Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-14
F(z,w)=0 is naturally a Riemann Surface

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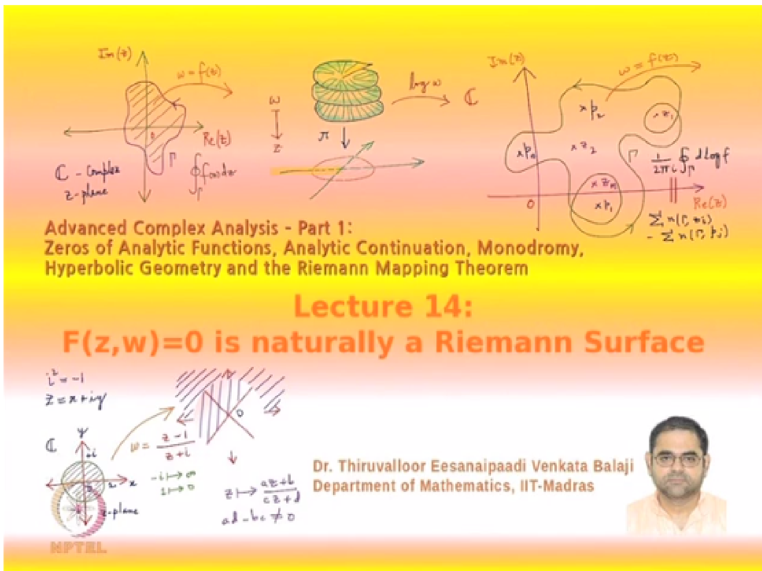

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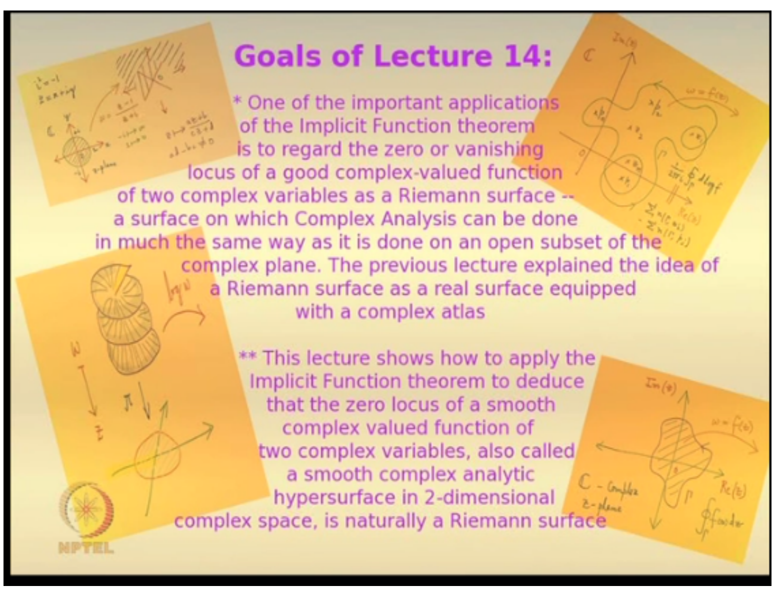

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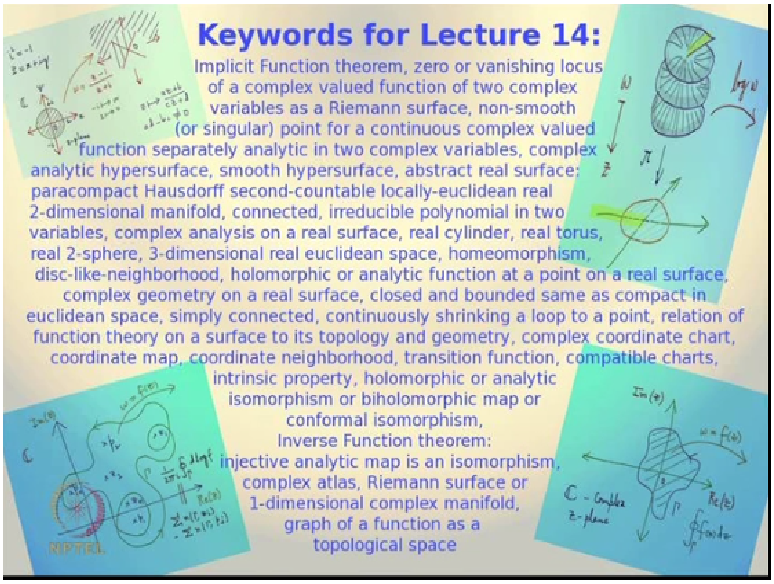

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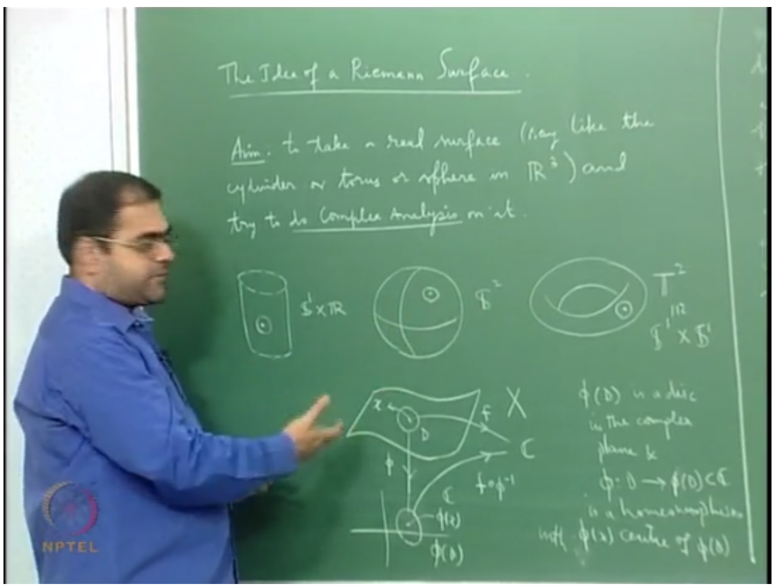

Okay, so this is a continuation of previous lecture and my aim is to explain why the implicit function is function theorem is important namely that it helps you to think of the 0 locus of a function of 2 complex variables as Riemann surface. So, what I told the last lecture was what the idea of Riemann surface is namely it is it is a you may call it a or it is a structure that allows you to do complex analysis on a surface okay.

A surface that you can for example imagine 3 space like the cylinder or the sphere or the torus and the method is that you try to define when a function on at a point of the define on a neighborhood at a point of a surface is holomorphic or analytic and you that by composing the function with a coordinate chat at that point.

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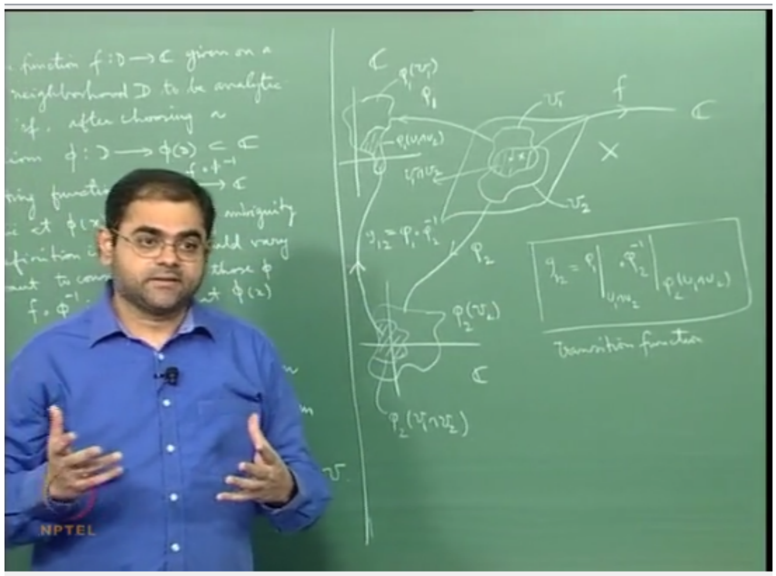

And then to make sure that the resulting definition of an analytic function is consistent, you only use chats collection of chats which have the property that whenever 2 chats intersect the corresponding transition function is is holomorphic which will which is equivalent to saying that the transition function is a holomorphic isomorphism it is a holomorphic function which has an inverse is also holomorphic okay.

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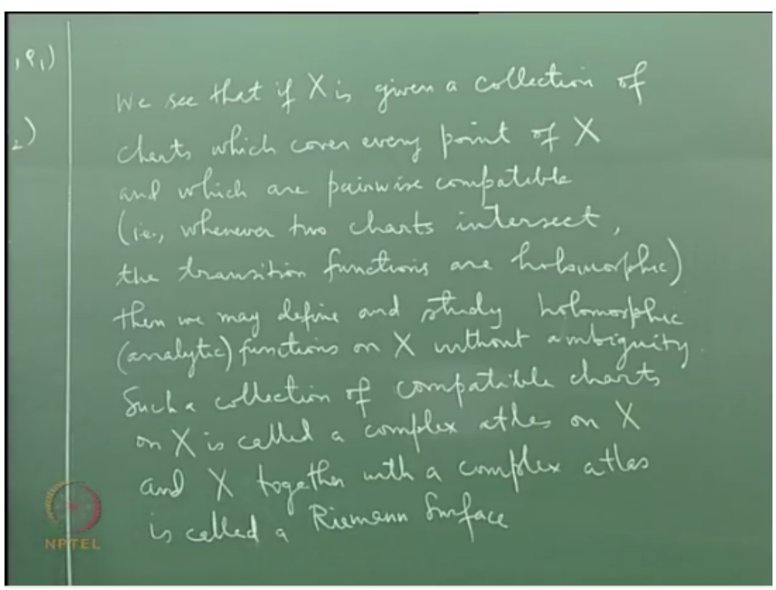

And the moment you give x the surface x the collection of chats that covers every point and which are pair wise compatible then you make x into Riemann surface okay. And then you can study you can define and study holomorphic functions analytic functions on the Riemann surfaces. And you expect that by studying the properties of analytic functions on the surface.

You will be able to get some information about the geometry of the surface or you expect the geometry of the surface to be reflected where you study the analytic functions on the surface or the holomorphic functions on the surface. So, of course the application that I have in mind is to look at the implicit function theorem okay for a function of 2 variables.

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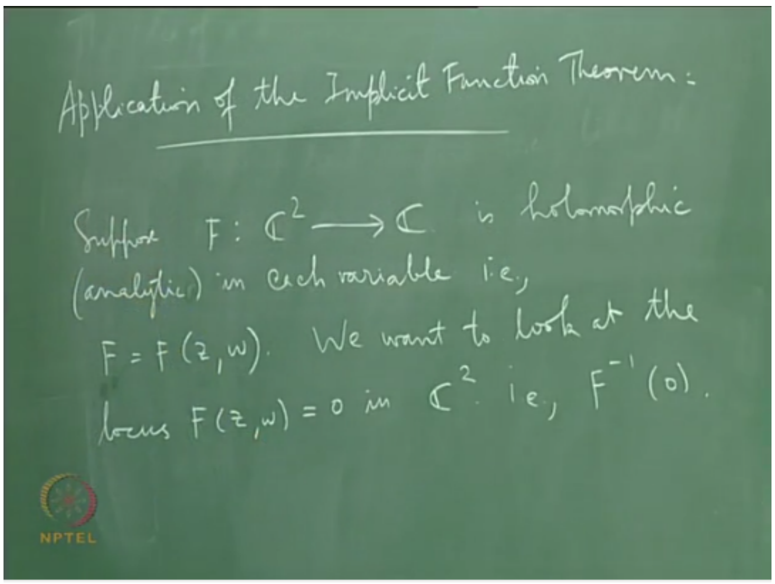

So, let me do that right away application of the implicit function theorem okay. Suppose F from c2 to c is holomorphic means the seem an analytic in each variable okay what this means is that you write F , so what it means is that you write F=F of z, w, z is the first complex variables w is the second complex variable and F is of course you know of course you are assuming have to be continuous right.

And for each fixed value of w the resulting gn of z is analytic and for each fixed value of z the resulting function of w is analytic you assume that okay. So, it is analytic in each separately in each variable right and we want to look at the locus F of z, w=0 in c2 okay. So, you are looking at this the points in ce zeros of c2 which are zeros of this function you are looking at the 0 locus of this function okay.

So, that is it is just F inverse of 0 is inverse image under F of 0 all those which goes to 0 under F okay the first thing we should realize is that if you think very naively cubistically c2 is 2 dimensions and if you look at the locus F=0 okay you are looking at you are cutting down by 1 equation, so the locus must be 1 dimension it should be 1 dimension less from a space of 2 dimensions you are looking at solutions of 1 equation.

So, the dimension has to come down by 1, so the resulting should be 1 complex dimension okay. So, it should be a surface, so you expect it to be a surface which it is provided F is good in okay, so now what I am going to do is suppose we look at a point a point in the locus say z0, w.

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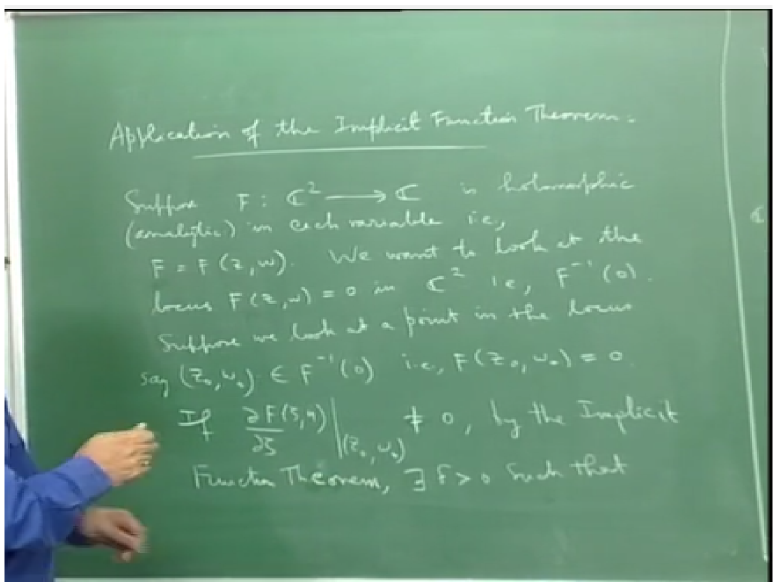

So, this is a point where F is 0 okay, suppose you are looking at a point like this okay and then what I do is that I look at the conditions that I need to apply implicit function theorem okay. And what is the implicit function theorem say the implicit function theorem says that your function the explicit the implicit function can be solved to give an explicit relationship for a variable in terms of at the other variable provided the partial derivative with respect to that variable is not 0 okay.

So, if tow F zetta, etta by tow zetta alright at z0, w0 is not 0 suppose this is not 0 alright this means that you know you are looking at F as a function of the first variable keeping the second variable=w0 okay and then you are taking it is derivative and then you are evaluating that derivative at z0 okay. This is same as d by d z of F of z, w0 at z=z0 right and if that is not 0 the implicit function theorem says by the implicit function theorem there exist delta greater than 0.

Such that so let me write that down basically it says that since the partial derivative with respect to the first variable is not 0 you can solve for the first variable explicitly as a function of the second variable at that point okay, so in other words there is a delta greater than 0.

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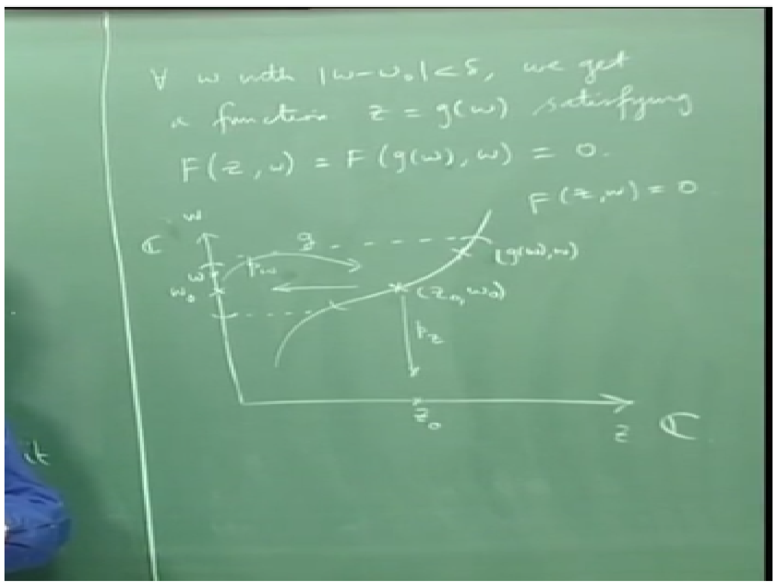

Such that for all w with nod w-w0 less than delta we get a function z as a function of w satisfying F of z, w namely F of g w, w=0 this is what the implicit function theorem says okay**.** So, you know I am going to draw a schematic graph I mean schematic in the sense at I am going to draw which is well which you really cannot draw in 3 space okay .

But then you have to imaginate with some imagination, so here is c and here is c okay, so the each I have drawn only a line okay but I wanted to think of this is c and this also a c alright and then I am actually drawing the locus F of z, w=0 okay. So, this is c2 this is the z coordinate and this is the w coordinate okay, this is the c2 plane this is F coordinate this is w coordinate.

If you take any point on the plane here is with it will 2 coordinate z0, w0, z0 will be the first coordinate which is what you will get when you project under Pz projection under the first coordinate you will get first coordinate at z0 and then if you project under the second coordinate Pw you are going to get the second coordinate okay, so every point has 2 coordinates.

And I am taking a point on the locus 0 locus of this equation, so that means I am taking a point of F of z0, w0 is 0 right and now I am looking at now I am assuming that the first partial derivative at zo, wo is not 0. So, let us interpret this. So, I think this spelling is wrong this is the okay . So you know I have a neighborhood of w0 okay.

So, you must think of this neighborhood of w0 as the disk on the complex plane the w plane centered at w0 radius delta okay. So, this is that neighborhood which is given by mod w-w0 is less than delta it is actually a disk okay but I cannot show that here right. Because I am thinking of this is an axis single line and what is happening is it I have a what is the implicit function theorem say it says that you know I have a function w going to gw.

So, there is a function like this which goes from this neighborhood okay such that if I draw the graph of this function, the graph of this function will be this locus namely if I take any small w in this neighborhood and I take the point w, gw then rather gw, w the way I have written it because w is the second variable formula if I take the corresponding point gw, w that point lies on this locus that is what it means say F of gw, w=0, for every w in this neighborhood.

So, in other words what I am saying is that locally the implicit function theorem says that the locus where F vanishes is actually the graph of a function okay is the graph of the function and now the beautiful thing about a graph is that the graph of a function if you project it to the free variable okay, that is always a an isomorphism okay. So, in other words you know.

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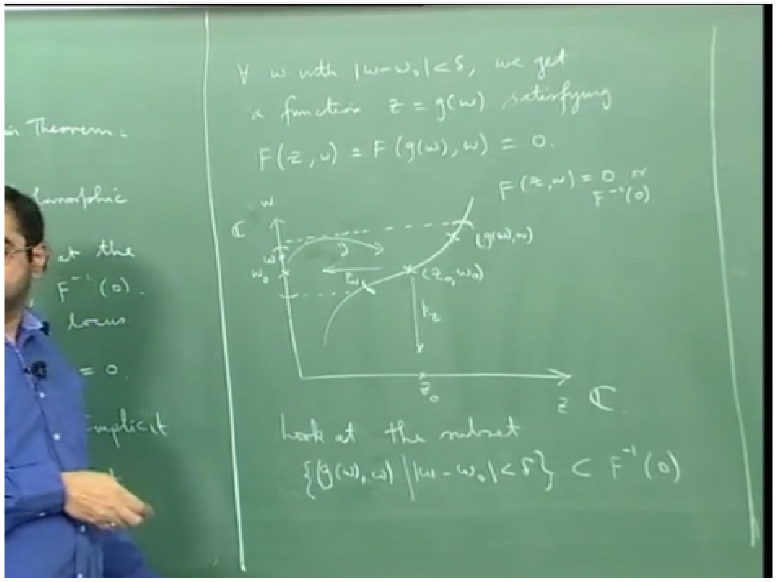

So, let me draw this correctly so this is a dotted line and this rounded arrow is g which is a map from here to here and of course this is a projection this is Pw okay. Look at the set look at this subset gw, w such that w mod w-w0 is less than delta look at this subset okay actually I am just writing I am just this is just the graph of g okay.

Only thing is that normally the graph you write the variable and then the function but then I am writing in the other way I am writing the function and then the variable, normally you write the graph of F of x y=F of x is x, Fx okay. So, ideally the graph of g I should write as w, gw but I am writing it as gw, w because of this diagram okay. So, this subset mind you this is inside the locus F inverse is 0 right.

Because gw, w satisfies F of z, w=0, so it is in this locus alright then I want to say that this is the image of this disk under so I just want to say this is isomorphic to the disk actually because of g. So, **so** let me say that I am just using the fact that you know if I apply the projection, so you know from.

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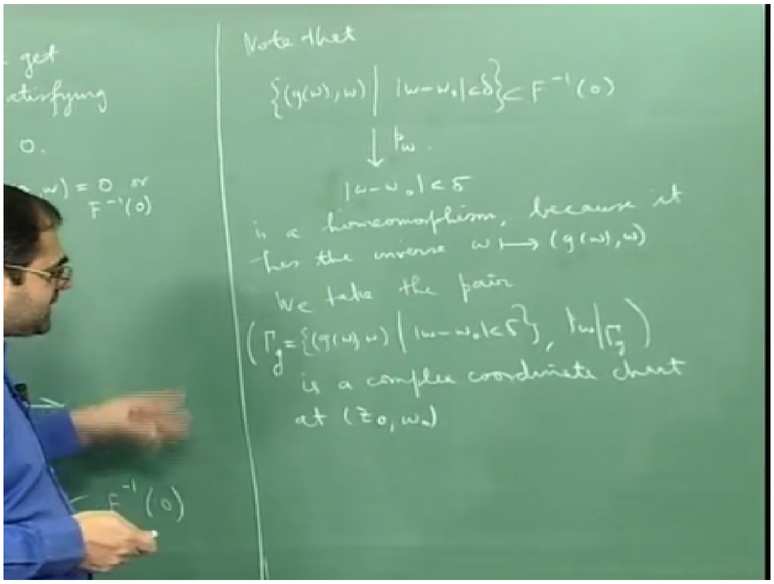

So, I have F inverse 0 that contains this, so you know this subset I am writing here is this portion of the of the locus which is the graph of g over this disk right expect that I have switch the order of the variables right. So, this is the set of all, so let me rewrite that here gw, w such that mod w-w0 less than delta this is the subset of this and now what you do is you apply projection onto w what you will get is the set of all you will get the set mod w-w0 less than delta.

Because you know if I project it under pw I will get back my disk and what I want you tell you is that this map I claim it is a homeomorphs is a homeomorphs note that this map is a homeomorphs okay. I just want to say that whenever you take a graph of whenever you take the graph of function from the graph to the free variable okay that is always a homeomorphs right.

So, I just want to say that this pw is is a homeomorphism which means of course this is a restricted to this piece of the curve okay I tell it is a homeomorphism to show that I will have to show that this is both injective subjective and then I have to show it has an inverse alright and what you must understand is that it is bijective because it has the inverse, the inverse is actually g.

It is just it is given by g namely is given by w going to gw, w okay because it has the inverse w going to gw, w this is just the graph map except that I have written the dependent variable first in then the independent variable, now may in the graph you write first the independent variable and then the dependent variable you write F, Fx for the graph of x okay.

So, I should ideally write w, gw but does not matter I am writing it as gw, w okay in this case it is very clear this is the continuous function okay and because this is the inverse of pw it follows that pw is bijective with this as the inverse and well, so the moral of the story is that for the point z0, w0 I have found you know I have found an open set mind you this is an open set now, the reason why this is an open set is because it is homeomorphic to an open set, this is an open set on in the complex base.

And this is the homeomorphism okay therefore this is also an open set homeomorphism will always carry open sets to open sets. So, what is going to happen is that this piece of the graph that I have marked okay is actually an open subset of this locus okay. And it is homomorphism to this disk okay, it is homeomorphic to this disk and it is so it is a disk like neighborhood of the point z0, w0 what is the disk like neighborhood, it is an open subset which is homeomorphic to a disk.

So, you see I have this open this piece of the graph which is an open subset containing the point z0, w0 and that is homeomorphic to the disk under the projection and the you know the moment I have something like this I have a chart because I have identified a point on this locus along with the neighborhood of that point with disk okay. Therefore this gives me a chart so moral of the story is we take the pair gw, w.

Such that mod w-w0 less than delta, pw, so I will use some notation I will use gamma g, pw is restricted to gamma g I am using gamma g because I mean by capital gamma g the graph of g okay. So, capital gamma sub g is the graph of g except that mind you I have switched the order of the variables. So, this is the graph so you take this graph along with the projection restricted to the graph that is a chart is a complex coordinate chart at z0,w0 okay.

So, what is the moral of the story the moral of the story is if you look at the locus where F vanishes at a point of that locus where F vanishes if the first partial derivative with respect to I mean if the first partial derivative with respect to the first variable is not 0 then I can get a coordinate chart okay at that point okay. Now a same argument will tell you that if instead see it may happen that I may not be lucky perhaps the first partial derivative with respect to the first variable might vanish okay.

But then there is still hope if I have you know that the second I meant the first partial derivative with respect to the second variable does not vanish then again the implicit function theorem will tell you that I can solve for the second variable as function of the first variable okay and then what I will get is there also I will get a chart okay. So, the moral of the story is throughout this locus at every point where either the first partial derivative or the second does not vanish I will always get a chart okay.

So, you know if I put the condition on this function F the condition called smoothness which is that there is no point on the locus of zeros of this function where both partial derivatives vanish I mean that is a having such a point is very bad it is called a singular point it is called singular because I cannot apply the inverse the implicit function theorem to apply the implicit function theorem I should have at least one of the partial derivatives non vanish.

If both partial derivatives vanish that is the singular point and I do not want to consider functions which whose 0 lows I have singular point does not vanish okay. So, you know if I am working with a smooth function okay namely a function such that there is no point on the 0 locus of the function for which both partial derivatives vanish then at every point I will get a chart because of the implicit function theorem.

The chart maybe projection on to the second coordinates it is the first partial derivative does not vanish and it will be a projection on to the first coordinate is a second does not vanish okay. So, I can cover it by charts now the beautiful thing is these charts are automatically compatible okay, these charts are automatically compatible and therefore we make this locus into a Riemann surface that is a beautiful thing.

So, the moral of the story is if you are looking at a smooth function okay of 2 variables okay then it is automatically a Riemann surface it is a Riemann surface which is sitting inside c2, it is a Riemann surface which is embedded inside c2 okay. So, let me also write let me write that down, so you know so this is the importance of the implicit function theorem I can when I am studying 0 locus the 0 locus of a function of 2 variables if it is smooth function I am already looking at a Riemann surface okay.

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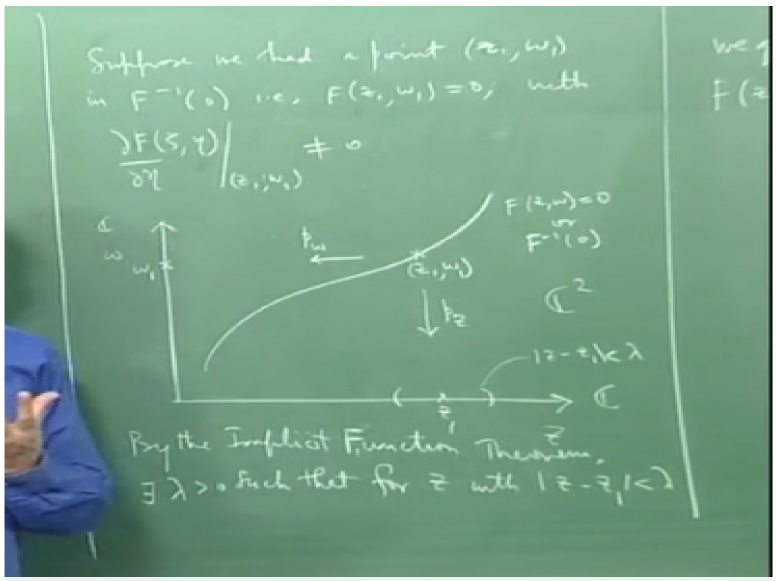

there are some technicalities which I will try to explain very soon, but let me see the following thing suppose we had a point z1, w1 in f inverse 0 that is F of z1, w1=0 with tow F sorry tow F zetta, etta by tow etta at z1, w1 not equal to 0 okay. So, I am considering another point where the you know this first partial derivative with respects second variable does not vanish and the point is again on this locus, 0 locus of the function okay.

So the so if I draw another diagram it should look like this, so here is my complex band this is z coordinate this is another complex plane this is the w coordinate and here is my locus this is F of z, w=0 which is otherwise F inverse of 0 and of course all this is happening in c2 the hole space is c2 that is where everything is happening c cross c okay and now I am having a point z1, w1 alright.

If I use projection onto z I get the point z1, if I use projection onto w I get the point w1 okay and I am assume that the partial derivative with respect to the second variable does not vanish okay at that point. Now again the implicit function theorem by the implicit function theorem there exist lamda greater than 0 such that for z with mod z-z1 lesser than lamda okay. We have we get a function of z we get a function omega in terms of z . Omega=h of z okay.

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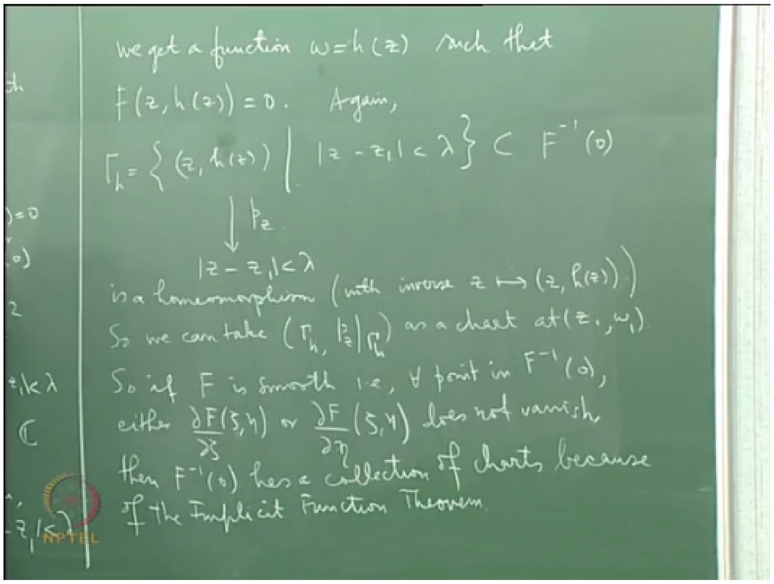

Such that F of z, h of z=0 okay that is in other words you are just saying that if the implicit function theorem just says that you know if the partial derivative with respect to the to a particular variable is not 0 then you can solve for that variable as a function of the other variable. So, the partial derivative with respect to the second variable w is not 0.

So, I can solve for w which is a second variable with respect to the partial derivative is not 0 in terms of the other variable is the first variable okay. So, I can get a function w= h oh z for z in an neighborhood of z1 okay which satisfies this equation, so I get an explicit solution to this implicit equation okay. Now what does it mean if you draw a diagram similar to that now I will get a I will get the neighborhood here I will get this neighborhood here.

This neighborhood here will be mod z-z1less than lamda okay I get a neighborhood namely a disk in the complex plane the z centered at z1 radius lamda and I will get a function of z I so I get a h I will get a function h like this okay.

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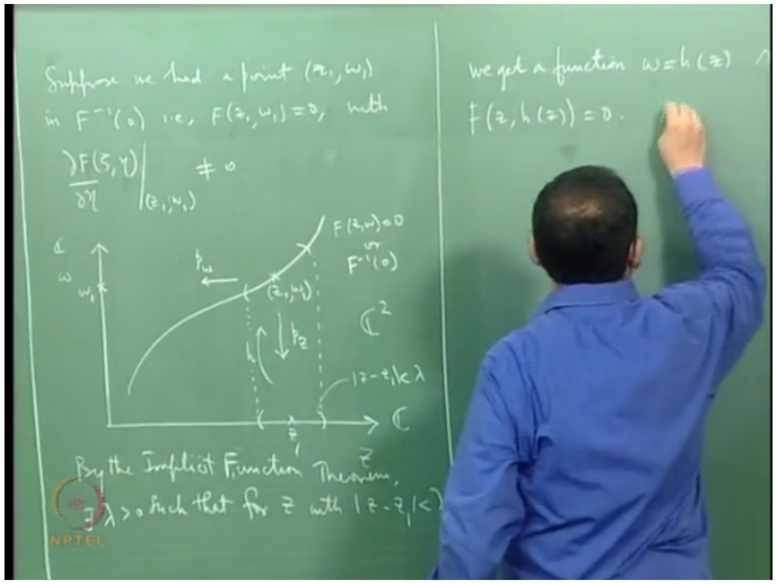

And If you draw the graph of that h I am going to get this piece of the graph okay. So, I am going to just get this piece of the graph okay. So, again what will happen is that we will have again we will have that the set of all the points in the graph of h gamma h is set of all points z, h of z such that mod z-z1 less than lamda this will be a subset of the 0 locus and if you projection to the free variable z okay onto this neighborhood at this center surrounding z1 radius lamda is homeomorphism.

This will be a homeomorphism because we it will have inverse z going to z, h of z okay. So, it will be homeomorphism, so what this tells you is that I get this so in this case I get this graph of, so this piece of the of this locus is actually graph of h it is a graph of h and that is an subset it is an open neighborhood of the point z1, w because it is homeomorphic to an open set in the point.

So, it is a disk like neighborhood and the homeomorphism is given by the projection onto z restricted to that open set alright. So, that is a chart, so I get a chart at the point z1, w okay. So, we can take gamma h, pz restricted to gamma h as a chart at z1, w okay.

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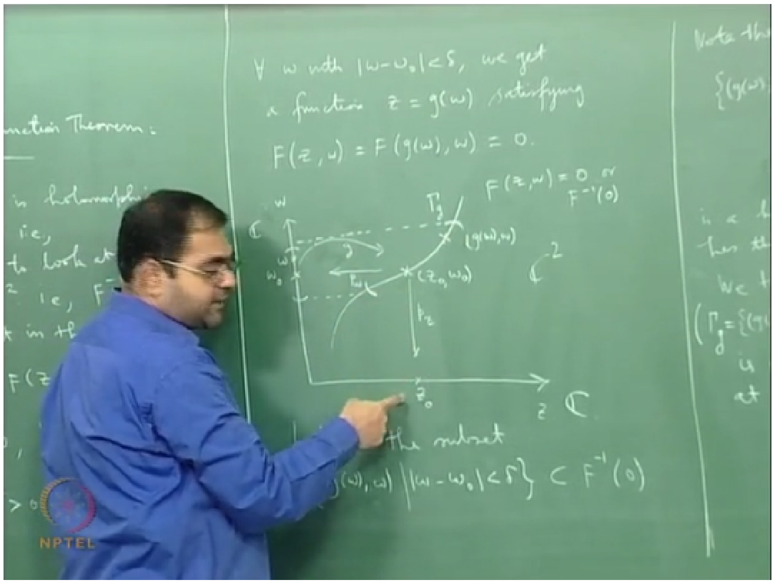

So, in this case this is the graph of G okay you got a function G from the w, disk on the w plane and this is this becomes graph of G this is what happens if the partial derivative with respect to z does not vanish and if the partial derivate with respect to w if does not vanish then this piece of this locus will become the graph of h where h is a function from a neighborhood of the point with the first coordinate right.

And so in any case are if you are curve is smooth okay namely if the function f is smooth then you get you automatically you get charts like this and all the charts come because the implicit function theorem okay. So, if f is smooth that is for every point in f inverse of 0 either tow f by tow zetta or tow f by tow zetta does not vanish. Then f inverse 0 has collection of charts because of the implicit function theorem okay.

And now the natural question you will ask is that well does this collection of charts which come naturally because of the implicit function theorem does not make it into a Riemann surface and what is it that you have to check to say that may it makes it into a Riemann surface you have to only check compatibility okay and all I want to tell you is that the compatibility is trivial okay what is the compatibility the compatibility is that the transition function should be holomorphic okay.

If you take a point if you take 2 nearby whose charts are in this direction okay then the transition function will go like this and come back okay and that will be identity map on w which is of course holomorphic okay, so if you have 2 charts of this type which overlap then the then the transition function is just w going to w which is of course holomorphic as a function of w.

Similarly if you have 2 charts of this type then the transition function is z going to z that is the identity function here that is of course holomorphic, so you get compatibility the only thing you have to case check is you if you have a chart in this direction intersecting the chart in this direction okay. And if you have a chart in this direction intersecting a chart in this direction you know if you go like this what you will get is z will go to you see if you go like this okay.

You will either get g or h okay which are both analytic, so then also the transitions functions will become holomorphic. So, for example you know if a chart like this overlapped with a chart like that okay and if I took the transition function like this then I am first going by , so I am going z to z, h of z okay and then if I project onto w I will get h of z. so, the transition from function will become h and h is of course holomorphic why is h holomorphic.

Because that is because of implicit function theorem, similarly if I go if I take the other transition function namely if I go like this and then come down via that then it will be w going to gw, w and then if I take first projection I will get gw, so it will be w going to gw which is simply the function g and g is also holomorphic again because of the implicit function theorem therefore you see automatically all charts are compatible all the charts are automatically compatible just because of the implicit function theorem.

So, automatically this becomes a Riemann surface okay it automatically becomes a Riemann surface that is the beautiful thing.

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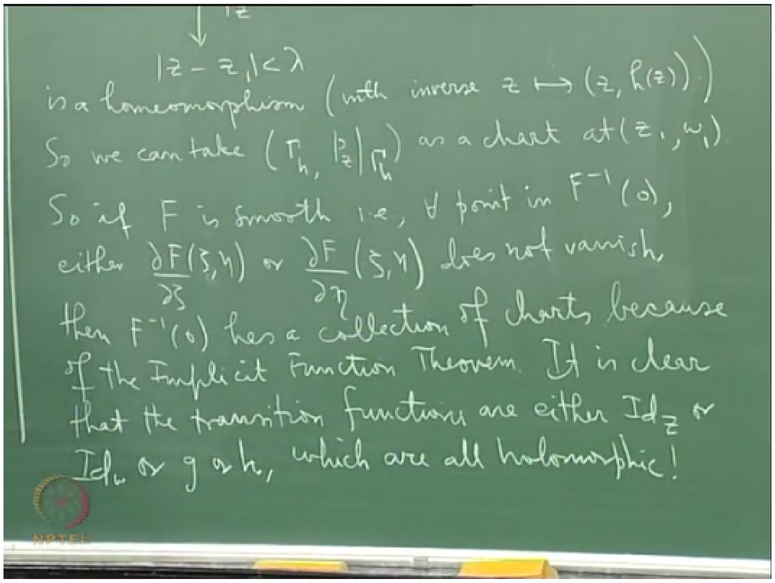

So, let me write it is clear that that the transition functions are either identity z or identity function on w or g or h which are all holomorphic. Because identity functions are of course holomorphic because identity function are holomorphic and the g and the h that you get their functions that you have gotten by the implicit function theorem they are holomorphic okay. So, the moral of the story is that F of z, w if F is a smooth function F of z, w becomes a Riemann surface okay.

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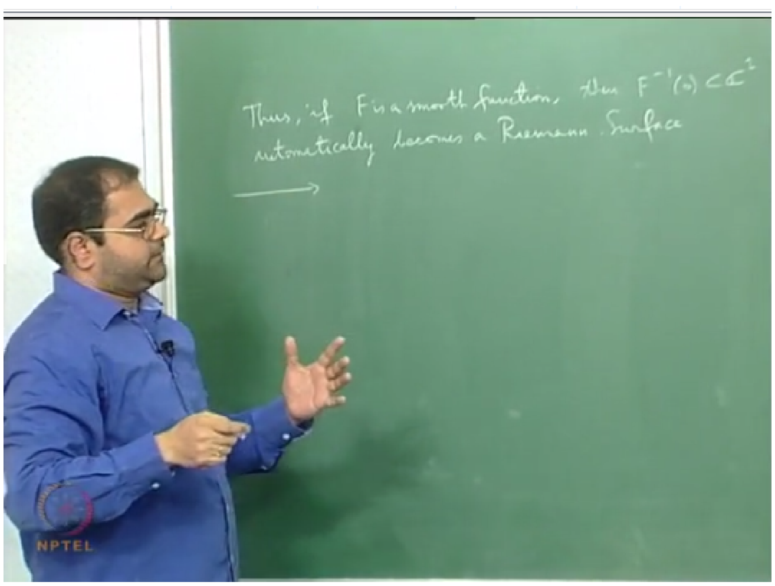

So, let me write that thus if F is a smooth function namely that either the first or the second either the partial derivative with respect to the first variable or the partial derivative with the with respect to the second variable does not vanish at each point where F is 0 okay. Then f inverse is 0 in c2 automatically becomes the Riemann surface this automatically becomes a Riemann surface okay.

So, now what I am going to do is I am going to tell you a little bit of technicality okay, so so this is of course I have conveyed the main point that one very important application implicit function theorem is that you can look at the 0 locus of a smooth function as Riemann surface okay. So, you can look you can do complex analysis on this surface okay very naturally alright.

So, the technical point is a following see when we define a Riemann surface if you want to give me the abstract definition of the Riemann surface you have to first define what an abstract surfaces okay. So, the so let me quickly recall these facts which you can try to understand if you do a little bit more re of reading. So, what you need is basically you start with a topological space x okay which is (()) (42:16) okay and which is second contribute okay namely you assume that it has a countable you know it has a countable basis okay.

So, you start to the topological space which is (()) (42:30) and which is second countable alright and which locally looks like the plane alright the complex plane or the real 2 plane such a topological space is called as surface it is called a real surface okay. So, you know the spear, the torus, the cylinder they are all the real surfaces okay and we also put the extra condition that you work only with connected topological spaces okay.

So, I when I define Riemann surface here I told you a Riemann surface is a surface x with a complex atlas namely with the collection of compatible charts but I need not tell you what that x is I told you for example x could be you know you can think of x as a spear or the torus or the cylinder but in general what can x p the answer to that is x should a topological space which is connected which is house dwarf which is second counter.

And which locally looks like the plane the fact that it locally looks like the plane is what tells you that every point of x has a neighborhood which looks like a disk okay and that is the only way of saying that it is a surface, a surface is something that should locally look like the plane okay. So, this is the technical definition and the point I want to make is that if I want to really with respect to that definition if I want to say that this the 0 locus is the Riemann surface.

I will have to verify that this is house dwarf I will have to verify this is connected I will have to verify this is a second quantum okay and the truth is that I mean the house dwarfness in the second accountableness are not so difficult to verify okay slightly more technical thing is the connectedness okay to say that for example if you take a polynomial if F of z, w the simplest kind of function 2 variables can think of is a polynomial into variedness okay.

And if you want to ensure that the 0 locus of the polynomial is connected that is this graph that I have drawn here it is actually a connected set in c2, 1 nice condition is that the polynomial should be irreducible that is the polynomial cannot be factored into a product of 2 different polynomials, 2 non-constant polynomials okay. So, the proof of this fact is not so easy but you can take it as a statement as a theorem that if f of z, w is a polynomial which is irreducible.

Then the 0 locus of f is actually connected okay, so you get connectedness you and I told you that it is house dwarf and second countable is something that you can verify okay because that is already there for c2, c2 is of of course house dwarf c2 is of course second countable alright see Euclidian space. So, the moral of the story is that in the with this formal definition of a Riemann surface also if you take for f and irreducible polynomial in 2 variables which is smooth.

Then the 0 locus will be actually an Riemann surface okay and so this tells you that the formal side of the picture is also correct okay, if you want to think of Riemann surfaces the formal senses house dwarf second countable connected topological spaces endued with a complex atlas. This locus is automatically one such Riemann surface okay and all this is just a beautiful corollary of the implicit function theorem okay, so I will stop here.