Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemannn Mapping Theorem

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Lecture-16
Constructing the Riemann Surface for the m-th root function

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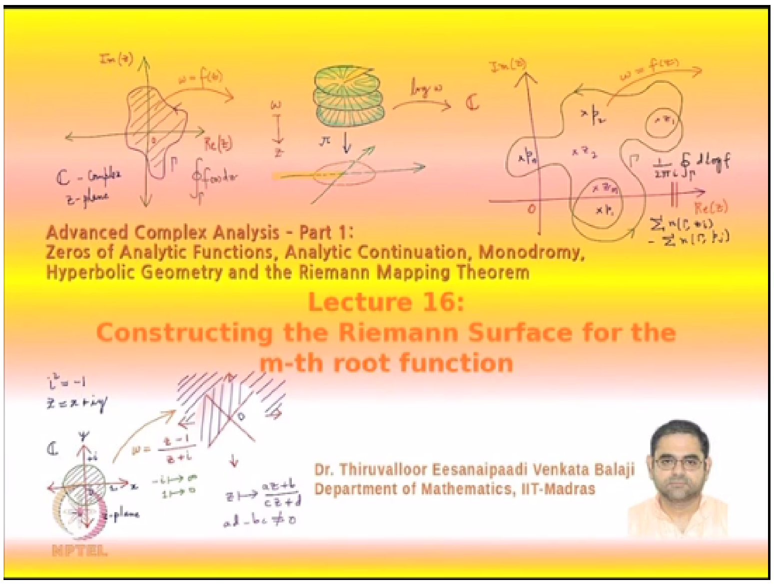

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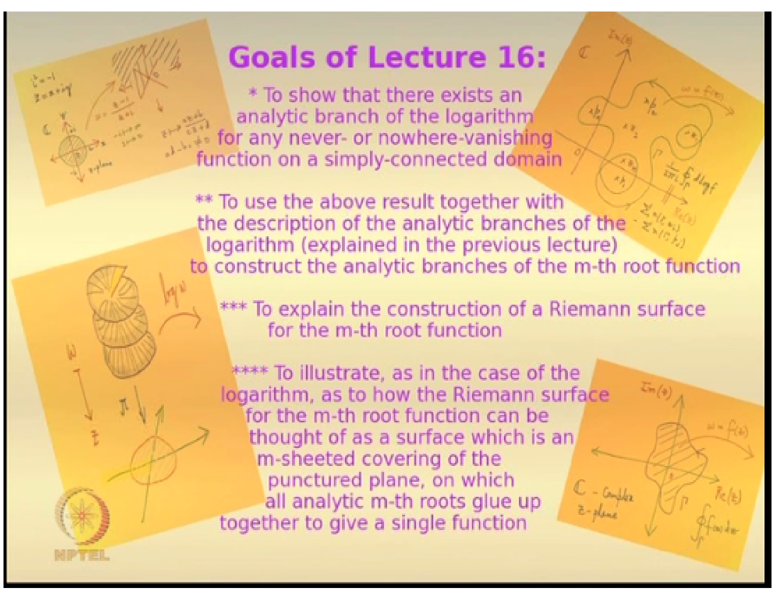

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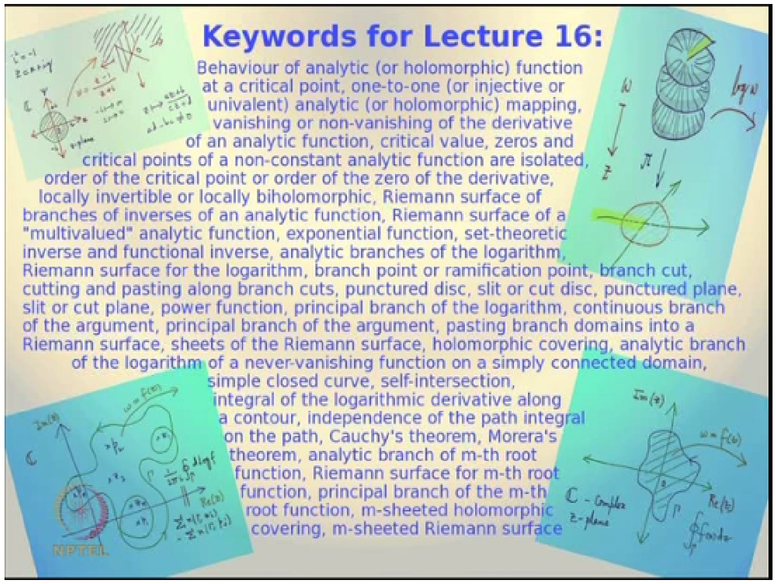

Okay so, what we discussing now is trying to understand the behaviour of a function as a mapping in a neighbourhood of a critical point. So, if you recall a critical point is point where the derivative of the function vanishes okay. And of course you have already treated the case when derivative does not vanish. We have already seen that the derivative does not vanish then thee is a small neighbourhood about the point where the function is even

And you can invert the function okay so, we come to the case when the attempt or the case of a point where the function has vanishing derivative. And to study the behaviour of the function at a point where the derivative vanishes we needed to understand the idea of branches of a function especially branches of a logarithm okay. And that is what I explained in the last lecture.

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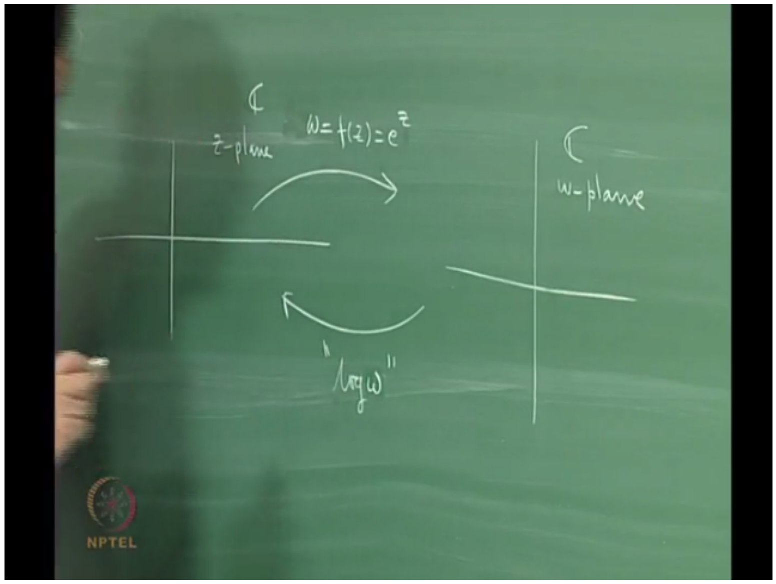

So, let me recall so, the so, we have this so, we recall that is we had the complex plane and we had the function F of z is equal to e to the z is a exponential function. And this takes values in the complex plane and the point is that we wanted to think of an inverse to this function. And the inverse function would be where the logarithm logz okay.

And of course how this log is defined first of all we should understand in a set theoretic sense that e power z is not a many I mean sort of 1 to 1 function. So, it does not have a set theoretic inverse okay. It is a many to 1 function okay and in fact if you take any value of z which is translate a z by and integral multiple of 2piei the exponential function will take the same value.

Because e to the 2piei is 1 okay so, therefore it is not a 1 to 1 function and intensive l one should not think about one should say that one is inverse function okay. Because it is you talk about a inverse function only if a function is injective okay. But we want a function which is inverse in the functional sense namely function that can undo the effect of exponentiation so, the logarithmic function is suppose your function that under that cancels the effect of exponentiation okay.

But the fact is that when you try to write out function in this direction okay then you get not one function you get several functions okay. And these are called branches over logarithm and if you recall how we defined the branches was like this what we did was.

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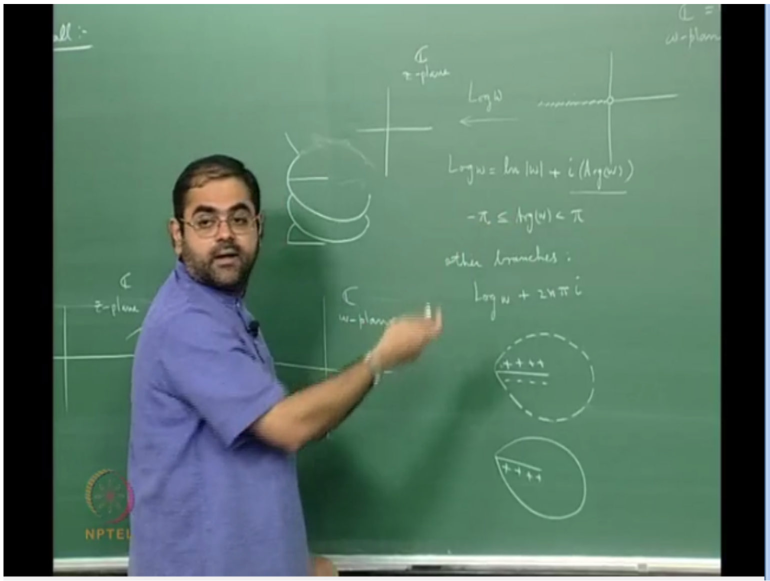

We took we defined first of all what is meant by the principle branch of a logarithm okay and so, well let me not say let me call this plane as the z plane and well then this becomes the omega plane or the w plane where w is a f of z. And well the inverse function should therefore the log w okay and so if I follow that convention here this is the principle (()) (04:14) logarithm.

So, this is from the complex plane which corresponds to the w plane and what we are going to do is to define the logarithm first of all you cannot define the logarithm of a g of this of the 0 contest number. So, you remove this okay so, what you do is you look at the c star which is c-origin you throw out the origin because the exponential function misses out the value 0 it never takes the value 0.

So, there is no logarithm for 0 so, you throw that out and what you do is well how do you define the principle points as the logarithm well you say the principle are the logarithm is given by take the natural logarithm of the nanic the positive real number mod w. So, modulus of w and add to it and that is the real part and the imaginary part is the principle argument of w and what is the principle argument of w.

It is this angle from –pie less than or equal to argument of w (()) (05:17) okay and if you define it like this log w make sense as a function and takes values in complex numbers. And if you and this is the z plane and well the fact is that this is an inverse functional inverse to e power z. Because you take z and then you take e power z and you plug e power z for w and you will get back z okay.

So, in that sense well it is an inverse function alright now well the problem is that we want an inverse function which is you know we want an inverse function which is not only is not a just a function we want it to be continuous we want it to be analytic. Because we are interested only an analytic functions now the problem is that this log w.

It is define on the whole punctured plane okay that is a plane-the origin. But is not continuous because the continuity the problem with continuity is at this negative real axis. And that is because the continuity is with the imaginary part of the function which is the argument the argument function is not continuous verse argument function on the negative real axis in just at points below the negative real axis.

The argument is closed to –pie and just above the negative real axis it is +pie and this as the jump in the argument of 2pie nearly. And this jump tells you that the argument is not continuous across the negative axis. So, what will have to do is well you cut this out okay there two ways of doing it one is you cut out this and by cutting this out I am declaring that the portion of the negative real axis above negative axis is far away from the portion that consist of the negative real axis and the points below it.

And when I say they are far away then I do not have to check continuity across the negative real axis okay. So, in that sense the cutting helps okay so, if you cut the plane so, has to separate the portion of the negative axis above it from the portion of the from the negative real axis and the portion below it. Then it is called a slit plane okay and on that plane log is of course continuous.

And continuity on points above the negative real axis is means continuity in a small disc about such a point. And for continuity the same will whole for continuity of at a point below the negative real axis. But for a point on the real axis it will be continuity on the boundary okay. So, the neighbourhood will only be a disc only the the neighbourhood will consist of an open disc only the lower half of the open disc centred at that point on the negative real axis okay.

And the upper part of the disc will not be considered as part of the neighbourhood. Because it has been cut off okay or it has been separated alright so, the point is that if I take log w and slit this plane okay mind you. When I slit this plane I am not throwing out the negative real axis and keeping the negative real axis okay only I am separating the negative points on the negative real axis with all the points above it okay with that kind of slit plane okay log becomes the continuous function alright.

But then I am not happy with a that we would like to have log as a an analytic function. But you see but to have a function which is analytic you know the definition of analyticity is that it is defined at a if a function is to be analytic at a point first of all the function should be defined in a whole neighbourhood of the point okay. So, what happens is that if I take a point above the negative real axis or below the negative real axis certainly.

There is a small disc surrounding that point where the log function will be an analytic okay. Because that you can verify by any number of names for example you can check the easiest way is you know that these two the real and imaginary parts you know for example satisfy coefficient equations. If you want you can verify that and verify that the first partial derivatives are continuous sense so on and so fourth.

So, it is very easy to check that this defines an analytic function of w if you throw out the points on the negative real axis okay. And but for points on the negative real axis is a problem because they are boundary points okay if for the slit plane the points on the negative real axis is a boundary okay. And at a boundary point you never define an analyticity because analyticity is always defined at a interior point.

Because function is a analytic at a point the definition already requires the function should be defined in a open neighbourhood containing the point. So, you never define an analyticity at a boundary point okay but then so, you if you want think of log w as a not just continuous function but as an analytic function. Then I will have to delete the negative real axis I have to delete the real axis.

And I will have to and then log becomes you know it becomes an analytic function alright. So, I will have to throw out the origin along with an negative real axis then it becomes an analytic function. And the point is that well this is not the only functional inverse to the exponential function there are many others. And the other branches of logarithm are given by well.

Other branches are given by taking actually the various branches of the argument function which before by fixed integer multiple of 2pie okay. So, the other branches are given by the principle branch+2npiei so, I just add 2npiei and get the other branches okay. So, all these put together will give you the various branches of the logarithm alright.

And well now there are two issues that one has to worry about one issue is that I have so many branches. And I am not able to see them as it will be nice I can see as one function okay. The other issue is that I have problems with defining logarithm on the boundary on the boundary I mean the negative real axis that I have throw out okay.

So, I mean well you know the negative real axis had to be thrown out because I chose my argument to be from –pie to+pie and that creates a problem at –pie alright. But instead of there is nothing special about the angle pie okay accord of chosen any ray okay. And I could of angle as theta and I can replace this side by theta and that side by theta+2pie alright. And then I would get then I would get different branches of logarithm which for which the slit will be a along that ray alright.

So, there is nothing special about the points on the negative real axis okay. So, for the modular story that I would like to really if necessary I would like to do have you even a branch an analytic branch of the logarithm on the negative real axis with which I can work okay. And how is all this possible all this is possible by looking at the Riemann surface for logz. So, I told you that what we do is that well.

You know we pace together as I explained in the in the last lecture we pace together you know so many copies of the plane carefully along the well along the slit negative axis. So, what I do is that I have several copies of the slit plane and mind you I am not throwing out the negative real axis okay. But what I am doing is that well take I identify the upper part and the lower part.

And you know I stick it in such a way I take two such copies of course you know this is I am just boundary bit there is no boundary. This is just to tell you that this is a piece of the plane which looks like a well disc right. But it is a whole plane that I want to you to match in and I cut it across the mind you I would not deleted the negative real axis. I just cut along the negative real axis in such a way that I have separated the negative real axis.

And the portion below it from the portion above the negative real axis and the reason is because I want to joined the portion above the negative real axis to the negative real axis. The portion below it in another piece okay so, this these thing which have denoted with +signs on one piece. Well I will you know attach them to this piece okay and well the these the portion on the negative real axis.

And below it will be attach similarly to the portion above the negative real axis in the piece above okay and these said the various pieces which corresponds to the different branches of the logarithm okay. And therefore the real the resulting picture will look like this the resulting picture will be that of a Riemann surface.

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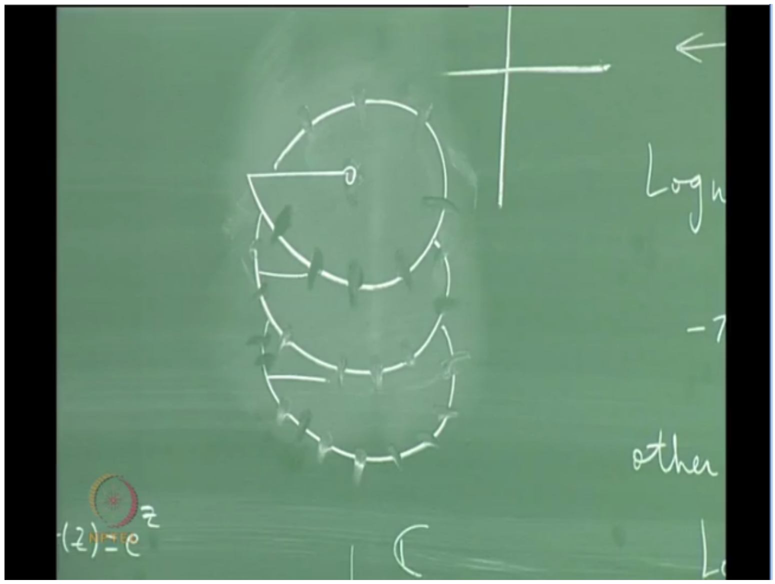

It will be so, it is going to be I will have one piece like this and well and I have attached yeah it is a little difficult to draw it. Yeah so, it something like this say if I keep joining it along the negative real axis I will get something like this okay. I will get the diagram that looks like this and in this diagram what has happened is that I have taken so many copies of the plane.

The which is just so many copies of the punctured plane I have no thrown out the negative real axis but I have done the cut and paste along the negative real axis in such a way that I get a continuous surface alright and if I do this continuously what I get is Riemann surface okay in the sense it is a surface namely it is locally homeomorphic to the plane okay. Because it is considered it is concuss it consisting of various sheets as you may think of it.

Each sheet being a copy of the punctured plane okay so, it is locally homeomorphic to the punctured plane alright and it is certainly house dwarf and it is second countable and therefore it is immediate that what you get here is the by definition of Riemann surface, a Riemann surface is suppose to be well topologically at least at the suppose to be something that is locally homeomorphic to the plane.

And which is house dwarf and second countable and of course on top of that you also need a collection of compatible charts to be able to do complex analysis on that to be able to decide when a function defined at a point neighbourhood of the point can be called an analytic function but then you get the charts because these homeomorphisms will allow you to identify the certain sheet with the punctured plane.

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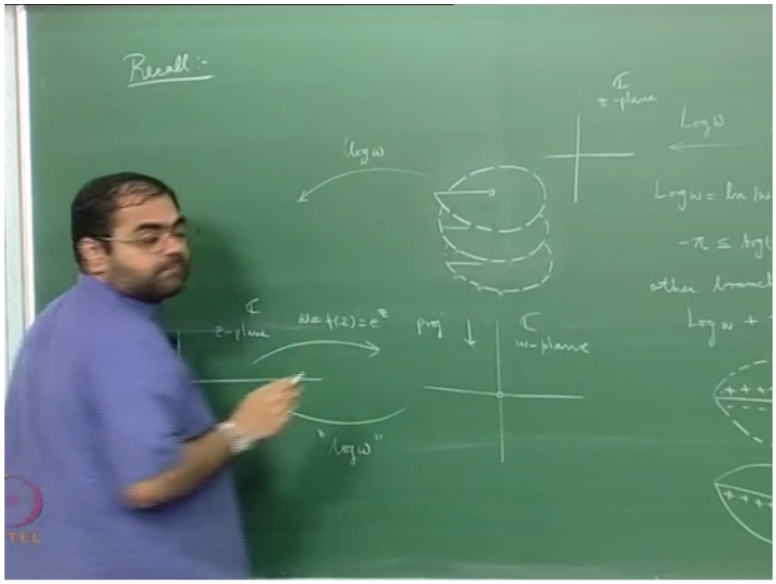

So, there is a projection like this, so here is the projection, this is the projection map, so we say that this is a covering of the plane of the punctured plane, so you know I can remove the origin here and what happens is that the this is a covering which we say is a countable covering because there are as many copies as a integers and because each copy corresponds to choice of m which is a domain of the branch of this logarithm alright.

And the moral of the story is that once you take this so this Riemann surface which is the Riemann surface for log z. Then the fact is that from this Riemann surface if you take the function log I can define a function log w on this surface which is the single valued function okay mind you this surface extends in infinitely in both directions I have just gone a 3 sheets okay

But it exchange infinite on both each corresponding so but the point is because I pasted in a augmented function on one piece if I take only for example if I take the principal argument it this continues at the point it function on these becomes well if I did not separate the cut the plain just for the point on the relative augmented -5 below above the value +5 but then what i do join this position of the negative realise and the another function of abut close to -5 and the -5 and do this what i done is extend the augmented.

Function so, the looming functions from each of these which correspond in the imaginary parts point is because the nice ways the augmented and the modular stories is ones take it the surface log said and then the fact is that on this surface. Single value function long function okay on each its corresponding to and all put the single augmented to the explanation function.

So the modular stories that if u have a function is an analytical function and then u have t minded extends defined the function but externally in the function so, both the point is because nice in the point way the augmented function is on this define its continue its becomes well if i did not but it generally get several inverse function so they branches will leave on remain surface what are the branch resize branch point of the functions

We choose for the branch for the branch cuts the logarithm of the function because the branch point of the in the negative on the cut of the various of the remain surface on which all the branches the also as the single function so, continuously function long double at the mind is the remain function from every he is are every sheet you have the projection which identifies it with the complex plane okay.

And if you calculate the transition functions that the transition function will only be a translation by multiple of 2piei and you have translation up to holophormic. So, the moral of the story is that these homeomorphism is that identify each sheet with the punctured plane they give charts. And the compatibility of charts is automatic. Because if you calculate the transition functions.

They will simply be translations by you know multiple of some multiple of 2piei but you know every translation is holomorphic. And the condition for charts to be compatible is that you know the well the condition for charts to be compatible is that the transition functions are holomorphic okay. So, the moral story is that you get a Riemann surface.

So, you on this surface you can ask whether a function defined on the surface is holomorphic and answer is if you take logw on this surface it is holommorphic okay. And it least throw out the surface in it is holomorphic okay and that is the that is what happened here and that is what happened for general function also sufficiently good function you will see that function will when you try to write out the inverse function you will get several branches.

These branches will be well defined and holomorphic if you make branch cuts along so, called branch points. And if you branch cuts properly and do cutting piece in process then you will get a Riemann surface on which all the branches will define a single valued function okay. So, this is the general philosophy alright and the point that you will have to understand is that well .

One important thing is the non-vanishing nature of the derivative of the function that you sort to invert. So, in this case I am trying to invert e power z is derivative is e power z that never vanishes okay. So, the non-vanishing of the derivative is something very very important alright. And so, what happens it more generality is the following so, well the more general situation is like this.

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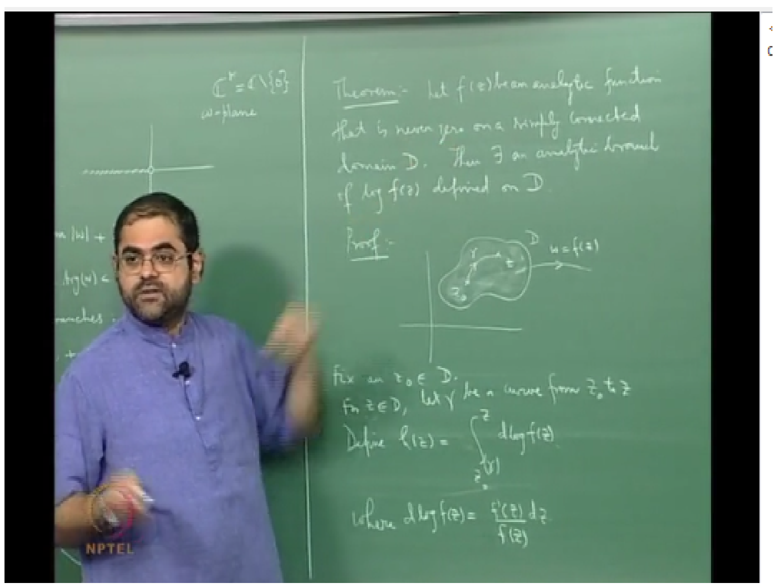

So, here is the theorem which we often will gives the theorem is that so, let me stated let me stated like this let F of z be an analytic function that is never 0 on a simply connected domain D okay. Then there exists an analytic branch of logf defined on D okay. So, see here I am what I have done here is I am trying to get the logarithm of a complex number alright that means I am trying to invert e power z alright.

But more generally suppose I have a analytic function that never vanishes alright suppose I have analytic function that vanishes never vanishes on domain and assume that the domain is simply connected that means the domain has a property that it has no holes geometrically it means that given any loop in that domain that is the simple closed curve in the domain. You can continuously swing to a point without going outside the domain okay which has essentially means that.

You draw any simple closed curve of course means simple means that here most self intersection okay. So, a closed curve is a u continuous image of an interval with initial point equal to the final point okay and you take any such simple closed curve like the fact that you can vision continuously to a point is same as saying that the region that in encloses is also we inside domain there is no part of that region which goes outside the domain which means that it is a wholes in the domain.

So, there no hores that is what simply connected this means and the point is that if you have a function which is nowhere 0uh not that which means is not 0 at any point of a domain is simply connected then you can find an analytic branch of logarithm of the logarithm of that function of that domain okay. So, the proof of this is well pretty easy probably you already seen this in first person complex analysis.

But again I can recollected for you so, you see so, here is my well here is my simply connected domain I have drawn a bounded domain but again it need not be but the domain is the this region which has been holds okay. And well and you know I am what is given is that there is a function w equal to F of z which is never 0 on this domain alright.

So, what I do is I do the following thing what you mean by saying that I have a logarithm of a F of z and an analytic branches of the logarithm it means it is an analytic function which is a logarithm of f of z namely that function is what if exponential that function you will get a f of z that is what it is alright. So, you have to find a function h of z which is analytic in this domain okay.

And such that h of z is logfz but what is that mean it means that e power hz should be fz okay. So, if exponentiation that function that trying to find you should give you f that is what it use. So, well how do you construct such a functions is very very easy what you do is you fix a point is not in the domain alright take any other point z. And it is very simple take any curve gamma from to z to z0.

And you define an integral fix an z0 in D let for gamma for z in D let gamma b ea curve from z0 to z of course by curve we mean a continuous image of an interval alright define h of z to the well integral from z0 to z along gamma. So, I will put this in brackets and of course I am going you know what I am going to write. I am going to write D log fz okay which you know where of course.

You know what D logfz means but D logfz is suppose to stand for f dash of z, f dash of z by f of z is okay. So, I define this function alright this mind you f of z does not vanish okay. And f is analytic alright therefore f dash is also analytic. And f does not vanish so, f dash by f is also analytic alright therefore the integrant is analytic function right therefore this integral is well defined first of all.

You must understand that first of all for integral to be well defined I mean the integrant should be continuous function. And the integrant here is f dash by f okay the continuity will not be a problem for f dash because f dash is a derivative of analytic function. And you know derivative of analytic function and you know derivative of a analytic function is also analytic. So, it is a also continuous but the problem is there is an f in the denominator.

And that should not vanish okay but then we have assume that f is never 0 on the domain therefore there is no problem about f being the denominator therefore you have a questioned of continuous function therefore it is continuous. So, in particular is also continuous on the path gamma therefore this integral is well defined there is no problem about this integral okay now what I want to say is that.

This integral is independent of the path okay see if I why is that true because if the integral is dependent on the path. Then this h of z will not be properly defined because to define h of z I have chosen of path okay which is the path that connects z0 to z of course inside D I am not going outside D alright. But the but this h of z should not depend on the path if it depends on the path.

Then it is not well defined function but I claim it does not depend on the path why because you know if I choose some other path say gamma prime alright. Then if I you know that I go along gamma and if I come back along the reverse of gamma prime alright. Then I get a closed loop and on that closed loop if I integrate this if I look at the integral of the D logfz that is f dash of z by fz I will get 0 by Cauchy theorem.

Because f dash by f continuous to be analytic there because you see f dash is already analytic f is also analytic the quotient is analytic wherever the denominator does not vanish. But the denominator never vanish the denominator is f it never vanishes therefore the modular story is because of Cauchy theorem this integral does it depend on gamma. It does not depend on what path you choose to connect z0 to z. So, long as you make sure the path lies inside your domain okay.

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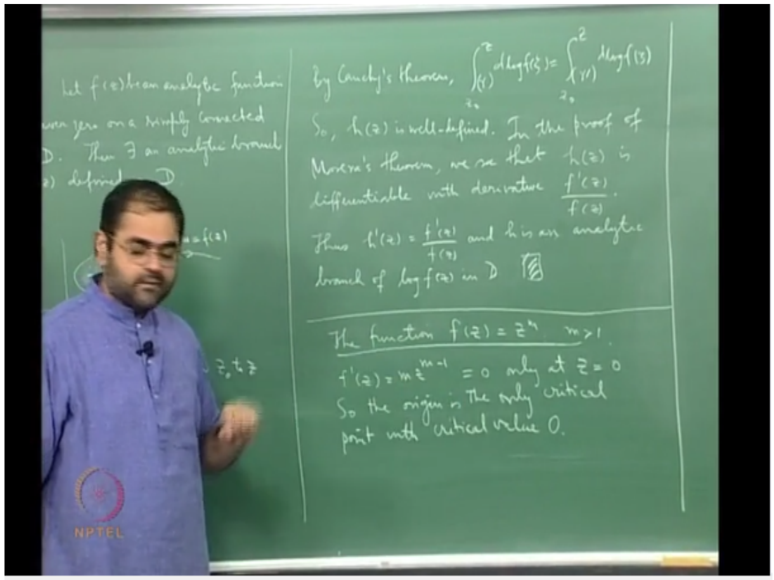

So, by Cauchy theorem you know integral over gamma from z0 to z D logfz is the same as integral over gamma prime from z0 to z D logfz well actually you know I should be a little careful because of use z here I should not use z in the integral. You know maybe I should use zeta alright it is better to use zeta. And then I will also change this to zeta so that it is easier.

Because z is supposed to be the fixed value the fixed point where I am trying to defined function h alright. So, let me change this zeta let make a difference between the variable of an integration and the limit of the integral. So, well here it is this is better notation okay and by Cauchy theorem this is equal to this. Because the difference is 0 by Cauchy theorem okay so, this tells you so, h of z is well defined.

So, it tells you the h of z is well defined now you see now look at h of z now again let supply another theorem namely morares theorem to conclude that h is actually analytic. You see if you look at the function so, you know morares theorem is a kind of converse to Cauchy theorem okay. What does Cauchy theorem say it says you take a function is analytic if you integrate it on a closed simple closed curve.

And assume that a function is analytic inside the curve and also on the curve then integral 0. Now the converse to cautious theorem will be if you have a function for which whenever you take a close curve the integral is 0 then is that function analytic that is the expected converse but for the expected converse is more or less same you need to put the extra condition that the function you are starting with is already continuous.

So, you see in this case you see I am integrating the function that I am integrating is f dash by f okay and f dash by f if continuous throughout the domain right and it is integral over any close curve is 0 because is analytic therefore it satisfies the continuous of (()) (35:13) theorem and therefore this integral is analytic function okay. so, let me write that in the proof of Morera's theorem we say that h of z is differentiable with derivative f dash of z by z of z okay.

And so the moral of the story is so in particular the proof of the theorem tells you that h is analytic okay. so, thus h dash z is f dash of z by f of z and h is a branch h is an analytic branch of a log f in D okay **.** So, it is easy define I mean the looking at the logarithmic derivative it is easy to define a branch of the log and a branch of the log which is analytic and simply connected domain all he need is a simply connected domain alright.

And okay I need this now what we will do is that we will try to study the function z going to z power m at the origin okay and what we claim is that you know this is how any function will behave at a critical point up to you know translations and up to some simple transformations okay any function at a critical point will behave like the mapping z going to z power m at the origin okay that is what I want to explain.

So, **so** the function f of z=z power m and I will assume that m is greater than 1 okay. so, look at the function of course if m is equal to 1 is the identity function those in much to say at you want to look at this function and if you look at if the then the derivative f dash of z is mz to the m-1, m is at least 2 and this is 0 only at z=0 namely the origin. So, it has only one critical point so the origin is a critical point is the only critical point with critical values 0.

The origin is the only critical point and the critical value 0 okay and of course I would like to study the function this function at the in a neighbourhood critical point namely neighbourhood origin. And the reason I am doing this is because I have this theorem about the availability of a logarithm for an non-vanishing function are simply connected doing.

I can use this to study the behaviour of any analytic function had at point which is a critical point okay. So, this is a foundation so, let us understand this case alright so, well the first thing that you can see is well I have so, I have this so, I can draw a diagram similar to the one here okay that where I have consented Riemann surface for logz okay namely a surface which covers the punctured plane on which log is all the branches of the logarithm leave together is one function okay.

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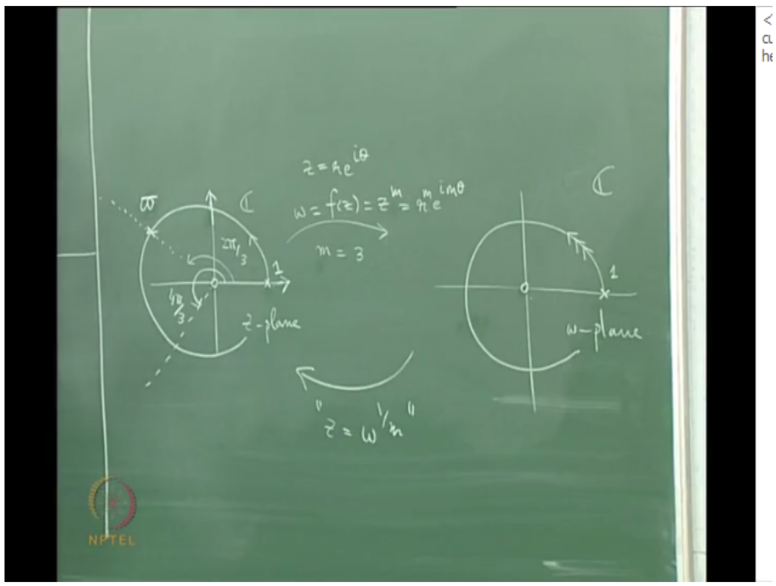

So, I am going to do the same thing here so, what I am going to do is I am going to just take the function ow is equal to f of z is equal to z power m okay. And this of function that goes from the complex plane to the complex plane and this is the z plane and this is the w plane. And well of course the critical point is the origin alright so, I will single that out by putting the circ a small dot circle a dot here.

So, the origin goes the origin and you know that the inverse function if I think of the inverse function it has to be z is equal to w to the power of 1/m which is what the inverse function should be which should be the un m th root of the variable that should be the inverse function alright. Because this way the function rises the variable to the power m the function on in the other direction should take m the root.

But the point is you do not have 1 m th root you have in fact m of them and these have different branches. And the branch point is the origin and the branch cuts are all rays the branches are defined on sectors alright. So, let me draw the diagram for let us take m equal to 3 okay. If I take m equal to 3 then you know pretty well that you know if I take this ray which is well it is 2pie/3 is this angle alright.

Then as z for this whole domain namely this whole sector with angle 2pie/3 is going to be map down to the whole plane alright that is because if I write it in power coordinates z power m if I write z as r e to the I theta then you know z power m will become r to the m e to the im theta. And if m is equal to 3m I am going to get e to the i3theta as theta varies from 0 to 2pie/3 3theta will vary from 0 to 2pie.

So, this whole sector okay is going to be map to the whole plane okay. And similarly you know if I take the other sector namely if I take this which is going to b e4pie/3 okay. Then this sector beginning from here to here that is an angle of 2pie/3 and that is again going to be mapped on to the whole w plane okay if you remove the origin then the origin there is also removed right.

And well similarly this sector which again is the angle of 2pie/3 alright that is also going to be mapped on to the w plane alright. And of course so, you know this whole copy of the z plane is being mapped thrice on to the w plane. When m is 3 and you know when it is m is going to be mapped m times and it means that you know if you take the image of if I for example if I take the unit circle here getting the unit circle here.

And I right travel along the unit circle and you know I look at it is image there what I get it is well I get the unit circle. But it will be travels three times if nh3 and it will go around m times for general n okay. And this also tells you something for example you know if I have so, you know if I take this point 1uh the unit circle will of course go to the unit circle only thing is it is going to go here 3 times as I go there 1 around once .

You get actually if you look at the point 1 it goes to 1 but then if you try to take the inverse function then you get the cube roots of unit right. Because you are taking the when I take m equal to 3 I am trying to raise I am trying to look at values of 1 to the 1/ 3 the which are the cube roots of unity. And so, what I will get is the these 3 values I will get 1 value if you call it as omega the other yeah.

The problem is this omega conflux with the w there so, probably I will use so, I will use a funny omega this is the omega with the dash on top fit. So, this is an omega with the dash on top of fit which is a complex cube root of unity. And of course you know the other one is omega square okay. And all the 3 points 1 omega and omega square all th e3 will be mapped to the point 1 okay.

So, you know what is how going to happen you know that there going to be 3 branches for this function alright. They going to be 3 branches for this function and the 3 branches again are going to how do you how are you going to write them down. So, that they are analytic they are continuous so, we make use of this theorem okay. We make use of this theorem so, let me go back to so let me rub this for diagram. And go to some heuristics when you define powers okay.

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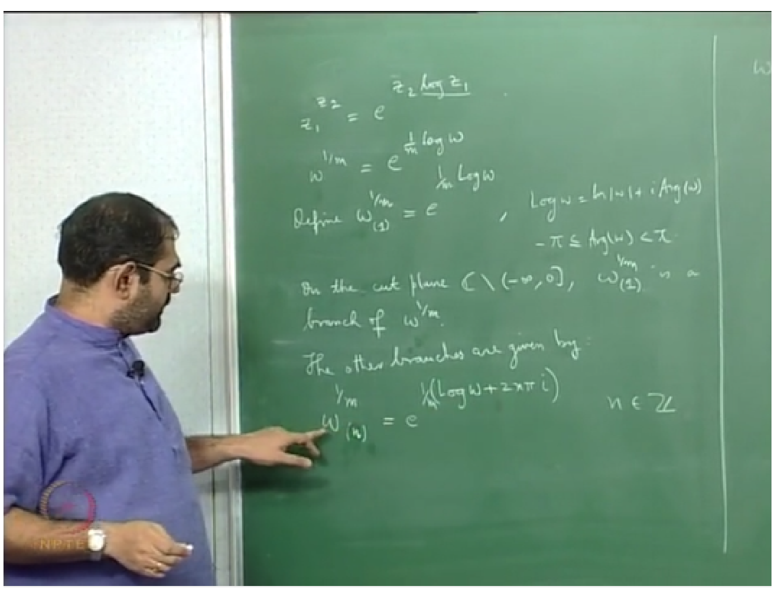

So, you define z1 to the power z 2 how does one define this complex analysis are more generally analysis you define as x e to the u log z 2 logz1 may this is how you defined okay. So, because if you know the rules z2 logz1 should become log of z1 to the power z2 okay and e and log has to cancel. This how you define for example if a and b are real numbers okay you would define a to the b like this.

The only problem is that when you define this log has several values. And therefore if you take z1 to the power z2 you get several values okay. And the those values come because the various values of this log. So, in the same way you know if I want to follow that example and I want to define the function w to the power of 1/m okay. Then how do I do it I would define in the same way w to the power of 1/m to be e to the /m logw.

This how I would define it okay and now the question is how do I define log properly you know that there is so many branches of the logarithm. And so, and you know all these branches of the logarithm they all differ from the principle branch by adding multiples of 2piei okay. So, what you do is that you define one branch using the principle logarithm okay.

So, what you do is so, this tells you that you define w to the 1/m okay and when I say this and maybe I will put here I will put a lower round bracket one to say this is the first branch this one branch I am defining. And I am defining it as e to the 1/m log w okay again define it like this where logw is the principle branches of the logarithm where logw is defined as lon mod w+i times principle argument of w.

The argument of w of course varying from I make a well branch cut from –pie to this okay. And you know that if I throw out the negative real axis you know that principle logarithm is a is you know an analytic function. And therefore I get an analytic function which will be one branch of w to the power of 1/m okay. So, on this found the cut plane on the cut plane c-negative real axis which is well –infinity to 0 okay.

I throw out the negative real axis w to the power of 1/m subscript 1 in brackets is a branch of w to the 1/m okay it is a branch of the m throughout function alright and now so once you have this 1 branch then how do you get the other branches are given by well I need some notation so I think I will get let me look at it for a minute w to the 1/m.

So, you know you are going to get m branches right in the case of w=z cube you get 3 branches okay 1 branch under each of those branches the image of 1 will be under 1 branch it will be 1 and the other it will be w this is a complex omega which is complex unity and under another branch written as a third branch it will be omega square okay, so you get 3 branches.

So, if it is m you will get m branches and what are how do you write those branches they are you can write them as e to the 1/m log w+ 2 pie 2 okay maybe I should not use i because is always a complex cube root of unity and it it is a root of square root of unity. So, let me use j or better I use even n alright, so then I will have 2 2n pie by i okay these are the various branches of the logarithm alright.

And you will see that essentially you will get only n distinct once you will get only m distinct one okay you will get only n m distinct ones because well it is you know if I start with , so you know if I start with .

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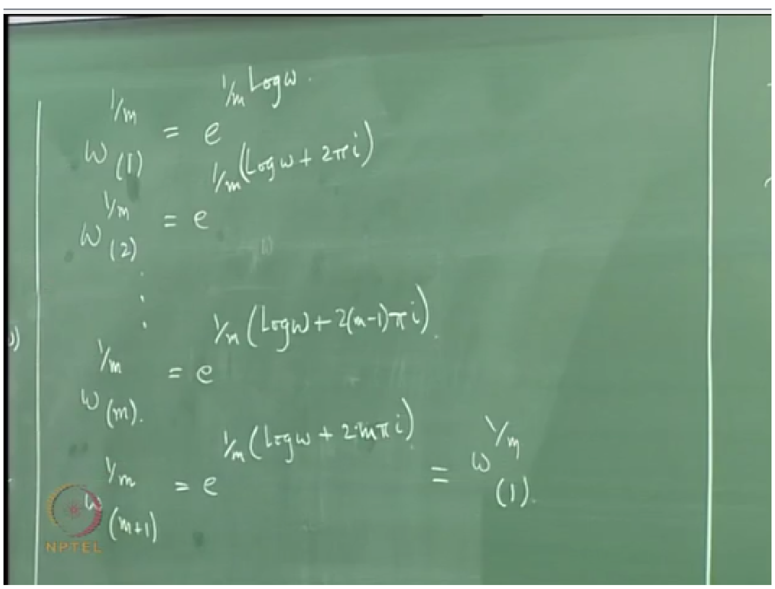

So, I will start with w to the 1/m if I take and here if I put n is an integer okay then these are the various branches of the logarithm and therefore I should the various branches of the m throughout function but mind you then infinitely many branches of the logarithm there are as many branches of the logarithm as there are integers alright. But then if you take the m throughout function there are not so many there are only m branches.

And the reason is because there is 1/m that is appearing outside this n has to read mod m because it will repeat n mod m will repeat okay, so you will get only m branches. So, you let us write it down see for example when I put w0 1/m is my original notation 1, so this is really bad , so I just change it so that I my w1 is actually w1, so which means I will have to put okay.

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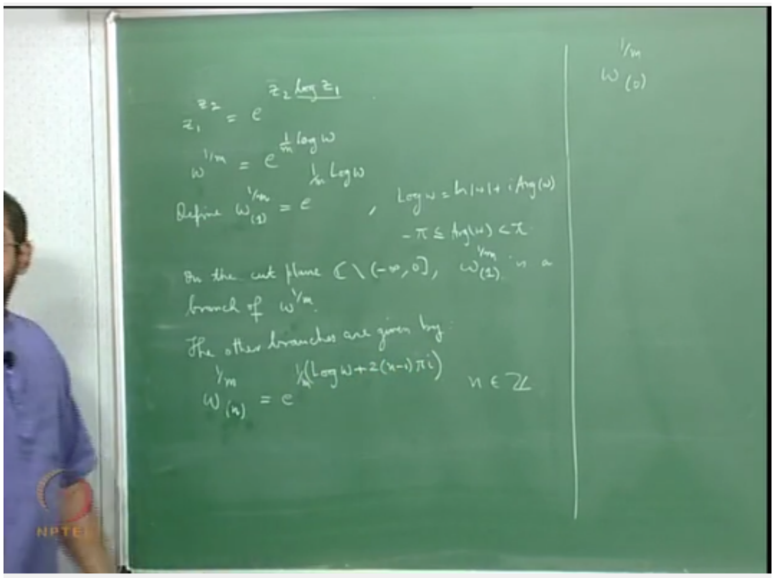

So, I will have to make the following adjustment I just put it as 2 into n-1 pie i if I do this alright and I put n=1 then w w1 is going to be well my w1 it is going to be e to the 1/m log w alright this is what I get when I put n=1 alright. Then I put n=2 I am going to e to the power of 1/m log w+2 pie i alright and this go on up to w to the power of 1/m the well 1 to m when I put m what do I get I will get e to the power of 1/m log w+ 2 into m-1 pie i I will get this.

And mind you all this are distinct they are all distinct because there is the e to the 2 pie i by m which is getting multiplied alright the this function is this function multiplied by e to the 2 pie m every successive function is a previous function multiplied by e to the 2 pie i by m okay these are all distinct functions they are all mind you they are all analytic functions, they are all branches of the root function alright.

But then if I put m+1 I will get back w1 okay because we have put m+1 you know w1 to the if I calculate w 1 to the m m+1 what I will get is I will get e to the 1/m log w+2 pie 2m pie i okay and then you know if I simplify e to the what 1/m into 2m pie i will give me the 2 pie i **i** and that is 1. So, this will again go back w to the power of 1/m the first branch okay, so you will get a reputation.

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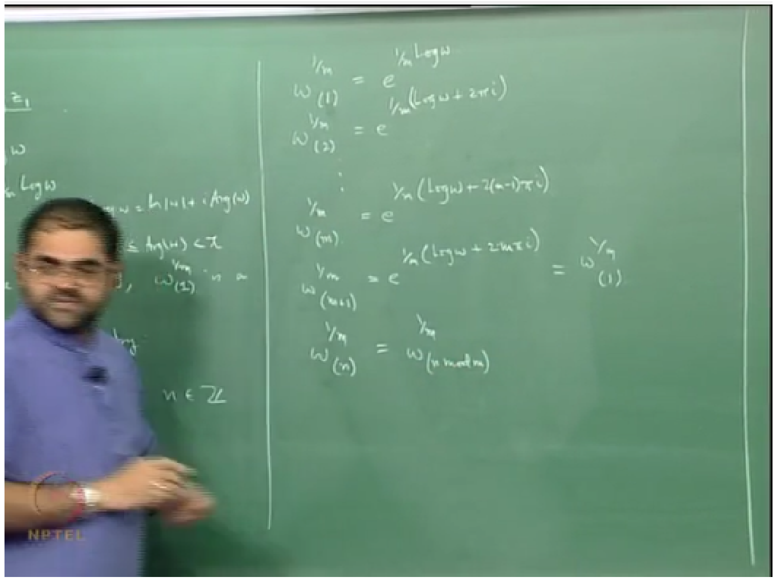

So, the moral of the story is that if you calculate w to the power of 1/m n what you will just get is you will get w to the power of you will get the branch that corresponds to n mod m this is what you will get as n repeats I mean as n varies over all integers. So, essentially you get only these m branches you get only one you only get m branches and of course the problem with all these branches is that at the at wherever you cut it on the that is on the negative real axis they are all not going to be analytic okay.

And again if you want to think all of them and the reason why they are going to be analytic is because well the even if you try to make the argument continues by separating the lower the negative real axis in the portion below it from the portion above it if you do that you can actually make the argument functions continuous but even then you know you are not going to make the function analytic on the negative real axis.

The only way to do is to construct the Riemann surface for the root the mth root function okay and the Riemann surface will be what you will get is the Riemann surface will be as many sheets as there are branches okay. So, in the case of the when you consider the logarithmic function which is the inverse function to the exponential function then the logarithmic function all the branches of the logarithm could be realise the single function on the Riemann surface for log z which is which has as many sheets infinitely many sheets as many sheets as there are integers.

Because these are the as many branches of a log that you have but here you have m branches of the mth root function. So, what you will get is you will get an m sheeted Riemann surface.

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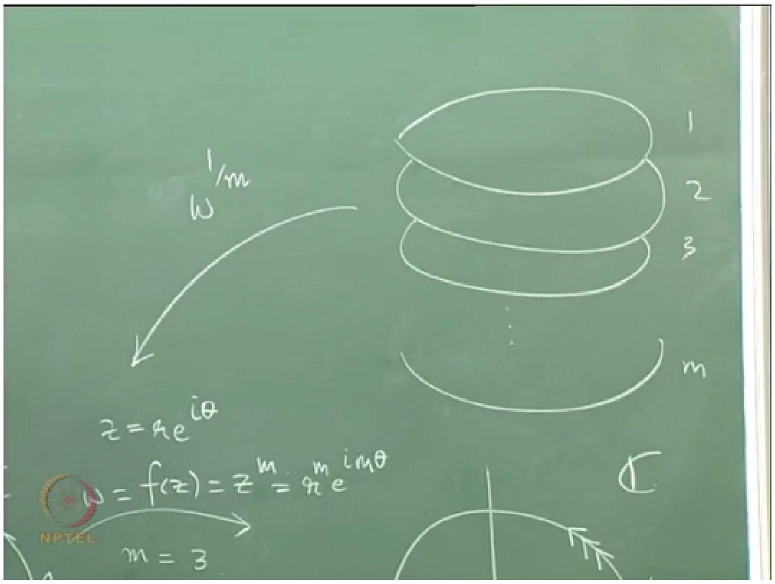

So, I will be content to draw it like this and well and you will get on this Riemann surface the function w to the 1/m will make senses a single valued function okay which is analytic and you can see all the m branches of the mth root function living here as a single analytic function okay.

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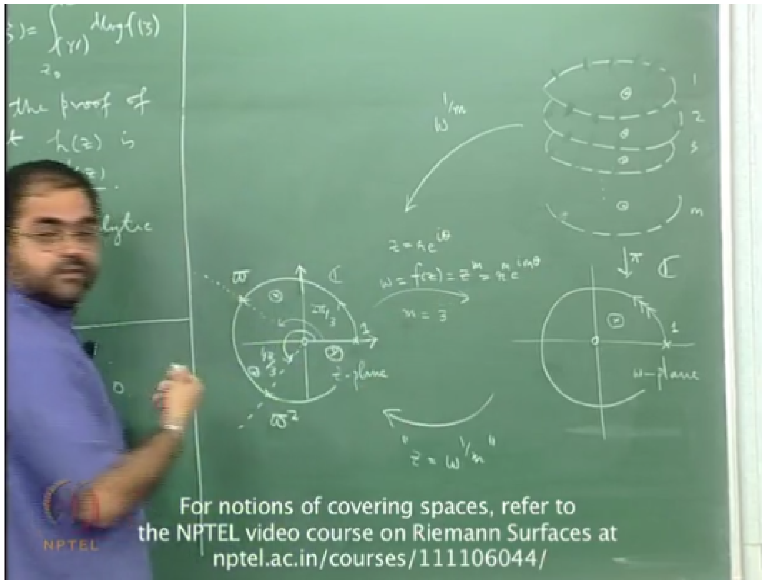

And of course I should tell you that this is of course not m disks and again I should tell you that this is a this m sheeted Riemann surface is an m sheeted cover of the punctured plane okay is what is called a covering space okay in the sense that if you take any point here need to take a sufficiently small neighbourhood then it is inverse image will be m points with m neighbourhoods which are homeomorphic to this under the projection map.

And if you call these as a projection map and for example in this case you can see them as you know you can see them as 3 points here for a point there corresponding to the images of the 3 branches and this is what so+3 this show +3 separate neighbourhoods of about in the coverings the covering space okay which is a Riemann surface for that the function w to the 1/m okay.

So, now I will end with the statement that this behaviour of the function f of z at the critical point z=0 is will be the basis for the behaviour of all of any analytic function at a critical point the behaviour will be like this up to a conformal change okay that is what I will explain the next lecture.