Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-17

The Riemann Surface for the functional inverse of an analytic mapping at a critical point

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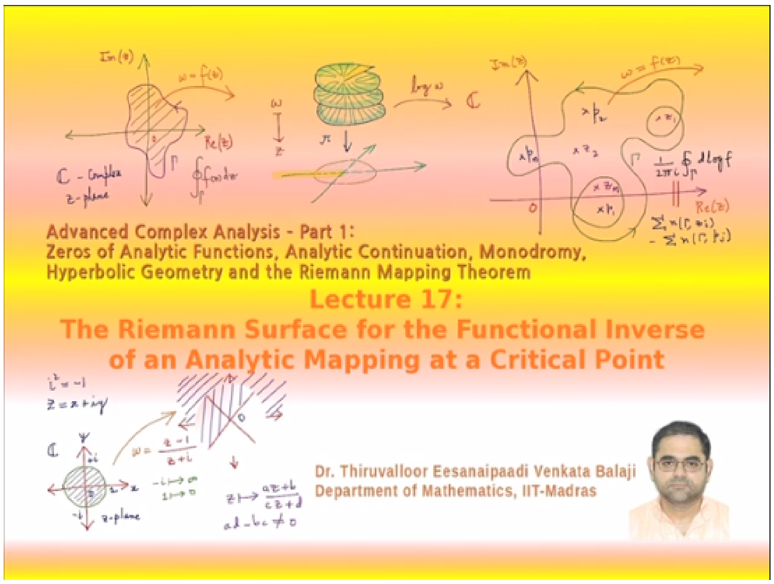

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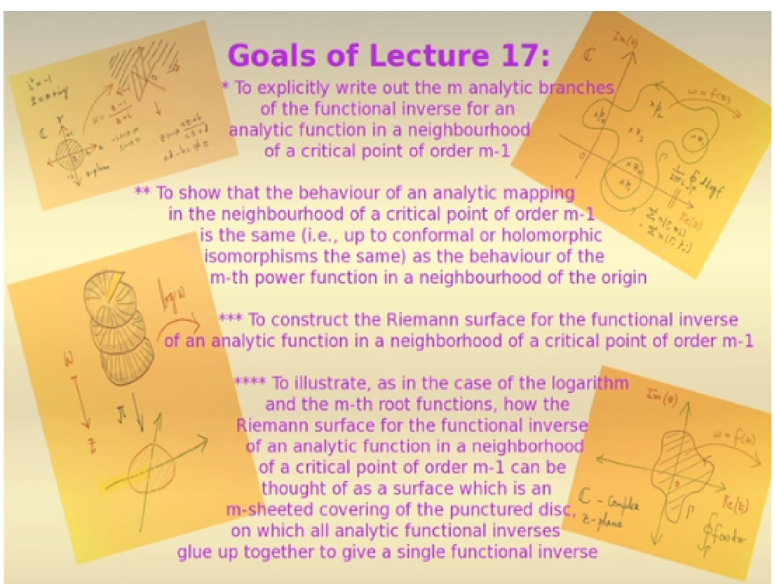

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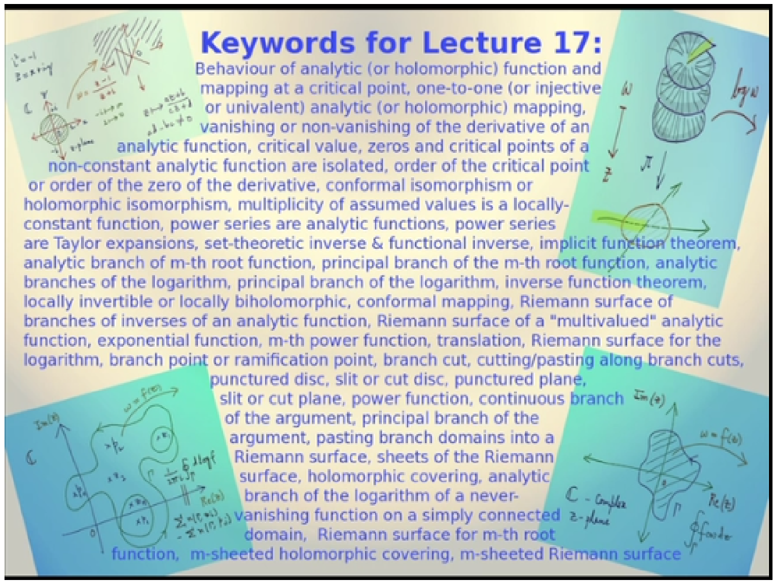

Okay, so let us continue with our discussion of studying the behavior of a function and a neighborhood of a critical point. So, we have discussed in the previous lectures you know the how to look at the logarithm the various branches of the logarithm and various branches of the power function okay and for a fraction of power alright.

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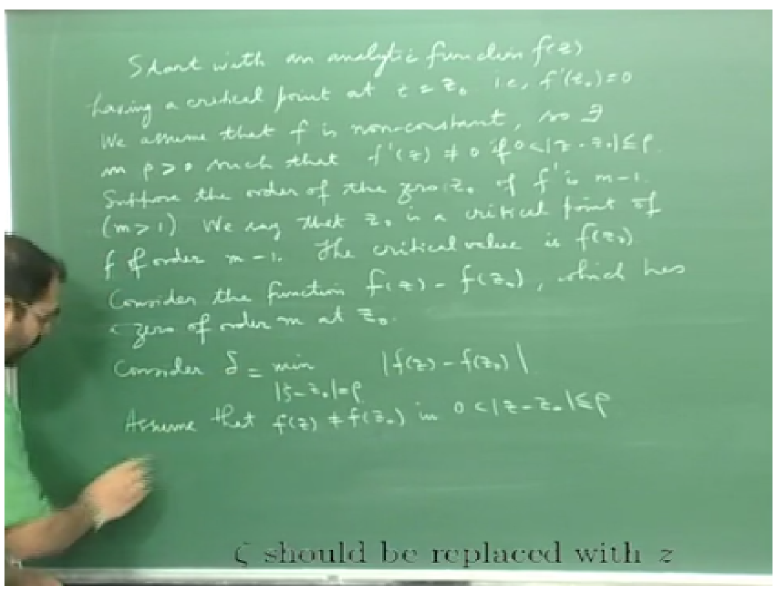

So, what we have going to do now is start with an analytic function f of z having critical point a critical point at z=z0. So, that is f dash of z0 is 0, the critical points are the points where the derivative vanish of course we assume that f is non constant on the domain on which is defined is which includes the point z0 and this will tell you that f dash is also non constant.

And well on the other hand f dash is an analytic function also and z0 is a 0 of that and you know the zeros of an non constant analytic function are isolated. So, I can write that so there exist and F long greater than 0 or a row greater than 0 such that well F dash of z is not 0. If mod z-z0 is angle row okay. So, z0 is o it is isolated it is an isolated 0 of the derivative.

So, there is a deleted neighborhood and of course I should not include the value z0, so I should put this greater than 0, so for every point in this deleted disk closed disk stated at z0 including the boundary except for the centre at every other point derivative does not vanish okay. So, any other 0 of f dash will lie outside this disk apart from z0 okay, z0 is only 0 that in the closed disk.

And that is just the due to the fact that the 0 is of an non constant analytic function and isolated right. Now suppose the order of the 0 of z0 of f dash is m-1 and of course I am assuming is m is greater than 1 right, suppose this the order of 0 z-1 the reason for m-1 I could have taken it as m but the reason I am taking at as m-1 is because you will see that it helps in our argument.

Otherwise throughout my argument I will have to keep saying m+1 which is not very interesting so well so suppose order of the 0 of f dash is m-1 and z0 okay . We say that z0 is a critical point of f of order m-a okay. So, the order of the critical point is the order of the 0 of the derivative okay and of course the critical value is the value at the function at that point the critical value is f of z0.

This is the critical value corresponding to z0 alright and now the reason why I took m-1 is because we can consider the function f of z-f of z0 okay, consider the function f of z-f of z0 of course this also an analytic function because it is the analytic function f- a constant alright. So, this is also an analytic function but the point about this function is that it has a 0 at z0 okay.

And all it is derivatives the first m-1 derivatives at z0 to also vanish because the derivatives of these are the derivatives of f okay because this is differ from f only by a constant therefore what happens is that z0 becomes a 0 of order m for this function okay which has a 0 of order m at z0 okay that is a reason why I took m-1 here because I get m here if I had taken m I would have got m+1 alright.

Now this is an argument that we have seen many times so what you have is that f takes the value see if f takes the value z0 f of z0 at z0 with multiplicity m okay that is the same as saying that fz-fz0 takes the value 0 at z0 with multiplicity m okay and you know we have already see this argument that in a sufficiently small neighborhoods surrounding z0 the multiplicity is constant okay consider delta to the minimum over mod zetta-z0= row of mod of f of z-f of z0.

Look at the minimum value of this mod okay and so you are on the boundary of that circle centered at z0 radius row alright I want to say that f the modulus of fz-fz0 is always positive on the boundary circle okay and if not I want to modify it so that it that becomes true perhaps it is already true let me do the following thing let me assume let me also assume that I not only assume that f dash is not 0 on this.

But I also assume that I will also assume that mod of f of z-f of z0 that is also non 0 okay. Because I that is also non 0 on this deleted neighborhood okay, the reason I can do that is because z0 is 0 of the analytic function f of z-f of z0 and that 0 is also isolated because f of z-f of z0 is a non constant analytic function because f of z is also a non constant analytic function.

You take an non constant analytic function and had a constant to it the resulting analytic function is also non constant. So, **so** let me write that down assume probably it is not necessary probably it will follow but anyway let me assume it for safety assume that f of z is not equal to f of z0 in 0 less than mod z-z0 (()) (09:31) okay. If which means that you know I have already chosen rows is that the derivative does not vanish.

But maybe if that row does not work I can make the row smaller I can choose a smaller rows, so that this is true and that will happen because z0 is an isolated 0 of f of z-f of z0 because f of z-f of z0 is a non constant analytic function okay. So, that means that on the boundary also this is not 0 I mean this is not to equal to this which means that the modulus of the difference is non 0 okay.
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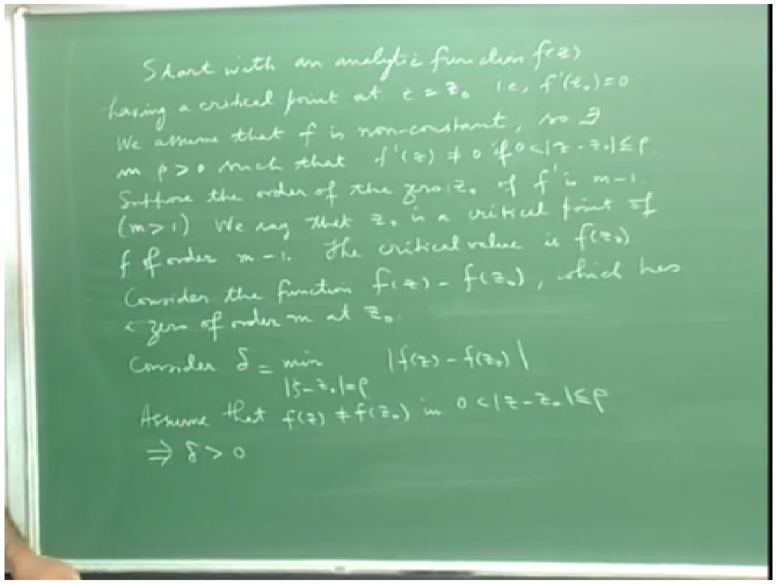

So, it means that this delta is positive, so this will tell you that delta is positive, delta is a positive number.

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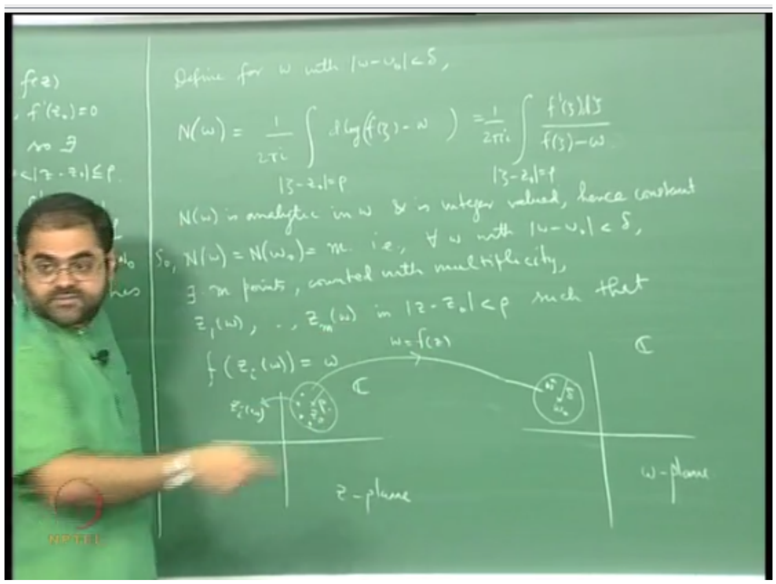

And then you know once I have this I can define for w with mod w-w0 less than delta there is a reason I need that delta we have been through this argument many times but anyway it define it to be the number of times f takes the value w okay. If you put w=w0 which is f of z0 I have not mentioned what w0 is probably I should do that now the critical value is f of z0=w0.

So, I should tell you what w0 is so when I put w=w0 I will get N of w0 okay and but then I can change w inside this alright because in this disk this is not going to vanish okay and except yes f of z-w is certainly going to be greater than or equal to delta alright on the boundary on this boundary. So, this integral is well defined right, so **so** the moral of the story is that this number which we have seen many times before is going to give you the number of times f of z takes the value w okay.

And we have seen before that N of w is analytic in w and it is integer value hence constant okay, this is an argument you have seen many times before. So, N of w=N of w0 and N of w0 is number of times f of z takes the value w0 which is f of z0 and that is m, so this is equal to m. so, what this tells you is it tells you the following thing for every w with mod w-w0 less than delta there are n points in mod z-z0 is less than row at which f takes the value w okay.

So, let me write that down that is for every w with mod w-w0 strictly less than delta there exist N m points counted with multiplicity, counted with multiplicity means some points maybe repeated okay z1 of w etc., z m of w in mod z4-z0 strictly less than row such that f takes at each these points f takes the value w okay. This is what N of w=m means right.

So, the diagram is something like this I will okay, so the diagram is something like this , so here is my complex plane this is the w plane I mean this is the z plane the source plane and there is this is the target plane which is again in the complex plane and this is the w plane and I have this well I have this disk here centered at z0 radius row and I have this function f of z, w=fz which is a mapping.

And it maps into and I am looking at the at a disk centered at w0 with radius delta okay this is my mapping and what I am saying is that if you give me a point w here then you get all you get m of these points here okay they maybe there maybe reputations it may it means that all the m may not be distinct point it some point maybe repeated with multiplicities okay.

They could all due 1 point with m times multiplicity okay which is what happens if w=w0 if w=w0 then the only z for which f of z is w0 is z0 okay. But it is not to be thought of it is one point rest me thought of us n point m points because it is multiplicity m the 0 of f of z-f of z0 at z=z0 has multiplicity m, so you should think of z0 as being repeated m times even though it is a same point.

So, when I draw these points there many of them put have be one at the same but these are the z I have w okay. So, you know if you look at it it looks a little like it is looks a little the implicit theorem see I what I am saying is if you take the equation w=f of z okay and you take this critical point z0 then I am getting a disk such that I try to solve for z from w=f of z okay.

Then I am getting z=n values z1 of w etc., zm of w and these are solutions of w=fz namely if I plug them in f of z I get w okay. So, I am solving for z from w=f of z okay, so what it tells you is it is you are getting m solutions for the equation w=f of z in the neighborhood of a critical point z0 that is what it says okay and you know the way you should think of these .

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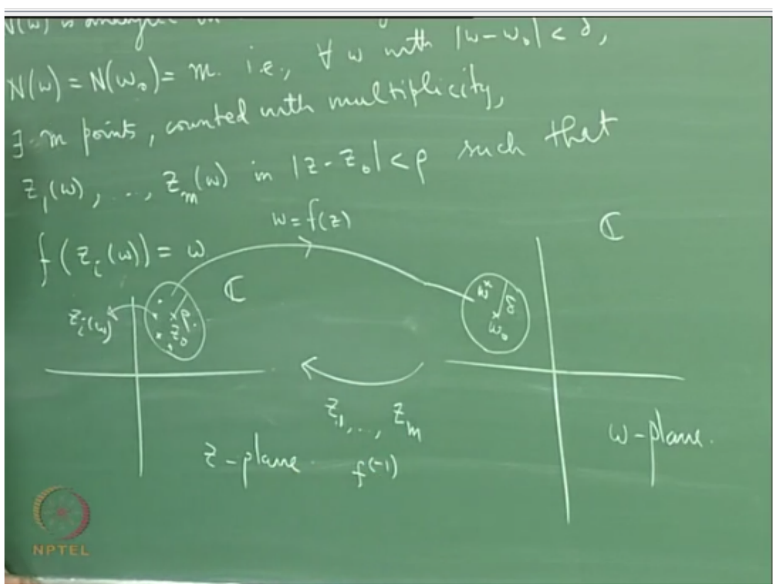

So, you know there are these, so we have these functions z1 etc., up to zm there are these functions and the way you should think of them as well you should think of them as you know inverses of f okay. So, I am putting a I am not writing f inverse you the convention is to write f inverse only when it is set theoretically an inverse at least in which has f has to be injective alright.

But here f is certainly not injective okay because derivatives has vanished at a point you cannot expected to be injective at all and therefore I should not write f inverse, so I am putting f and putting the inverse in the bracket to tell you that you know this inverse function okay the inverse function has m solutions z1 through zm and these zi's are functions of w okay**.** And they all solve the equation w=fz alright.

Now we just want to understand what these functions zi is what these functions are, are they continuous, are they analytic etc., etc., okay. And we also want to know how this mapping looks like how can you draw this mapping or how you can visualizes this mapping okay. So, what we are going to do is we are going to break this down using what we have something that we have already seen namely.

We have going to use the fractional power function branches of fractional power function which have which comes from branches will logarithm where m branches for the 1/mth power of a variable okay and they come from the logarithm by choosing the various branches of the logarithm we will use that to break this map down okay and see how it looks how this map looks and the point is that up to confirm conformal equivalence.

That means up to a holomorphic isomorphism this map really looks like z going to z power m, this map from here to here looks like z going to z power m that is a whole point alright that is a reason why we studied the mapping z going to z power m and it is branches in the previous lecture alright. So, let me explain that, so what we are going to is we are going to do the following thing. So, you know, so let us study in a neighborhood of in this neighborhood of z0 what is happening.

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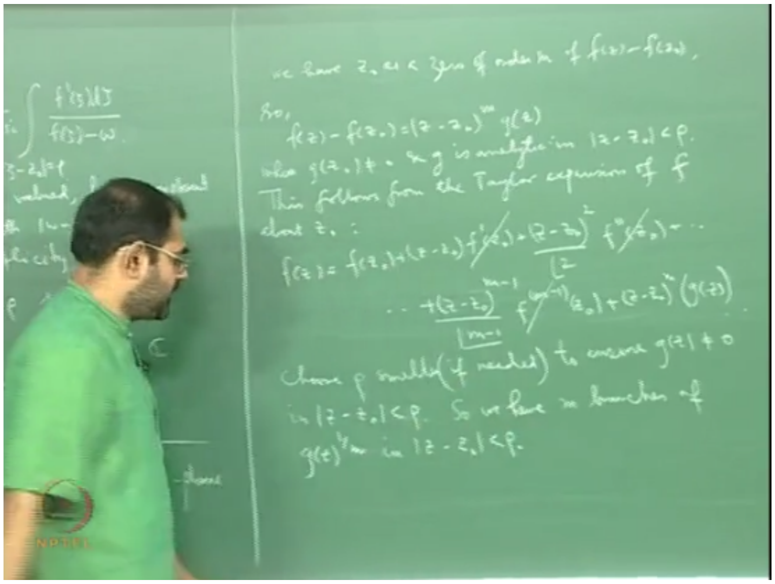

See we have z0 as a 0 of order m of f of z-f of z0 I mean which is f of z-w0. So, what you are going to get is f of z-f of z0 is z-z0 to the power of m times some g of z okay where well g of z0 is not 0 and g is analytic in mod z-z0 less than row okay this is very simple this is just by the Taylor expansion of f okay you have this follows from the Taylor expansion of f about what is the Taylor expansion.

The Taylor expansion is you know it is f of z is f of z0+z-z0 into f dash of z0+z-z0 the whole square by factorial 2 by f double dash of z0 and so on you go on up to z-z0 to the power of m-1 by factorial m-1 f to the m-1 derivative z0 and then I will get z-z0 to the power of m times something I will call that something as g of z this is what I will get alright and well the you know from the power m onwards I am whatever is there I am z-z0 power m outside the brackets as common.

And whatever is inside is there anyway power series it is an analytic function, so I call that as gz okay this is the Taylor expansion and mind you all these terms will vanish. Because z0 is a critical point of order m-1 okay, so that means f dash has a 0 of order m-1 at z0. So, you know f dash f double dash the all will vanish alright, so the so f of z-f of z0 will become z-z0 power m into g of z that is exactly what I have written here. So, this is simply from Taylor expansion alright.

Now and mind you g of z0 is not going to be 0alright that is because f mth derivative of f with respect to z0 is non 0 right. Because it is a the order of the 0 of f dash at z0 is only m-1 is not m alright, so well now what we going to do is going to do the following thing choose probably it already is true but maybe what I will do is a choose row smaller if you want to make sure that g does not vanish in mod z-z0 less than row choose row smaller if needed to make to **to** ensure g of z is not equal to 0 in mod z-z0 strictly less than row okay.

I want g0 is equal to 0 the reason why I want g0 is equal to 0 is because you know I want to write g I want to write branches of g of z to the power of 1/m I want to do that alright and you will see why I want to do that alright that is why I want to g0 is equal to 0 right and I am saying g is not equal to 0 it is already true in fact I do not have to choose row smaller that is because I have already assumed that f of z is not equal to f of z0 in this okay.

So, this sentence unnecessary anyway let it be there , so well now so we have m branches of g of z to be 1/m in mod z-z0 strictly less than row okay. So, you see this is again a I proved a lema the previous lecture saying that you know if you have a function which is non vanishing on a simply connected domain then you can find the branch of the analytic branch of the logarithm of that function okay.

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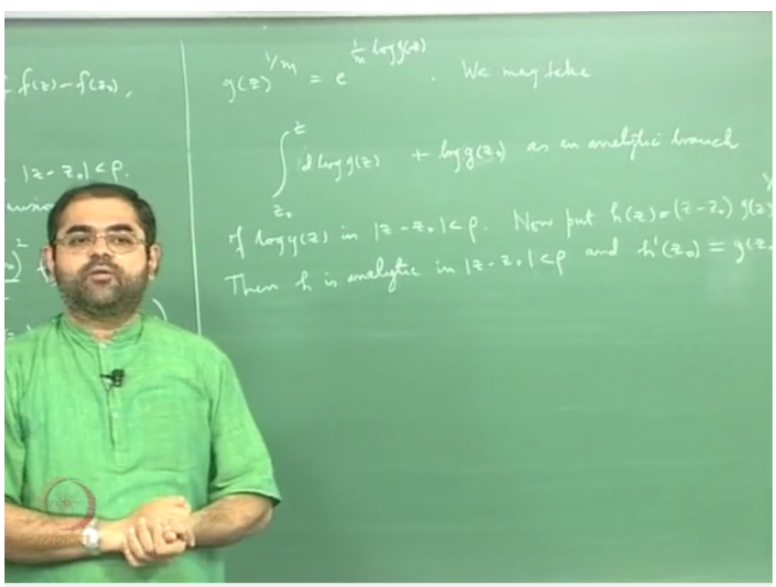

And what is g of z to the 1/m see g of z to the 1/m is e to the 1/m log g z this is what is row by definition. So, you know and you know I if I have an analytic branch of log g z then e to the 1/m log gz will give me an analytic branch of gz to the 1/m alright. So, well of course there are there are going to be m branches but nevertheless let me choose 1 branch so let me write that down .

We may take so if you we may take integral from z0 to z of d log g of z well this will this suppose to give me log gz-log gz0 and if I want log gz I have to add log gz0, log gz0 to this as an analytic branch of log gz in mod z-z0 strictly less than row, this is an analytic branch of the logarithm okay and of course where when I do this integral function z0 to z I can choose any path okay .

The integral is independent of the path this is what you saw last time. So, this is a branch of logs analytic branch of log g log gz and once you have this branch I can write a branch of gz to the 1/m as e to the 1/m into this branch okay. So, that is how I get an analytic branch of gz to the 1/m alright. Now put h of z to be well you know look at this expression z-z0 power m alright.

Now this gz can be written as gz to the 1/m whole power m and I can take an m common okay and so I can write I can take the m through then write that a h of z, o I am writing h of z=z-z0 into gz to the 1/m where gz to 1/m is an analytic branch of the logarithm as I would it is an analytic branch as I would defined it here okay, you put h of z is this, this is already this gz to the 1/m is an analytic function.

And z-z0 is analytic function those is so there this is a therefore an analytic function of the disk and the point about this function is that if you calculate it is derivative at z0 it will not vanish okay. So, then h is analytic in mod z-z0 strictly less than row and h dash of z0 is not equal to 0 because you know if I differentiate this I will differentiate it using the product rule alright.

And when I differentiate where there is I keep z-z0 constant and differentiate that gz to the 1/m and then if I substitute z0 is going to vanish because there is this z-z0 outside and for the other I have to add it to the this term kept constant and the derivative of this which is 1 okay and if I calculate it I will get g of z0 to the 1/m. so, in fact h dash of z0 is actually it is actually equal to g of z0 to the power of 1/m this is what it is.

And g of mind you g of z0 is not 0 g does not vanish where I have written that g of g does not vanish z0 g does not vanish at z0, so g of z0 is a non 0 complex number, so it has a logarithm and this one of and one of the logarithms is one of the logarithmic values will be pick by this branch let have chosen and that is the value here okay and therefore h dash is a non 0 number alright.

So, what this will now tell you will tell you the following it will tell you that you know if you it will tell you that h dash is going to be 1 to 1 I mean the mapping is going to be 1 to 1 in a smaller neighborhood of z0 probably in this neighborhood itself thy the statement .

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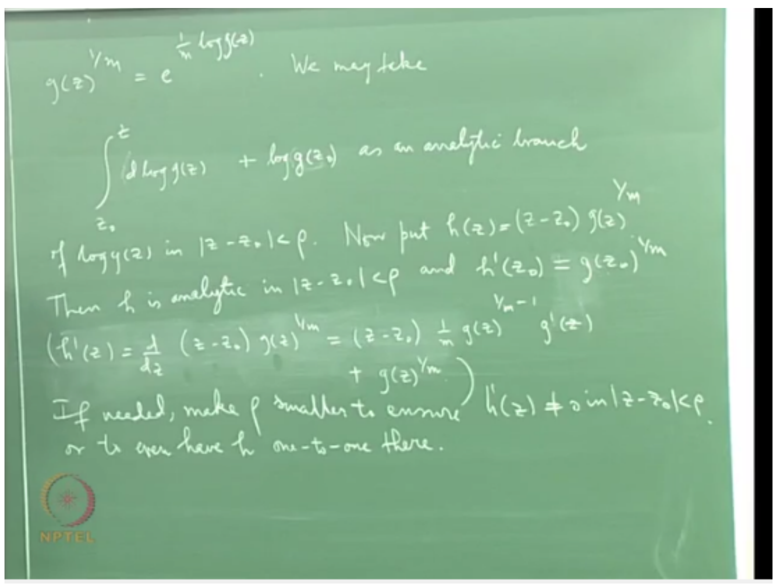

H dash of z is that let us differentiate that it is d by dz of z-z0 g of z to the 1/m and this is going to be at keep z-z0 constant and then differentiate this I am going to get 1/m g of z to the power of 1/m-1 into g dash of z+ I keep gz to the 1/m constant and I differentiate this and I just have to say that this does not vanish probably it does not but maybe let me do the following thing if necessary let me shrink let me make row smaller alright.

So, that I ensure that h dash does not vanish because after all h is an analytic function therefore it is derivative h dash is also an analytic function and so it is a continuous function if a continuous does not vanish at a point then there is a disk surrounding the point where does not vanish just by continuity okay. So, **so** you know well if needed make row smaller to ensure h dash of z is not equal to 0 in mod z-z0 strictly less than row okay

And well and you know and in fact you can even ensure that but probably I do not even need this what I really want is I want a neighborhood where h is 1 to 1 okay, so that I can invert h alright or to even have h 1 to 1 okay this I can do alright and I can do this because of the that is because of the inverse function theorem and using the inverse function theorem here, see I h dash is non 0 at a point.

So, it is non 0 in neighborhood and then inverse function theorem says that wherever the derivative is non 0 you can invert in a smaller neighborhood okay. So, you make row smaller if you want and h will become a 1 to 1 analytic function which you know is an isomorphism on to you it is image because that is what the inverse function theorems says a 1 to 1 analytic function is a isomorphism onto it is image the inverse function is also h inverse will also be analytic alright.

So, now what is advantage of this the advantage of this is at this diagram can now be split up okay how can it be split up can be split up like can be split up in the following way then we can understand the nature of these functions that 1 of w through zm of w alright.

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So, what we do is we do the following thing we have this so I have this disk here in the z plane so this is disk centered at z0 radius row and well what do I do I apply h, so I put a new variable I call zeta=h of z alright I apply h and when I apply what I going to get is well mind you h is 1 to 1 okay h is a 1 to 1 function h is analytic alright and I have chosen row small enough so that h is 1 to 1 okay .

And you know analytic function is a is conformal okay it is a it maps an analytic function with non 0 derivative okay is a conformal map alright in fact I have assume that of course h dash is also non vanishing there alright and that will follow h is 1 to 1 alright and therefore we mapping h is conformal which means that it will preserve angles between curves alright.

So, what I will get is I will get a if I take image of this is this disk I will get something like a disk alright as like this staff and if you want but then essentially it is going to look and it is going to look something probably like this I am just drawing it like this but then the point is that z0 is going to go to 0 okay z0 is going to go 0 because h of z0 is 0 because h is z h of z is z-z0 gz to the 1/m.

So, if I put z=z0 h will go to 0, so it will map the point z0 to the origin alright and well I can choose you know the small enough neighborhood here of the origin alright. So, this is how h is going to map and then you know now what I am going to do is that I am going to put another mapping here this is zeta=z to the m okay when I do this the combine map will be z going to h of z to the power of m.

And z going to and h of z to the power of m is z-z0 power m gz which is f of z-f of z0, see I am trying to come to f okay, so if I you know how this mapping behaves okay. So, you know this is what we saw last time then you have this disk and you know if I draw it for m=3 you know how it is going to look like if I start with ray like this alright then the inverse image is going to be 3 of this things and going to get 3 separated by angles of 2 pie by 3 okay.

And for any general m you are going to get the inverse image of a ray like this is going to be m rays alright separated by angles of 2 pie by m alright and you know that each sector here of factorial angle 2 pie by m is mapped by zeta going to zeta power m on to the hole disk alright. So, this is what you have seen and you know well and you know that there are branches of this here alright.

You know that there are branches what are the branches the branches are zetta to the 1/m these are the branches okay and you know what those branches are you have written down those branches etta to the 1/m branches are there are m branches and the branches are given like this in a given as e to the 1/m log zeta is the first branch eta to the 1/m second branch is e to the 1/m log zetta+2 pie i by m is the second branch.

And goes on like this until the mth branch which is e to the 1/m log zetta+2 into m-1 by i by m okay these are the branches of there are m branches okay in this case m=3 we have 3 branches for example if m=3 and this is the 0.1 until the 3 branches you will get the cue goods of unit alright in generally you get the m through of it is unity if this point where is the point on the real axis with coordinate one okay.

So, inside the branches we have seen this okay and all these branches live they are all you know analytic on the slip plane when I on the slip disk you have to cut out this portion of the negative real axis along with origin then you know that these are all become analytic functions and where they will all live together as a single analytic function it will be above in the Riemann surface for zeta to the power 1/m which is at m zeta covering over this over the punctual disk alright.

So, we have seen this last time and then now what you do is you now you know already when you go from here to here to here you already f of z-f of z0, so to get f of z a f of z you have to add f of z0. So, what you do is you take this mapping this which sense eta to neta+f of z0 this is a translation if you can this then you finally you end up the disk this is original diagram you will get this disk well you get an image.

The image will be something that contains this disk alright will be will do bigger but this point will now be w0 okay. So, this is the whole map w=f of z pictured in a neighborhood of small disk surrounding z0 and it is image containing a small disk surrounding w0 this is how the picture looks like alright and what one is to do is to look at the solutions alright, the solutions in the in this direction okay there m solutions for every w here there are m points which I call z1w, z2w etc., zmw okay these are m solutions.

And I want to tell you that these solutions are also it is always a D solutions will be you know analytic on split disk okay and you can write down now what these z1w through zmw are alright. In fact they are going to be so let me write them down maybe I have space here to do that.

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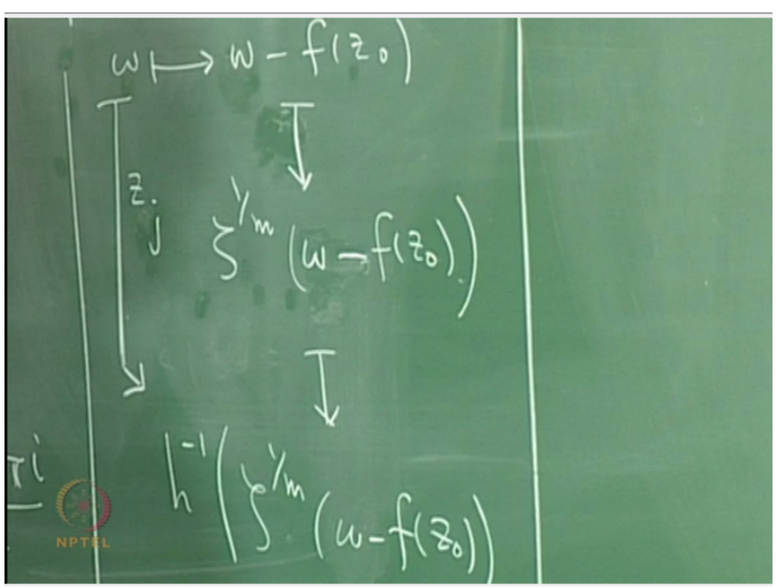

So, you know, so this is what I should get if I take the first one I will get this .

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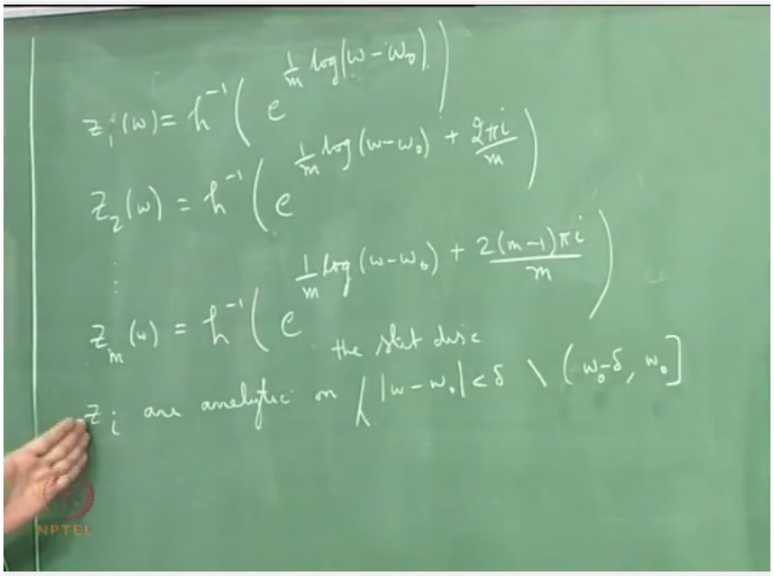

Then if I take the second one I will get h inverse e to the 1/m log w-w0+2 pie i by m this you can take out and write us e to the 2 pie by m times this and so on. I will get and I will get zm of w to the h inverse e to the 1/m log w-w0+2 into m-1 pie by second, so, these are the functions okay in this case for example you can take this to be principle branch if you want you can take it to be principle branch. So, this log I you can take it with the principle branch then you will get the other branches.

But the truth is that invert to be get the principle branch you can take it to be any 1 branch then the then you will get the remaining branches you can of course if you want take the rest of the branch there is no problem okay. And so you see you get these functions the point of these functions is that the zi are analytic on you see there will be analytic on the one analytic on the whole slit on the whole disk but I will have to split out the if you take the principle branch of the logarithm you will have to split out the version of the negative real axis.

From I mean the portion I mean you have slit out this piece okay you have to cut this out alright and so I will write it on I will write this is out mod w-w0 strictly less than delta-this is a line segment from w0-delta to w0 I throw out w0 and this line segment okay on this is a slit disk this slit disk okay these functions are all analytic on the slit disk and what happens is that they are all function of inverses for this.

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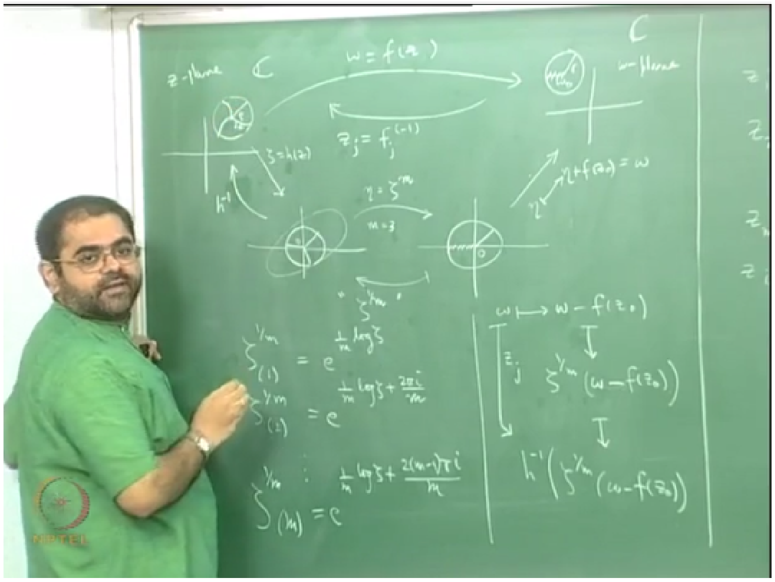

So, you know see you have this you f is of course not 1 to 1 it is m to 1 alright, so there are these functional inverses, so that is the reason I am putting f to the -1 in inside a bracket because it is not 1 there are so many of them and these are the z j s j=1 they are the functional inverses okay because this followed by this will give you identity if you plug in zj=if you plug in for z zj and here I should have written w=fz not fz0 it should have been w=fz.

See if you plug z, zj in f okay f of zj will give you back w I have been it will give you back cut it will give you back w okay all these zj are functions of w mind you because this is a w plane this is may complex plane this is the w plane these functions of w. so, you take this zj of w you plug it in f of z you will get back w, so this followed by this is an identity map that means you have m inverses you have m solutions for this equation w=f of z in a neighborhood of the critical point that is what is happening okay.

And the mapping looks to mapping can may describe like this okay, so in fact you know if I so you know if I take the inverse image of the disk here this is after all translation again I will get this disk centered at the origin okay and then if I you know when I go like this I am taking the fractional mth power okay. So, you know it is going to look like this and then from here to here when I go by if I go like this it is h inverse.

And h inverse is conformal therefore you know this will resulting something like this okay will get something like this, this is for m=3, so more generally if you write m if you take any m you will get m you know you will get m curves centered at z0 and going out radially okay and that is how the mapping looks like okay. So, you know if you forget this essentially this is just a distraction of this map.

This map is just a distraction of this map and what is this map this is z going to z power m it just a distraction of the power map. So, what you are saying is if a function f you look at a function the behavior of the mapping at neighborhood of the critical point of order m-1 you know up to a conformal twist okay it will look like z going to z power m that is what we wrote through okay and therefore it will have m branches the inverse function will have m branches.

And these are the branches alright and where can you make sense of them all as a single analytic function what you have to do is that you have to put a Riemann's surface over this which is a m sheeted cover on that all these will become analytic okay and they will become a single valid function, so the moral of the story is that you get a single valued inverse for a function even at a critical point. But the inverse lives on a m sheeted covering that is the point okay that is how it looks alright, so I will stop here.