Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodrama, Hyperbolic Geometry and the Riemann Mapping Theorem

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Lectyoure-15
Constructing the Riemann Surface for the Complex Logarithm

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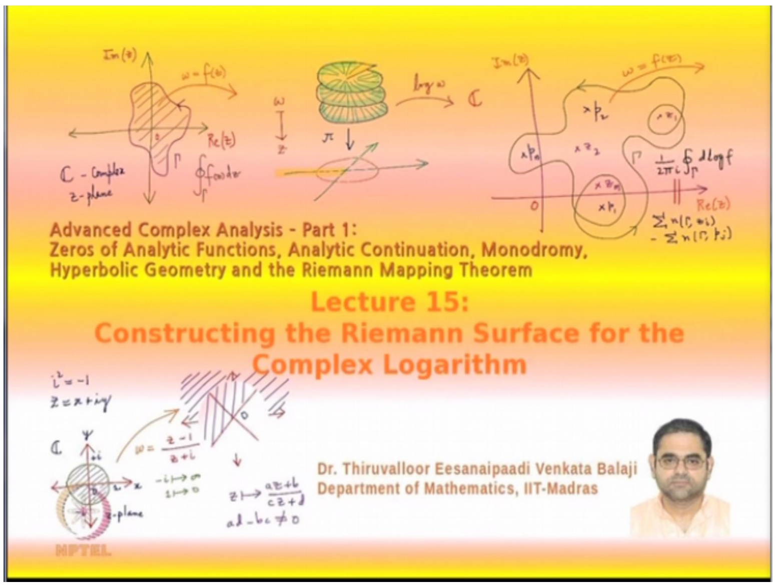

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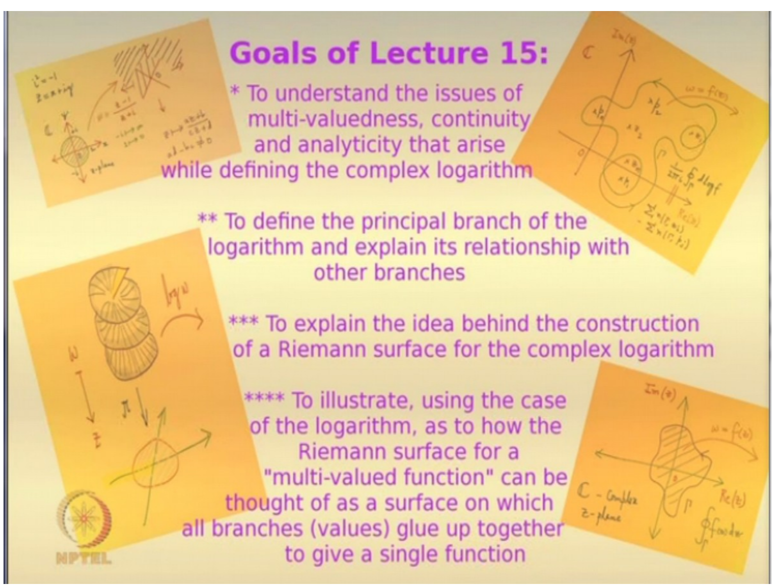

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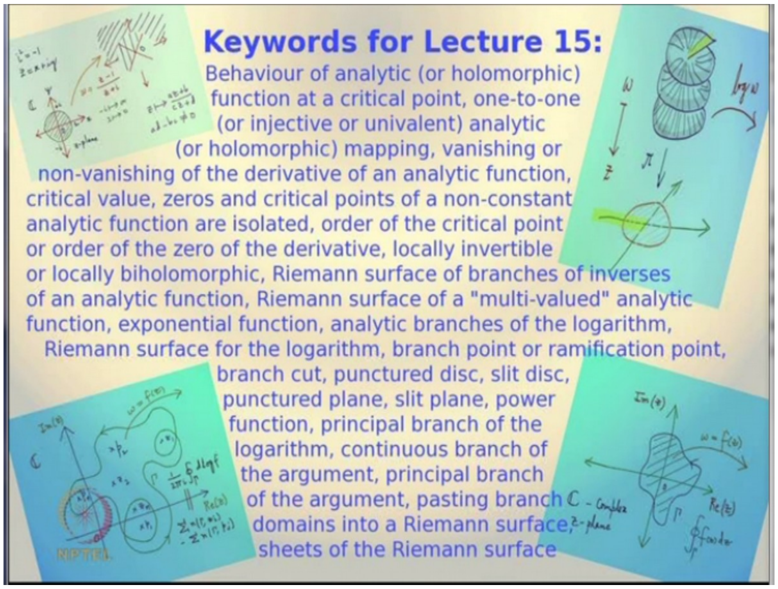

Okay so see we saw in the previous lecture that using the implicit function theorem you know you can look at the locus of function of complex valid function of two complex variables which is smooth as a Riemann surface okay. Now what I am going to talk about in this and then probably the next lecture is trying to understand what happens to a function at what is called a critical point okay.

So, basically what we have been looking at so far is the relationship between a few an analytic mapping being a one to one mapping and for example the derivative non vanishing okay So, what we have proved is if you have an analytic mapping if you have a analytic function which is one to one then we know the it is an isomorphism on to it is image may be the inverse function is also analytic.

It is holomorphic okay and on the other hand if the derivative of function does in vanish at a point then there is a of course an neighborhood by continuity there is a whole neighborhood surrounding that point where the derivative will not vanish and what will happen is in that neighborhood the function will be locally in by holomorphic. So, in other words for given every point in that neighborhood.

I can find a smaller at neighborhood where the function becomes one to one okay So, now the question that we turn to it is what happens when the derivative varnishes okay. And just as we have in functions of one variable the set of points where the derivative vanishes is called the critical set such point are called critical point and the functional values are these points are called critical values okay. And we want to study the function or the behavior of the function add at critical point right now so let me make the definitions let so behavior at a critical point.

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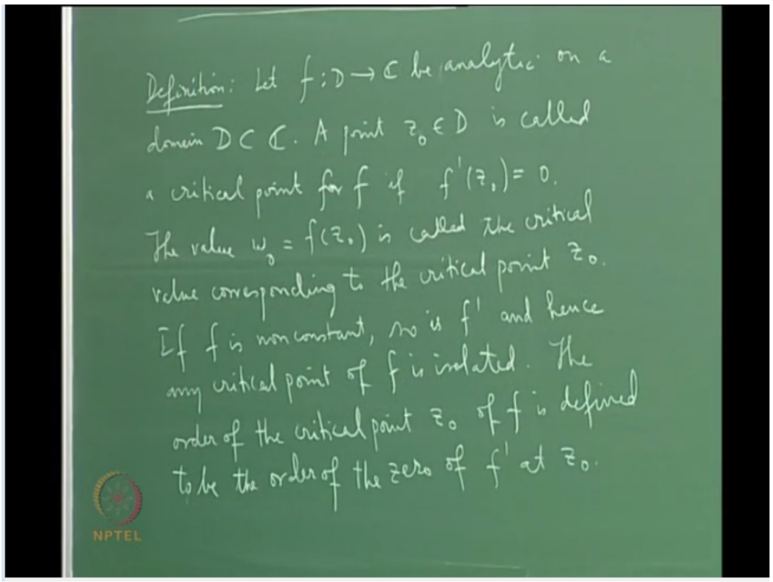

So, here is a definition let from f from D to C be analytic on a domain D, D in the complex plane and a point z not of D is called a critical point for f if f dash of z not equal to 0 okay. So, it is simply a point where the derivative or the function vanishes and of course the derivative exist because of function is analytic of course you know if the function is analytic then it is infinitely differentiable okay.

So, derivatives of all orders existed a condition for the critical for a point being critical point is a derivative becomes vanish alright. And the value w0 which is f of z0 is called the critical value is called the critical value corresponding to the critical point z0 okay. And so if we look at z0, z0 is a 0 of f' dash okay and of course f dash is a analytic function okay and note that you know of course I am suddenly not going to consider a constant analytic function okay.

I am summing that analytic function f is non constant because had it been constant then the derivative will be identically 0. Then so what happened is that in that sense it will be here you know every point will be a critical point and there will be a only one critical value namely the constant value of the function okay I am not considering that case okay so I am considering non constant analytic function.

So if I take a non constant analytic function then then f then if you look at f dash what happens f dash I am particularly intensely in the case when f dash is also non constant analytic function okay. And of course you know if f has critical point then f dash is 0 at that critical point and if f dash is the constant. Then it will you that f dash is identically 0 and that will tell you that f is constant.

Therefore you know f dash will be also non constant analytic function okay if f is a non constant analytic function and so if you look at this non constant analytic function f dash z0 is 0 of that and it has to be isolated you know the zeroes of a non constant analytic function are isolated okay. That means every 0 can be surrounded by a disc of finite radius where you cannot find any other zeroes right.

So, these this z0 is isolated 0 of f dash and what will this tell you is that the critical points are isolated okay so, the critical points of an analytic function are isolated points okay so there is a that is what I wanted to say and then you can also define the order of the critical point to be simply the order of the 0 z not of f dash okay because after all z0 is a 0 namely the critical point is a 0 of the derivative.

And you look at the order of that 0 and called that the order of the critical point okay. So, let me write that down if f is non constant so is f dash and hence of course if f is non constant f has a critical point then f dash' is also non constant and hence any critical point of f is isolated so the critical points are an non constant analytic function are isolated okay the order of the critical point z0 f is defined to be the order of the 0 of f dash and z0 okay.

So, this the definition of what a critical point is what a critical value is and the and what is a order of critical point okay. Now the of course you know if you take a point which is not a critical point alright. If you take a point which is not a critical point then a taking a point where the derivative does not vanish okay and that case we have already studied where the derivative does not vanish.

If a take a point where the derivative does not vanish then there is a whole neighborhood surrounding that the point where the derivative will not vanish. Because the derivative is continues and then in that neighborhood you know as t E given E any point in that neighborhood you can find a small smaller disc where the function is one to one and . the function is invertible okay.

So, you can locally get an inverse but therefore what we want to study the behavior of function in neighborho0od of a critical point in a neighborhood of the point where the derivative is 0. So, the first thing I want you to say is that you know that the technique of studying this has always being trying to look at only zeroes okay. The whole theme of the lectures of so far has been to look study only zeroes of analytic functions okay.

That is our theme and therefore you can see that even when I study a critical point I am studying this 0 of an analytic function namely the 0 of the derivative which also is an analytic function okay.

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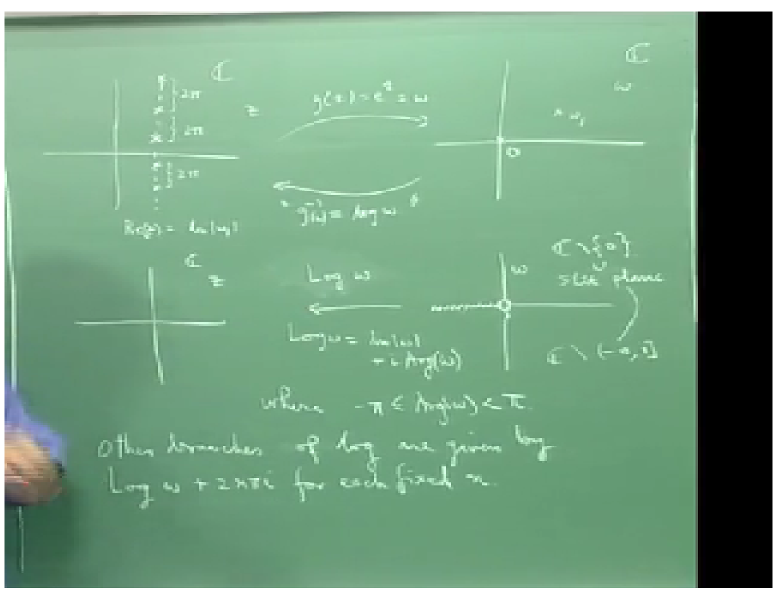

So, let me write this down we want to study the behavior of f in a neighborhood of a critical point z0 of order m-1 okay. So, I am choosing m-1 for a particular reason with critical value f of z0 is equal to w0 okay. So, see you see the what we do why this m-1 is because be instead of considering the function f of z if you consider f of z-w0 okay.

Then what happens is the uh a critical point for a function will be the same as for any other function which differs from the given function by a constant because when you take the derivate the constant is going to go away alright. So, if I consider the function f of z-w0 okay that function will still have z0 as a critical point because it is derivate will be the same as the f dash of z and which will vanish at z0.

But the nice thing now is z0 also 0 of the original function of this function okay so, note that z0 is 0 of order m of f of z-w0 okay. So, the method is give you reduce everything to studying zeroes of analytic functions alright. And of course you know we have studied the in the all this I am assuming that m is at least you know to okay note.

Because it is a critical point z0 is at least a simple 0 of f dash okay so, m should be at least two alright. So we if you take f of z-w0 then z0 is at least a 0 of order of two okay now see the to understand the what happens to the mapping f in a neighborhood of the point z0 okay it given this condition we will have to we need to understand few other concepts probably you have come across them in a first person complex analysis.

I am not sure if you have but any way I will recall them so, these are to do with looking at the Riemann Surface corresponding to various branches a giving the inverse of given function okay. These so, called a Riemann Surface of multivalued function as it is called okay of course I should born you that the terminology multivalued function is a **is a** misnomer in the sense that a function is not supposed to have take several values okay.

A function by definition strictly something that gives have well defined value for each value of the variable okay. So if it is multiple value is not a function okay but that is does not what it means is that you have several solutions to the inverse function. So, let me explain the point of few that when you try to write a function and write in inverse okay.

The problem is that you may have several inverses okay and they will give you various so, called branches of the function right. And these branches will be different functions basically but then if you want think all of them as one and the same function okay. Then that is possible on what is called the Riemann Surface of that inverse function okay. So, I will explain that so l let me write this let me write the title as the Riemann Surface of a multivalued function giving inverse inverses to a given function.

So, these something that means to understood, so I will start with the most the two most basic examples the first one is the logarithm which is the inverse to the exponential function okay. And then I will look at the power function z going to z power m and a why I am looking at the logarithm first because that is this source.

They if you understand that the logarithm has different branches and you understand how to define these branches. Then you can define branches for many other function just using the logarithm for example for the power function and then why I am interested in the power function I am interested in the power function. Because the final fact is that if you look at the mapping f of z in a neighborhood of z0 up to change of coordinates.

It will behave exactly like the power function z going to z power m where you know where m is the same okay. So, if you look at these function in a neighborhood of the z0 the behavior of this function will look like z going to z power m so, that the reason I want to study z going to z power m and to study the inverse of z going to z power m okay which are the mth roots of the variable.

They are there are you know m mth roots they are the branches okay and you get those branches from the logarithm the various branches of the logarithm okay. So, that is the outline of what I am going to do. So, start with f of z so, let me not use f probably I gives g of z is equal to e power z okay exponential function. So, this is an entire function means if you recall it is a function that is analytic on the whole plane okay.

It is define on the whole plane and it is analytic on the whole plane and you know Pretty well that the image of this function is the whole plane- the origin. The only value that the exponential function does it takes is the value 0 alright. And you know also Pretty well that the inverse of this function is given by a logarithm okay. So, you know how to write the logarithm if z0 if z is x + iy.

Then you know that we have you can define log set okay which is you define it as 1/2 natural logarithm ½ x square+y square +i into argument of set +2npie okay. This is how you define the logarithm of z if provided of course provided the z is not 0 okay. So, probably I should may be change notation let me do the following thing so I want to think of an inverse function okay.

So, I **I** would like to solve for e power z equal to Omega okay. In that equation I would like to solve for z so, I am thinking of e power z equal to w let me call it w0 omega. So, I have e power z equal to w I am trying to solve it for z as a functional w and what I will get z equal to logw. So, let me change everything to w. So, I will the source complex variable z **or** real and imaginary parts are x and y.

The target complex variable is w I put the real parts are 0 and v okay wu +iv so, if you do that I will get something like this. So log w so, I will get u square +v square and I argument of w +2npie provided w is not 0 okay. So, the point is that here the this augment of w something that has many values okay. the argument has many values and you get all the values by fixing one value of the argument and adding all possible .

You know integral all possible integer multiples of 2pie okay and this is a reason why you are getting many inverses okay. So, if you solve e power z w then the solution is z equal to log w that is the inverse function alright and the does many values log w has many values. The reason for so many values is because the argument of w is not well defined it is only up to a multiple of 2pie okay.

Now so, in sums in is not correct call this as a inverse function okay because the inverse function should be unique but we can make this into a function. So what we so do this is the the whole point the **we** define what is called a branch of the algorithm okay. We define what is called the branch of the logarithm that gives you a single valued inverse function okay to the exponential function which is analytic okay.

And the fact is that this single valued inverse function will not be defined on the target plane and of course when I take the inverse function you see this

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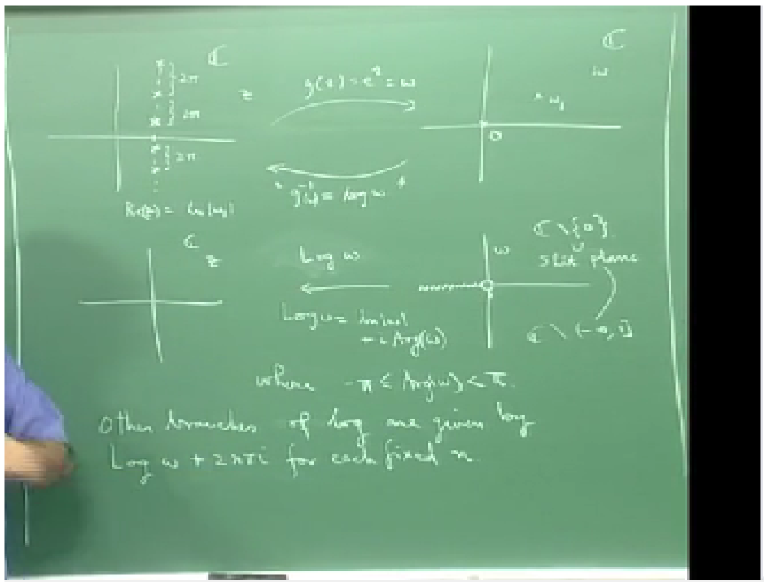

The situation is like this I have the complex plane here which is the variable is z and then I have the mapping it g of z equal to e power z equal to w and the well the images in the w plane okay and I am missing out. I will miss out the origin. Because e power z will not take the value 0 so, if at all I try to define an inverse function g inverse is log w.

And you know let me put this in codes why I am putting that in codes is because you know there is no if you start with the non 0 value of w. You now get a unique value of the logarithm you get so many values of the logarithm they get the several values of the logarithm. So, you know the way it is if I start with the value say w one let as say okay.

Then I get look at the log w1 alright I will get lot of values all the values will have the same real part which will be half lan mod w okay which is a say some value here so, this is this line. This line is given by real part of z is equal to half lan mod w okay maybe I should not try is just log mod w it should not write half okay. Because is already lan of mod w right and of course what are the imaginary parts the imaginary parts of this logarithm.

They all differ by multiple values of 2pie so, what is going to happens it well you are going to get various points you know so if we have a point here then you will have a another point here. And you will have another point here and so on and all these distances will be 2pie will get so many values okay.

So you will get so many values all having the same real part all right namely mod w alright. And the imaginary parts will differ by every successive point to nearby points. If you take to logarithms the imaginary part will differ by 2pie alright. So, in some sense you know the inverse mapping is trying to send w1 2 of course this should be w1 lan mod w1 okay.

So, the inverse mapping is trying to send w1 to this all these points so it is not a function okay. Because I do not have a unique value I cannot pick out unique value from this alright. Now what we do this we do the following thing we try to define a map in this direction and of course on this side you will have to take c-the origin okay. you to throw out the origin so which I will I draw it like this.

I will put at circle here saying that the origin is thrown out okay and I will try to I can try to think out of defining a branch of the logarithm so let me write this. So let not me write this so, let me not even say branch now for the movement let me say try to define log in this direction from the w plane to the z plane right. Now the fact is that zi there is no of course I am trying to define inverse function to an analytic function.

And I would like that also to be analytic okay I would like to define any function that I would like to study is something that would like it like that something that certainly I would like of that the function is that it should be analytic alright. Now in which means in particular does we continues alright now the continuity itself forces that you cannot define it on this okay and so, to explain that what will do is we will let us cut out .

The the negative real axis you it if you cut out the negative real axis it is called the slit plane okay that means you throw out the line segment from-infinity to 0 including 0 okay so, it not possible to define it on the punctured plane c–origin. But it is possible to define it on a subset of that namely the slit plane and what is the slit plane the slit plane is plane-the complex plane- the line segment from-infinity to 0.

So, this is the interval on the real plane that of the real axis extending from-infinity to 0. So, that is the shaded piece I am just three taking out this okay. So, the wait the picture it is that you know you must think of this line being cut okay and then what you do is you do the following thing. You define so let me define this so I will define for any w here alright.

I will define what is called principle branch of the logarithm right and so for the principle branches of the logarithm actually depends on the what is called as choosing a principle branch of the argument of w okay. So, what you do is you define log w principle points of the logarithm of w to be lan mod w + i items principle argument of w okay.

And what is this principle argument of w the principle argument of w is the argument that will take from it is angle from -pie to +to pie okay. But then I throw out one of these end points so, probably I throw out I want the what the convergent is maybe I throw out +pie okay. So, what we do is so, you put this condition where -pie less than or equal to argument of w strictly less than pie okay.

So, you see the what you must understand now is that then I do this I when I right this I will not thrown out the negative real axis okay mind you the argument w the argument of a complex number is always define for an non 0 complex number. So, **so** this w this argument always defined on the punctured plane for any complex number any non 0 complex number you can define the argument the only thing is that argument is an ambiguous in the sense that.

You can add any multiple of 2/2 to it okay that is because as for the angle is consent adding 2pie does not change the point on the plane alright. So, well so, what you must understand as this is define on the punctured plane okay. But the problem is it will fail to be continuous on the negative real axis because you know if you take a point below the negative real axis the argument will be close to –pie okay.

But you just push that point above the negative real axis and in fact it will exactly be equal to -pie if it is a point on the negative real axis alright. Because I have taken -pie to be the argument also as I have taken -pie also to be one of the values okay. So, every point on the negative real axis has argument -pie and the points below that will have argument closed to -pie but if you just go little bit above the negative real axis.

The argument will be closed + pie so, you see there is a jump in the argument which is a jump of argument nearly 2pie okay. And it is this jump in the argument that province this function from being a continuous function on the negative real axis. And you know if you think of log w of the function okay and of course ambiguity was what argument use suppose I use the principle argument okay to remove any ambiguity.

If I want this I want this log to be an analytic function so, I want it to be a continuous function alright and mind you if it is continuous then both the real and imaginary parts should also be continuous a complex valued function is continues if and only if real and imaginary parts consider as real valued functions are continuous alright therefore if you want this to be continuous.

Then lan mod w and argument of w should both be continuous function of course lan mod w will always be continuous okay. Because it is a natural logarithm okay it is a real value non negative real valued function there is no problem with this okay where as the problem is with the argument as you can see because the argument is not going to be continuous on the negative real axis.

On that is the reason why if I even though this function is defined on C-0 okay this principle logarithm as it is called which is define is the principle branches of the argument the principle logarithm define using principle argument where we usually use capital L and to show that it is a principle branch. So, the logarithm and capital A to say that, it is a principle branch of the argument or principle argument.

The point is that this function even though it is defined on punctured plane it is not continuous on the punctured plane to make it continuous you will have to cut out the you will have cut out the the negative real axis okay you have to cut it out. What that means is that what is advantage of cutting it out it mean cutting out does not mean remove it in a strict sense cutting it out means that.

You separate the portion of the you separate the Portion about the negative real axis from the negative real axis and the Portion below. You separate them and why do you separate them you separate them because once you separate them you cannot move from here to there. You cannot think of see what is the problem in this being continuous, the problem in this being continuous is that I can take a point here.

And I can I take a point which is a on the negative real axis or slightly below the negative real axis and then very easily push it to go above the negative real axis that is because the reason above the negative real axis is closed to the negative real axis and the region below it okay. Now by silting the plane what I am doing is a I am just making them far away okay. Therefore I want have this I simply cannot push a point on the negative real axis or below the negative real axis to above the negative real axis.

That is because I am cut it **I** I have purposely created a disconnection okay and once you look at the slit plane like this then you see that this becomes a continuous function. Because the only problem was continuity on the negative real axis, so it becomes the continuous function. And the truth is even because of an analytic function okay this becomes an analytical function and of course this function called the branch of the logarithm with the principle branches of the logarithm becomes an inverse function to the function e power set okay.

But mind you that inverse function as a function is defined on c-0 but as a continuous function can be defined only after make the slit okay. And then of course if you make the slit then it becomes not only continuous it is actually analytic right. And so, you get so the moral of the story by take this function exponential function even though the exponential function is not one to one.

You know because if I change it to z if I add any multiple of 2npiei I will still get the same value and apply the exponential. So, it is a many to one function so, obviously the if I take if I think of an inverse function, inverse function will be multiple valuehood in general we call this multiple valuehood function as log. We call it as the logarithm function but to make it a single valued function you have to take a branch of the logarithm that a branch of the logarithm.

For example in this case is the principle branch is defined on the punctured plane but even though it is defined on the punched plane it will not be continuous. If you want a think it is a continuous function then you have to slit the plane okay. You will have to separate the portion above the negative real axis from the negative real axis and the portion below it by making a slit okay.

And then if you consider the slit plane then this function becomes an analytic function and it will becomes the inverse to this function in the sense that you apply this function then apply the exponential function you will get the identity map on the slit plan. And this becomes the single valuehood function and it becomes analytic and it an becomes really inverse function to this. So, the moral of the story is you are able to get an inverse function only after making a slit in the plane okay.

In fact this slit need have been made here you can even make it along a radial line okay only think is on which our line make it you. You make the argument to you know very from there to I mean you just it is just you rotate this lecture by what our angle you want but slit along any radial line okay. That does it match that will also you have branch.

This is the principle branch okay now so, what are the other branches see the other branches are given by the principle branch+a fixed multiple of 2piei okay other branches is equal to or given by of log or given by the principle branch +2npiei for fixed n okay. So, in particular you know her I have taken the are the principle argument from –pie to pie alright but then I can add 2pie to it if I add 2 pie it to it.

My principle argument will vary from pie to 3 pie that is another branch of the argument that you give you the another branch of the logarithm that branch of the logarithm will be this principle branch of logarithm +2piei. So, I just add 2pie okay and these how you get every branch right from this from one branch now how the point is the where does a idea of Riemann Surface coming the idea of a Riemann surface comes in the following nice way.

We want to think of all the branches has one function you want to think of all the branches as giving one and only one function okay. The way to do it is of course they are not one and the same function they are different functions. So the idea is that you modify the domain of the function okay you change the domain of the function instant of it being a slit plane you make a Riemann Surface and on that Riemann Surface.

You will get a function which on each piece that is designated subset of that Riemann Surface you will get all these branches on every piece. So, how does one do it one does not in a very nice way see you have so let me explain so you have let me draw like this.

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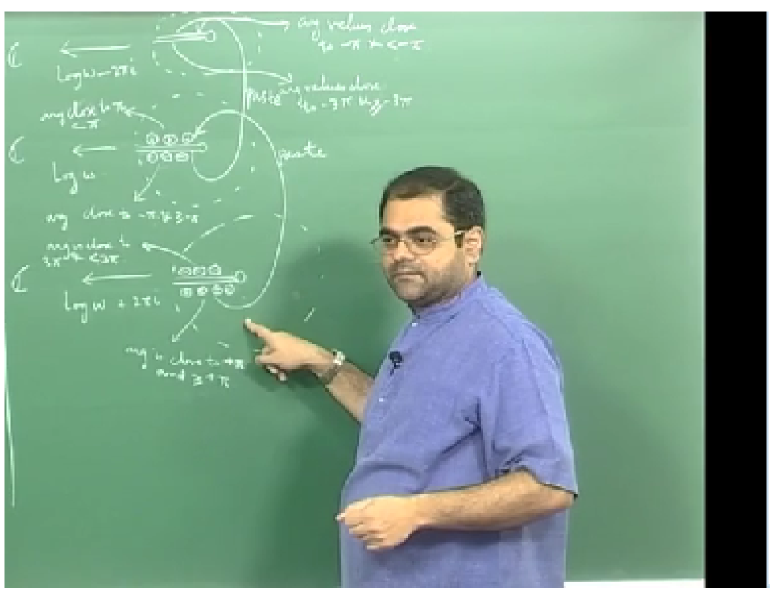

This is a slit plane and I just draw it like this and here I have log z log w okay and what you must understand this where if you if I then I what I have next is well. I have log w + +2piei these are other branch these also define on the slit plane okay. This is I have added 2pie2 to it alright. And so on it continues in the both directions alright.

So, what I should draw above will be something like this it will be log w -2pie that is another branch of the logarithm that again is going to come that is going to again analytic function on this slit plane. So, I have so many copies of the slit plane and I have for each copies I have an inverse function which is given by a branch of the logarithm all the functions they all differ only by integer multiples of 2piei okay.

Now what I do is I do the following think see I do a base thing construction and the basic construction is like this. You see look at log w and look at the imaginary part is the argument you see the argument is negative here it is negative at these points that just that is on the negative real axis and below it is negative and above it is positive okay. And in the negative on the negative side the values are closed to here.

The values are closed to argument close to –pie okay and above the argument is closed to +pie alright now look at these one look at these branch if we look at these branch the argument is going to now change from so, I valued 2pie. So, it is going to change from +pie to +3 okay so, what happens is here the argument is close to +pie and of course is lesser than +5 okay is closed to +5 and not lesser than in fact greater than or equal to +5 that is what it is.

Because I valued 2pie to this okay and if you look at it above if you look at points here the argument is closed to 3pie. And of course less than or equal to 3pie strictly less than 3pie because it changes from -3pie lesser than or equal to the argument of these branch strictly less than I mean +pie lesser than or equal to argument of this branch strictly less than 3pie okay.

So, below it is going to be closed to +pie and greater than or equal to +pie and above it is going to be closed to 3pie. But lesser than 3pie alright and now you look at these two slit regions okay this anything bothering (()) (49:00) you are right. I mean if you think of it as removed then I should write strictly greater than pie if I think of it has remove.

Then I write strictly greater than or equal to +pie okay. But if I do not think of it as remove then I write it is like this okay. But I want do the following thing what I want to do is I do not want to remove it because I want to paste it. So, what I am going to I am going to do the following thing look at see the thing of this piece you think of this piece like this okay.

So, this is the piece of the complex plane here the values of the argument are greater than or equal to pie okay. And above they are well going to close to 3pie but look concentrate below here. The values are greater than or equal to pie and that these are the values which are the values here. The values here are the argument is close to pie and strictly less than pie okay.

So what you do is you see you paste this piece of the negative axis to that this the lower piece the lower this slit plane you take the lower portion of the negative axis along with the negative axis. And so, simply join it to the upper portion of the portion above the negative real axis on this piece okay. So you know basically if I put them together I hope I will be able to I wanted if I will be able to draw a neat diagram.

So, you know just for the purpose of identification let me put something here so, I am I identifying I am pasting the lower thing here with the upper thing that is while put +5 here and I will put a – here okay and so you know so, **so** I am going to so, **so** let me write this here so it is from here to here I am doing a paste okay. Why I am doing this pasting because now if I paste these two slits together.

Then I move across the negative real axis then the when I am on this piece they arguments when I am on that piece the argument values are close to pie and lesser than pie they are approaching pie. But then after I paste it they will continue on this piece they will achieve on the pie and they will continue okay so, by pasting the upper portion of the imaginary axis with the lower portion of the imaginary axis.

And the imaginary axis together what I have done is I have got in a surface on which the argument see should be continuous even on the negative real axis. You see what is happening the argument has become now a continuous function including the points on the negative real axis just because I have cut the negative real axis the portion above the negative real axis from here away and I joint the portion consisting the negative real axis.

And the region below it to the portion of the negative real axis above okay So, and you know in fact in the same way what will happen is that you paste the if you look at the values of the argument here. The values of the argument here are close to –pie and of course and well greater than or equal to -pie that is what happens here alright. If you look at these okay I have taken away -2pie so, I will get if I take away -2pie from this.

I will get -3pie to –pie so, what will happen is here the at these points below a negative real axis and on the negative real axis. The argument will have values are close to -3pie and of course greater than equal to -3pie this is what I get okay and what will I have about I have values close to pie – 2pie which is –pie and lesser than –pie. So above I have argument values close to –pie and lesser than -pie okay.

And now you see here if I take the top portion of the negative real axis the argument has values lesser than -pie and tending to -pie and those values are achieved here in the bottom portion of the negative real axis and the negative real axis. The argument values start from -pie and then increase so, what you do is in the same way you paste this to this you make a paste like this okay.

So, what you do is you take the negative real axis and the lower portion of the negative real axis and paste it to the upper portion of the negative real the portion above the negative realax is in this piece okay. And that is you see that is just what you did here okay so, if you now look at this procedure. What you doing is for every piece what you are doing is you are taking the negative real axis and the portion below the negative real axis.

And you are pasting it to the piece to the portion of the negative to the portion above the negative real axis in the slit plane. I am in the piece above okay and you if you do this infinitely okay you can imagine it.

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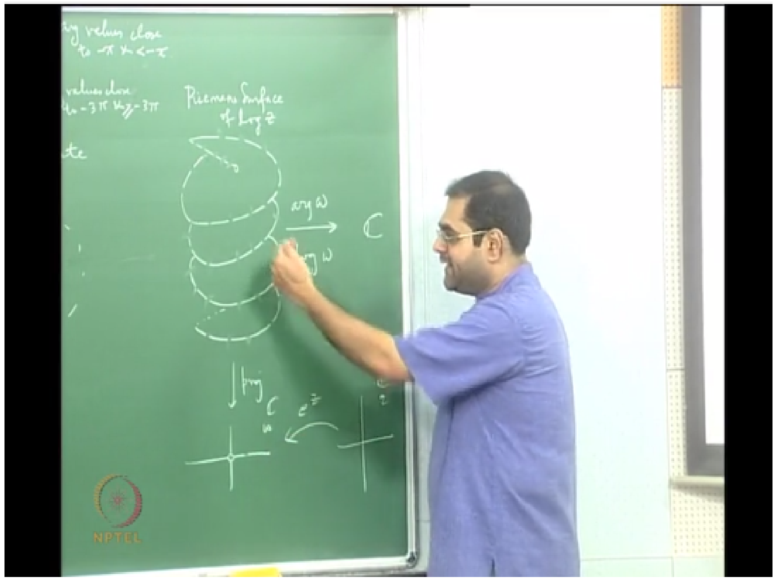

So, let me draw a diagram may not be the best diagram. You know if I do this I will get at any finite stage I will get something like this. I will get something like and of course you know the origin has been removed. So, there is a whole and you see I get something like a spiral continuously spirals that is way kind of thing alright. And so, I get a surface and this surface will of course if I do this of course.

You know I have drawn a boundary here this is no boundary okay in fact I should put dotted lines this is just this is the whole plane here. There is is no boundary but you know I am just drawing so that you can visualize it. It is little heart to visualize but with the little bit of thought you can okay. So, you what you get is you get you know copies upon copies of the plane okay being cut.

And join on the negative real axis okay with origin remove and you get a surface now this surface is a Riemann Surface why because what is your definition of Riemann Surface a Riemann surface something there should we locally homeomorphic to the plane. It should have a it should be as a topological space it should be you know connected it should be (()) (57:44) it should be second countable of course all these things are true.

The way I have pasted it is going to be connected it is going to be a of course host off because you take two points I can separate them by open sets. If they lie on the same piece of the cut plane then I can certainly separate them by two open sets if they lie on different pieces okay which are called so these various pieces of the plane the cut plane they are all called as sheets are the Riemann Surfaces okay.

So, you have infinitely many sheets as many sheets as there are integers okay because you have all the as you have as money branch of the logarithm as there are integers and every it is only by choosing an integer in that you get a different branches of logarithm from a principle branch by adding 2npie to it okay. So, you get the this thing is suddenly a Riemann Surface okay it is a Riemann Surface it is connected host of second countable.

And certainly I can make sense of there are natural chats because each piece is just the slit plane. So, I can define the and the way I have define the way of cut and paste it the argument becomes cutting the function. So, what you must understand is now you see if I can write arg said argw this is a continuous function argument is a continuous function argument which are originally not a continuous function on a single copy has now become a continuous function.

That is because I have carefully cut and paste various copies so, this argument which is a multiple valued function which is not continuous I have cut and paste the domain so, many copies of the domain so I get a function which puts together all these branches of the argument function so, you know therefore argument function is an continuous function on this and beautiful thing is the log function will also be a continues function on this.

And it will be analytic on each piece it will give you the branch of logarithm to be corresponding to that piece okay so, the beautiful thing is that you know so the I have I can write a projection here onto the w plane okay. And you know now you know from the z plane to the w plane I have the function e power z alright I have this of course when I project it the origin will be missed the origin will not be there.

Because the origin has been removed right so, look at this diagram I have the exponential function if I try to define the inverse function I do not get it here but I get it above I get it an Riemann Surface which it is so, which projects the on to this so, this is called Riemann Surface of log z okay this is called the Riemann Surface of log z I will let me put small l I will put small l.

Because it as it is now a single valuehood analytic function on this Riemann Surface it is analytic because analytic is local. And if you want to check a analytic on every sheet I know it is already analytic on each sheets so, it is an analytic function okay. So, it is a single valuehood analytic function it is inverse for this functions. So, the beautiful thing is here is a function is a many valuehood function okay.

The inverse you do not get from the target you get on a Riemann Surface that covers the target in fact this map in the language of covering space is called a covering map this is a covering map it is a it is called a infinite sheet covering of the punctured plane okay. And the moral of the story is if you try to look at the inverse to the function you will get so many branches and the if you want to think of the inverse as a really a function.

You have to go to a Riemann Surface of the inverse function there it will be a honest analytic function which will be an inverse to this okay. So, this is a picture so, this is one of the nice things about Riemann Surfaces they allow you to think various branches of inverse function as a single function on a surface okay alright so, I will stop here I will continue in the next lecture.