to here I make one more I apply another function which is zeta going to zeta power m which is neta okay.

And you know this will go to this will take the disc centered at the origin, so the same disc centered at the origin except that probably the radius of the disc will get you know diminished or enlarged depending on what the radius here is alright whether this radius here is less than 1 or greater than 1. But in any case it will go to a disc centered at the origin.

And now this function is function that you know very well this is the power function z going to z power m of course mind you m is greater than 1 I am at a critical point alright m is greater than 1 and you know that how you know that you we have studied this already and then to get from here to here, so if you come all the way from up to here it is h and the now here it becomes h power m.

But h power m is f of z-w0 alright, so to get f of z I just have to add w0 to it, so the transformation is neta going to netta+w0 that is a transformation it is just a translation okay. So, this disc will get translated to this okay, so the and well if you look at this picture now it is clear that if I so you know there are m branches of .

So, if I take the inverse functions here the inverse function will be w going to w-w0 that will be the inverse function here alright and which is of course neta okay and here the inverse function will be neta going to neta to the 1/m okay which is which has several branches it has m branches okay. And then of course from here to here that there is a unique inverse function and that is just h inverse okay.

The derivative of h does not vanish you can check that derivative of h does not vanish and so h inverse comes because of the injectivity of h and the inverse function theorem. So, you know if I compose all the way here the inverse function is unique here the inverse is unique because it is a translation it is an isomorphism this is also a holomorphic isomorphism because it is injective.

So, here the inverse is here, here the inverse is here also the inverse is unique the inverse is not unique in this in here because there are m m branches and each of this branches you have been follow it with this forward function will get identity here okay. So, the moral of the story is that you get functions you get several functions that so you get so you know when I have to define a fractional power okay etta to the 1/m is exponential of 1/m log neta.

So, I will have to get a fix a principle branch of log alright and of course I will have forget the origin okay. And to fix a principle of to **to** get a function that is analytic what I will have to do is that I will have to cut out the portion of the negative real axis okay. So, in this disc-this portion of the with the portion of the negative real axis removed each of these functions neta going to neta to the 1/m or analytic.

Therefore if I compose it what will what it will tell you is that you know if I take the corresponding slit disc here centered at w0 then the resulting functions will be analytic okay. So, you know I will get functions like this these functions are going to be zj of they are the zj of w okay and they satisfy the property that this followed by this is the identity here okay fj f of zj of w=w for j=1, 2 etc., up to m where zj s are the for.

So, what are the zj's I mean if you start with the value w here okay then this value w is assumed by f at m points counted with multiplicities and it is those m points which are called z1w, z2w and so on up to zjw up to zmw okay. So, you know this so the you get points here m points counted with multiplicities and these points are the zj of w and they depend they will change as w changes.

So, for every point here you get m points here which go to that point, so this function f is a m is to 1 mapping or outside z0 of course z0 goes to w0 okay and no other point goes to w0 but the point is z0 goes to **w** w0 with multiplicity m okay. Because that is what it means to say that z0 is the 0 of order m of fz-w0 okay, f assumes the value w0 which is f of z0 with the multiplicity m okay.

So, these so for any w here you get m values of z okay and these values of the z asses that when you plug them back in f you will get back the w okay. So, in other words these are solutions of w=f of z you are solving from the equation w=f of z you are solving for z okay. So, you must think of this as these are inverses of f inverses in the function of z okay.

That is the reason I have put a bracket -1 on top because strictly speaking f is a many to one function it does not make sense to talk about is a theoretically inverse. But these are functional inverses alright just like you know it is the same idea for example when you have the exponential function the inverse functions or the lone functions but the exponential function is not 1 to 1, it is many to 1 okay.

But then we but log is functionally an inverse to the because it un does the act perfect the exponential function. Similarly when you take the mth root function the mth root function is the functional inverse of the function which takes everything to the power m okay. The function which takes everything to the power m is not we have 1 to 1 function it is a m to 1 function again expect at the origin alright.

So, that do undo the operation of taking something to the power m you have to take the mth root okay. So, functionally it is an inverse taking variable to the power of 1/m is a functional inverse and how many such inverse is are there, there are m such okay. So, in the same way if you take any you take any analytic function you should look at neighborhood of a critical point what happens is that this analytic function has functional inverses.

There are m of them okay and the each one of them is actually analytic on a slit disc alright but if you want to actually see them as globally as an analytic function one has to go to the Riemann surface of this f inverse okay. So, you know what happens is that you we have we can construct an m sheeted covering sorry I draw it like this and so on, so there is an m sheeted covering with a projection.

And a projection here such that on this on this you really have a function which I can call as f inverse okay. And again there you see you should understand this f inverse is only a one sided inverse in the sense that you know this followed by f is identity as should say this no in fact I should say this f inverse is a functional inverse which on every sheet assume a corresponding value fj.

But the only what is the advantage of this f inverse okay maybe I even put a bracket here, so that you know you do not get confused at the point I want to say is that this f this the good thing about this when compare to these dash a these j are only analytic on a slit to it okay but the flow above is analytic on the whole Riemann surface this is the this guy here is the Riemann surface for f inverse the functional inverse okay.

And we have seen this already for the case of f the exponential function in this case we got the Riemann surface of the log function which is the functional inverse of the exponential function and also we have seen Riemann surface for **z** z to the 1/m which is the functional inverse for f of z=z power m you have seen 2 cases but more generally here is the point is that when you look at the exponential function.

You see the derivative of the exponential function is again itself and in an exponential function never takes the value 0 therefore it is derivative never vanishes alright. So, somehow there since the derivative never vanishes it is locally 1 to 1 alright. Therefore it was easy to write the inverse function itself which was given by a branch of the logarithm okay.

But the **poi** because exponential function has no critical values alright whereas we are now in a bad situation we are what you are saying is take a function which unlike the exponential function has critical values how even if you take a critical value in the deleted neighborhood of the critical value this function also admits functional inverses. The functional inverses they all can be put together as a single analytic function on a Riemann surface okay.

And they are they give the solutions for the independent variable in terms of and it is dependant variable. So, they are they solve z for w okay you get z as a function of w okay, so let me write this down, so the point I want to make is that .

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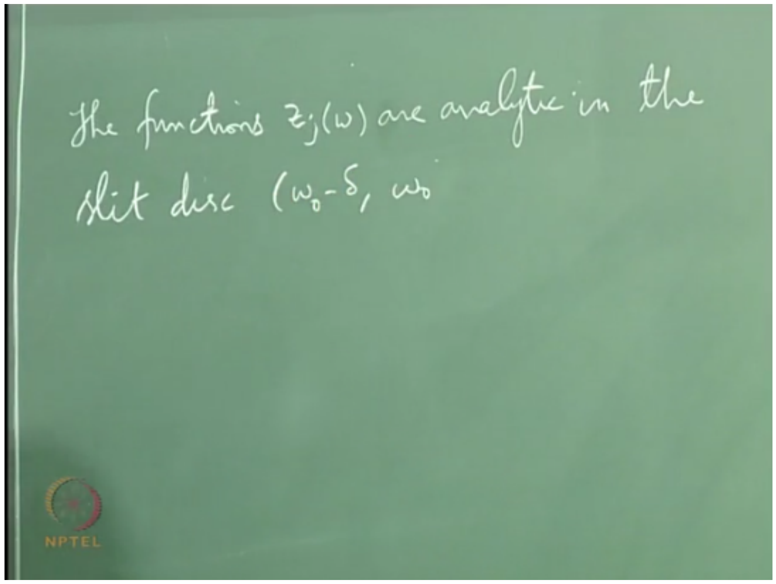

The functions zj of w or or analytic in the slit disc w0-delta w0 yeah is should this is what I should throw out.

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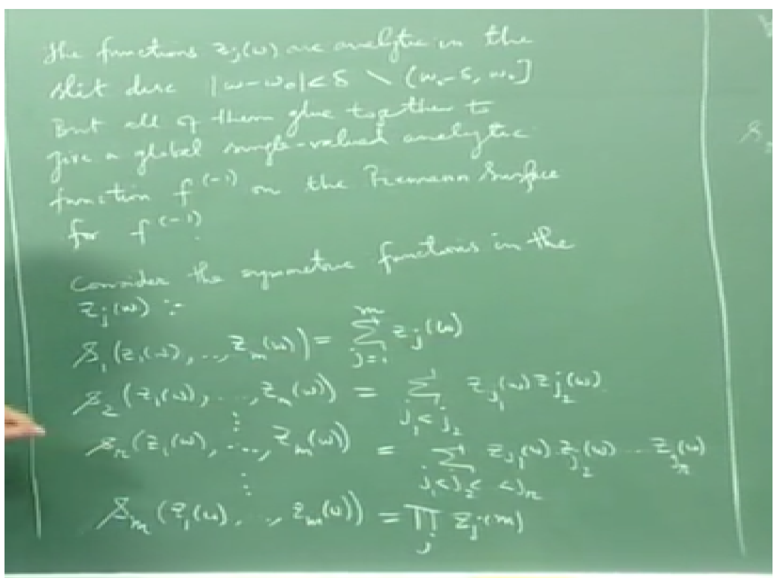

So, let us mod w-w0 less than delta- you throw out the portion parallel to the negative real axis from the to the left of and including the center w0. So, it is going to be w0-delta w0, so this is the line segment from w0-delta to w0 with the right end point w0 included and the left hand point w0-delta not included in this what you have throwing out from this disc to make it as slit disc.

And on this each of these functions are analytic okay and of course you know all of them glue together to give a gloval analytic function global single valued analytic function f to the -1 on the

Riemann surface for f to the -1 okay and the Rieman surface you know is obtained by pasting causally the upper you have to take m copies of this disc, this slit disc okay.

But do not throw out you take the cut disc not the slit disc okay which means you do not throw out in the slit disc you throw out the this piece but now you do not throw it out you just cut it and you know cut and pasting process where the portions above the that line segment in 1 disc or join to the line segment and the portion below in another disc and you do this repetitively okay.

And then the last disc you paste with the first one okay this is something that is very hard to visualize in 3 space okay but then you can do this and you get a surface of Riemann surface that is the Riemann surface on that Riemann surface these you get a function which is on each piece that you have pasted equal to the corresponding zj. So, on the first piece it will be z1 and as you moved to the second piece which is being pasted to the first piece the function become z2.

Then as you move to the third piece it becomes z3 and so on okay and of course the last piece is connected to the first one alright. So, you finally get 1 single function which are call as if you want one get all z=f-1w w hat okay and why I am calling this is w hat because this w hat is the is a point above the point w below okay, so you get a function on the stand a function here it is a function there alright.

It clears as a function on Riemann surface right, so it is nice to see that you are able to even at a critical point you are able to get inverse functions that is the whole function that is the beautiful thing that we have to notice okay, even at a critical point you are able to get inverse functions functional inverses for your function you are able to get okay. Now what I want to say is that I want to say more I want to actually say that these functions are actually solutions of a polynomial equation okay .

The solutions of a polynomial equation could quotients an analytic function okay and it is a that is a very deep fact and so let me explain that so let me do the following thing consider the function the symmetric functions in the zjw okay consider this. So, you know if you give me set of functions then for example if you give me set of variables.

Then you know how to construct the symmetric polynomial in those variables okay namely what you do is you the first symmetric polynomial is simply the sum of the variables, the second symmetric polynomial is a sum of variables taken 2 at a time okay. And then the third symmetric polynomial is sum of variables sum of products of variables taken 3 at a time okay, the second symmetric polynomial is sum of products taken 2 at a time.

And similarly the jth symmetric polynomial is you take a product of the j variables and take all possible such products and take some okay. And of course in this when I say product the product you do not worry about the order okay the variables are assumed to commute alright. So, in the same way if you give me any set of functions I can construct symmetric functions from the given set of functions while looking at the corresponding I us e the same method .

I use the same definitions that I use to construct symmetric polynomials in a given set of variables only now that I am thinking of the functions as variables. So, what will be s1, s1 of z1 s1 of z1 w etc., zmw the first symmetric polynomial will be just the sum of the ziw or zjw okay of course j=1 to the m. So, the first symmetric polynomial m variables is simply the sum of the variables alright.

The second symmetric polynomial in these will be now you have to take sum of products of variables 2 at a time. So, it will be in the form sigma z j1w, zj2wtoward j1 strictly less than j2, this is second symmetric polynomial and so on you can define the well the rth symmetric polynomial which is now you take the sum of r of them sum of the products are r of them.

So, which means that you write zj1 w into zj2 w and so on you go on up to zjrw you take r of them take the product and then you take the submission over all possible such r indices okay and if you go on like this the last one will get is the mth symmetric polynomial which will simply be the this should be the sum of products taken m at a time but there is only one product taken m at a time.

Because only m of them so this is will be only single expression not a sum of expressions it is a single product using the just product of it will just the product of all the zj of them over j okay. So, you will get these are the m symmetric polynomials in all the zj's okay and what is the property of the symmetric polynomial this probably as symmetric polynomial is that property of the symmetric polynomial these n variables is that if you change the variables by permutation with the value of the polynomial does not change okay.

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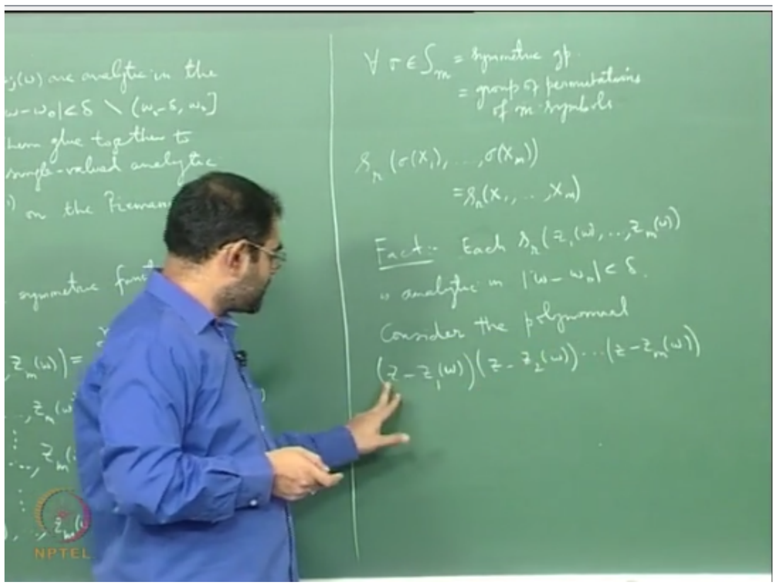

So, that is what symmetric polynomial means okay it is invariant under the action of the symmetric group alright. For every sigma is sm so s so you know I will okay so let me use small s because it is better, so that I can use capital S to denote the symmetric group okay which is just set of group of permutations of n symbols m symbols it is a det of all bijective maps from a set of m elements back to itself okay it is a group under composition the symmetric group**.**

And the point is a Sr of if you take sigma x1 etc. sigma xm is the same as Sr x1 etc., xm okay you apply the value of the symmetric polynomial does not change if you substitute for a variable for the variables a permutation of the variables and that is the reason it is called a symmetric polynomial okay. And fine and what is the advantage of looking the symmetric polynomials you see the way I would have define them these are also functions of w oaky**.**

And of course they are all based by taking products and then taking sums, so all these fellows they will all be certainly analytic functions on the they will all be analytic functions on the slit disc but the big deal is that they are not only analytic function of the slit disc they are actually analytic functions on the whole disc that is the amazing theory okay. So, the so here is a fact on each Sr of z1w etc., zmw is analytic in the whole disc.

This is the amazing fact okay and so let us grand this fact for the moment then what does it says, it says that it says the following thing so consider the consider the polynomial z-z1m, z1w into z-z2w and so on z-zmw consider this polynomial these are polynomial of degree m okay whose roots are these m functions I mean whose zeros are the same functions.

This polynomial becomes 0 whenever z is one of the zi's okay, so the zi's are roots of this polynomial alright of this the zeros of this polynomial alright but then if you expand it out the quotients will come to be as you would have seen any quotient algebra the symmetric functions up to a sign alright.

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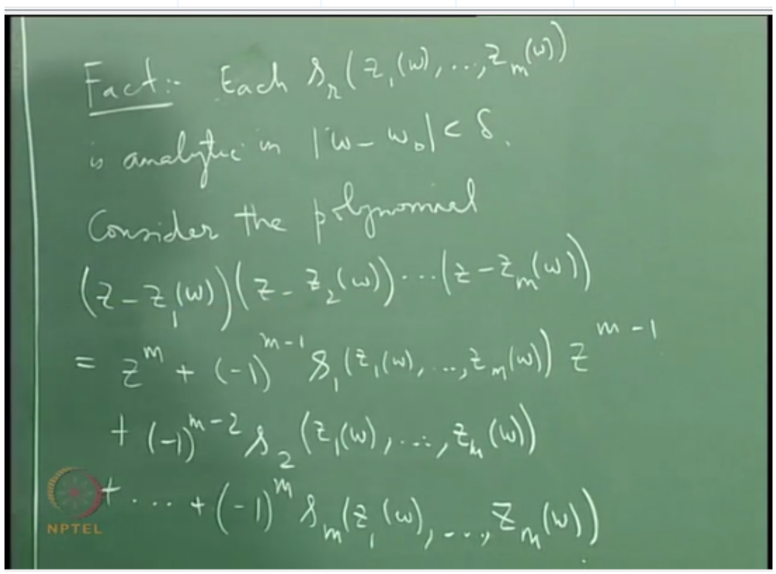

So, what you will get is this will become z power m+ -1 to the power of m-1 s s1 of z1 of etc., zmw+ -1 to the power of m-2 s2 of z1 of w and so on zm of w and so on it will end with a constant term which would be it is going to be – I have to worry about my signs in this case is going to be I choose I think let me adjust the signs.

This is going to be finally I take -1 to the power m sn of z1w and so on zmw only thing is I will have to take the sign for example and I have to add the quotients also here. So, here I want z to the m-1, so which means I will how do I get the quotient of z to the m-1 I take I choose m-1 of these to be just that and the remaining 1 I choose it to be – of certain zjw and then add it.

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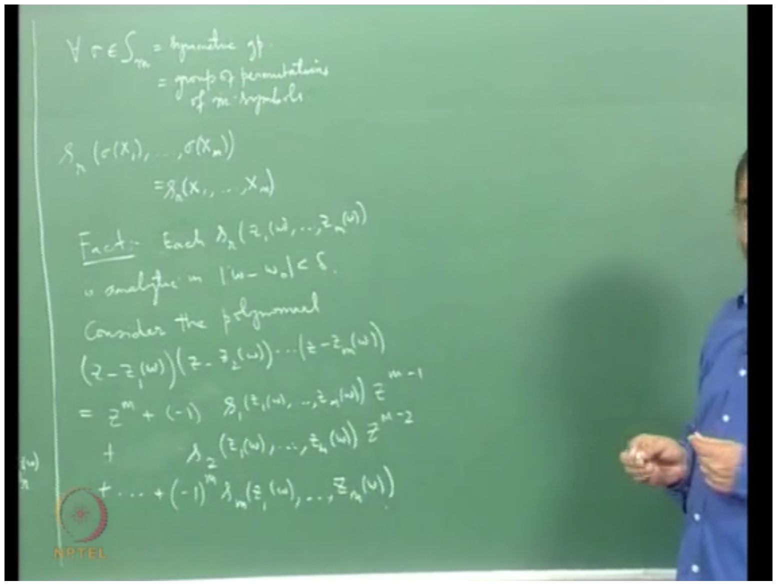

So, I will get a – here, so it will just be – okay and here I will get a + probably so this will be a – this will be a + and it will keep on alternative on this right, so this is what I get. So, and here I should of course z to the m-2 alright I should put z-m-2 the m-2 and so on of the and this is the constant term this simply the product of all the ziw-ziw's okay, so, this is what you get okay.

Now so what did I just say I said that you see the ziw's are zeros of this polynomial but what is this polynomial is the same as this polynomial but what are the quotients now the quotients are s1, s2 etc., sm up to sign but what would I say about s1, s2 etc., up to sm they were analytic functions. So, the m these m functional inverses for f are actually zeros of at polynomial with quotients are which actually analytic on the disc on the target disc, that is the fact that I want to say okay.

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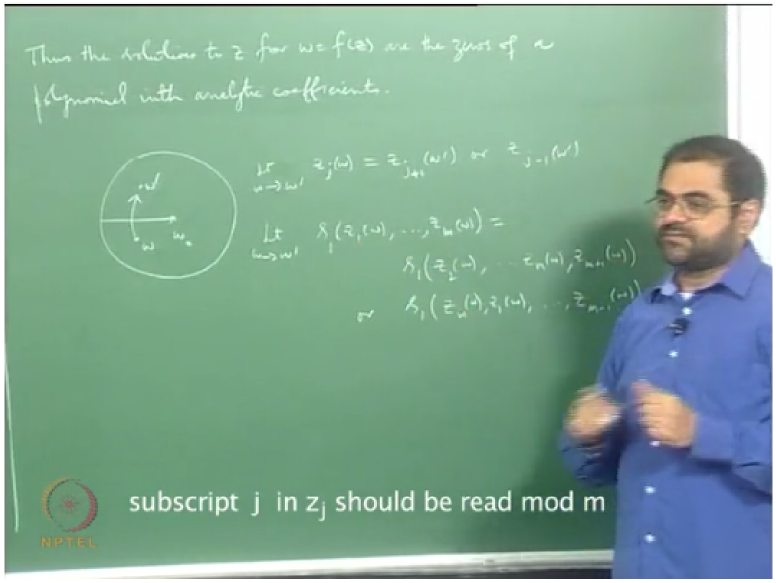

So, and it is a thus the solutions to z for w=f of z are zeros of a polynomial with analytic quotients okay this is the fact this is an algebraic fact okay it is an algebraic fact it says that you have found bunch of functions which are zeros of a polynomial with analytic quotients okay and in fact this is a great significance in Riemann surface theory alright probably I some point later at some later point of course I will try to explain that.

But now so you know so the moral of the story is that all these functional inverses are zeros of the nice polynomial okay the only fact that needs to be check these at each of these is analytic alright. Now to prove that it is analytic there are let me at least first try to begin by convincing you that they are at least continuous leave alone analyticity, see the problem is you know if you go back.

These functions they will you know if you if instead of throwing out this ray okay if you keep it but you cut the portion of the disc on the ray and below it and separate it from the portion of the disc above it then on that day it become continuous they will become continuous it is just like the log function you take the branch of the logarithm, the branch of the logarithm if you take a principle branch of the logarithm for which the imaginary part is a principle argument which varies from value starting from – pie.

And going all the way up to values strictly less than + pie if you take that as a function the principle branch of the logarithm. Then you know the real part is always continuous because it is the natural logarithm of the modulus of the complex number which is a continuous function the problem is only the with the imaginary part which is argument and what is the problem is the problem is that you know if for all points on the negative real axis.

And below the negative real axis the principle argument is you know is greater than or equal to – pie okay whereas for a point just above it above the negative real axis the argument is close to + pie and lesser than + pie. So, there is a jump of almost 2 pie on the negative real axis and therefore you have discontinuity on the at a very point on the negative real axis.

So, suppose I separate the points of the negative real axis above the points of the plane above the negative real axis from the negative real axis and the points below okay by separating it I am saying that you know the points above the negative real axis are not close to the points on the negative real axis and the points below that is what this cutting means okay and when points are not close I do not worry about continuity, continuity I worry about only one points are closed okay.

So, once I cut it if I try to check continuity at a point on the negative real axis I am only going to check in a neighborhood in a disc half disc which lies in the negative real axis and below it I am not going to worry about points in the half disc that plays about alright. So, **so** you know if I cut this instead of anything if I cut okay then these fellows are going to be continuous okay.

But then what happens as you go across across the negative real across this line segment if you go from below to above what happens is that the zj the values of zj will change from zj to zj+1 okay depending on whatever you are ordering is either they will go from zj to zj+1 or zj to zj-1 depending on the way you brought it okay and that is because of functional self changes.

And what you do is you cut and paste these various copies m copies of this cut domains not the slit domains a cut domains to produces Riemann surfaces in such a way that as you cross the this ray on piece okay the function zj changes to zj+1 which is the function on the next piece okay and that is the reason why all these functions agree and they become a continuous function on top okay.

So, the moral of the story is that as you the problem is with points on this ray for each of these functions okay, what is the problem as you move across that ray the zj will become some other zj+1 or zj-1 alright. But if you look the symmetric functions okay and you do the same business you see the symmetric functions will not be affected okay.

For example you know if I take the sum of all the zj's and if I take the w and move it across if I move it across this ray okay that I remove it across this ray what will happens each zj will be replaced by the corresponding zj+1 okay each zj will become the corresponding zj+1 okay. And but then the sum will remain the same okay, so what I will get is instead of getting zj you know if I start with so .

So, let me if you want let me draw it so that you understand what is going on so here is my ray this is my w0 and you know suppose I start with the w here w1 and I move across this ray and go to w2 okay. Then what will happen is that you know zj limit w times to so I I keep as a w and let me call this as a w prime w times to w prime of zj of w what you will get is you will get zj+ one of them.

You will get this is what is happening okay or you know it maybe zj-1 of w prime depending on how you wrote it alright this is what you get. But it will always be +1 or always be -1 depending on how you wrote it, so what happens is. So, you know if I take limit w times to w prime of for example the first symmetric polynomial z1w etc., so on zmw what I will get is I will get it back I will simply get I will get the first symmetric polynomial evaluated at.

If I of course symmetric polynomial is a polynomial ending variables, so it is continuous in the variables. So, I can push the inside and apply the unit each of the variables okay and if I do that I will get well I will get z2w and so on and you know and the point is that zmw will go to z1 because it is cyclic alright, so it will go on at some point I will get zmw and then I will get zm+1 w or I will get the other way round the index may come down, it maybe zmw z1w and so on zm-1w, this is what I will get.

But both these are the same as this in any case both are equal to s1 of z1w and so on zmw they are the same because after all the s1 is suppose to just add all thing and whether I add from z2 to zm+1 including zm or zm to z1 through zm-1 and then zm and I am going to get the same result okay. So, what this demonstration tells you is that the value of s1 of the first symmetric function of these m functions.

That can that value does not change okay, so it means that this first symmetric function is a continuous function of w.

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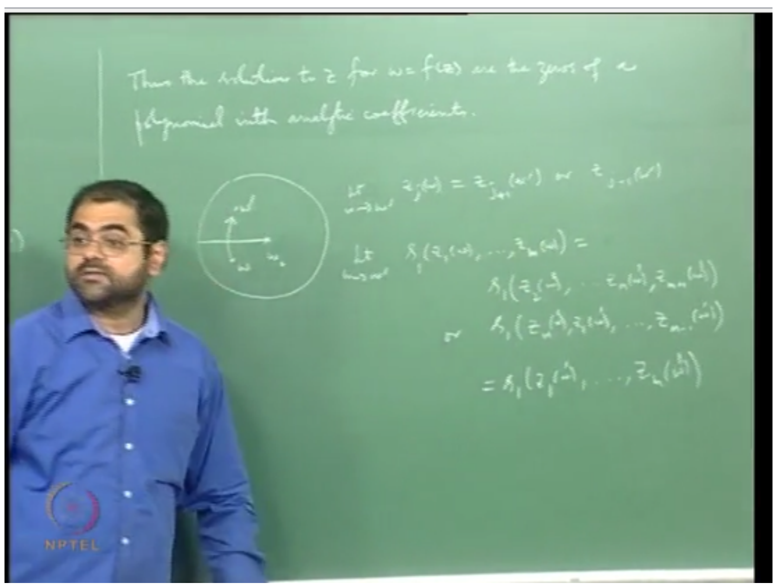

And of course you know I will have to put w prime since there is a here these are all primes and so these are also primes okay these are primes, so the moral of the story is the symmetric function will not be affected and I have written it down for the first symmetric function but you know for all the symmetric functions this is going to happen okay what is going to happen is when you cross the troublesome ray okay.

The symmetric functions are only going to be permuted by a permutation which is given by a shift in the index alright and but when you make a permutation the symmetric function is not going to change the value of the symmetric function is not going to change because it is invariant under permutations. Therefore what is going to happen is each of these si's each of these sr's they are all going to be continuous even on the even on this ray okay**.**

And therefore you can now believe it is very clear that each sr is therefore is certainly continuous on this **on** disc okay. They only issue is now that is left to be fixed is that it is analytic okay and the fact that is analytic can be for example it seen by what is called the principle of analytic continuation okay which is what I am going to discuss in the forth coming lectures okay.

So, I am going to next go on to discuss analytic continuation and then so called monodromy theorem alright and it will follow from that discussion that each sr is not only continuous as we demonstrated it is actually when analytic okay, so with that I will stop now.