Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-11
Proof of the Implicit Function Theorem
(Topological Preliminaries)

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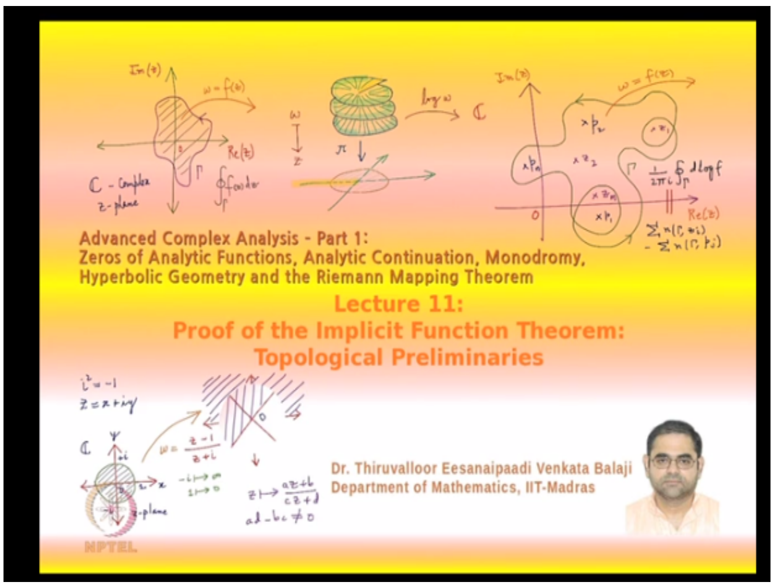

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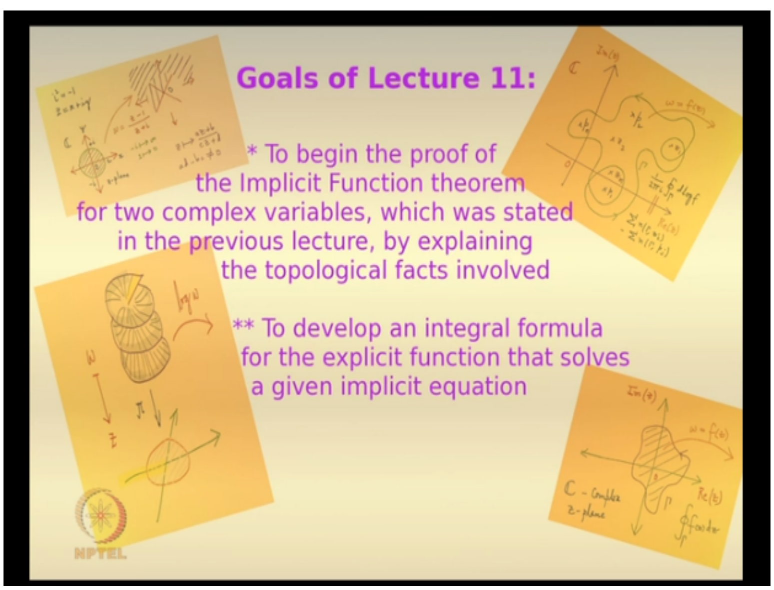

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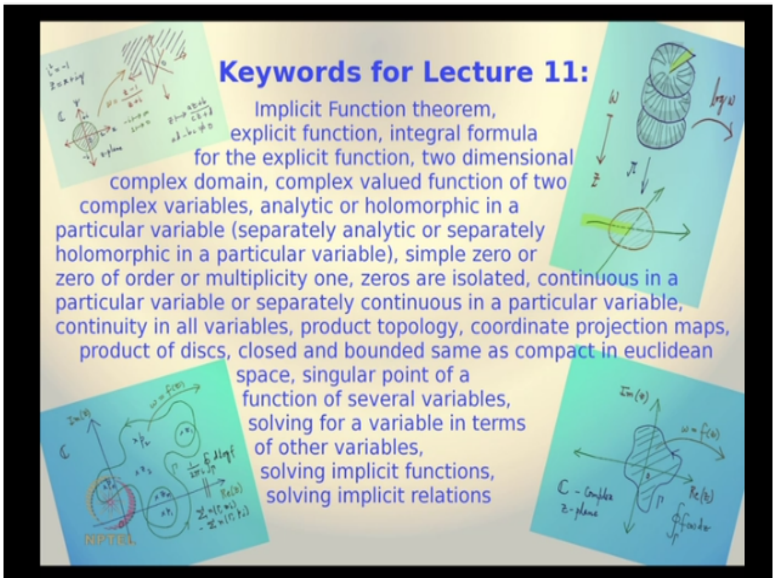

Alright so we will discussing so-called Implicit function theorem and basically implicit function theorem. For example in two variables says that if the derivative with respect to a particular variable is non-zero. Then you can solve the equation for that variable okay. So, I stated implicit function theorem and I will try to prove it. And I also told you that implicit function theorem is stronger than the inverse function theorem okay.

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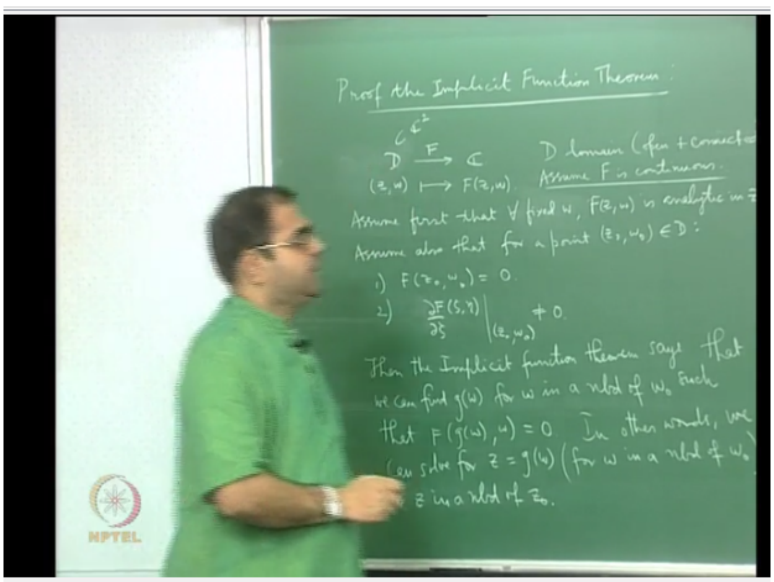

So, **so** let me write it has the proof of the Implicit function theorem, it is proof of the Implicit function theorem right. So let us we call what we started with, so we have say D is a domain c2 okay D domain which means it is a open connected sets. But mind you it is a subset of c2 okay and may be the arrow is not from c2.

But it is from D so let me D is sitting inside c2 okay. And I am having a function so I called the two variables that two complex variables Z and W okay. And I have a function F if find on this domain. So it takes the point z, w to F of z, w okay. So it is a complex value function of two complex variables alright. And assume first that for every w in the in D for every w in D fixed for every fixed w in D.

So let me write that again carefully for every fixed w in for every fixed w okay , F of z, w is analytic in z okay. So, you see if you fix a w fixed value for w. Then there will be values of z for which this function if you see if you fix a w then this because of function of z okay. And you can make sense of this is a function of z .

And what will happen is of course for every z at that point z, w is in this domain alright, of course is not define for all values z. It is only define for all those values of z, such that the point z, w is in this domain okay. Now for all those values of z this is the function of z okay and the fact it is an analytical functional set, there is a holomorphic function of z. This is what we assume alright .

Assume also also for that for a point is it not comma w not in D the following things happen number1 F of z not, w not is 0 okay that is uhh this is the point this point with coordinate z not in w not this is 0 for the function F. Number 2 you assume that dou F/dou zeta okay. Why I am writing dou is because of 2 in variables involved right. But actually I am fixing the second variable I am finding the derivative with respect to the first variable, which I get to because it is with respect to the first variable.

It is in analytical function, it is a holomorphic function which means is a differential function. So, I can derive, I can take the derivative and I take these derivative and evaluated at this point is it not comma w not. And that I assigned to the derivative is non-zero okay. I assume that the derivatives so, all I am saying is the function has a zero at the point z not comma w not and the first and the function .

And the derivative of function with respect to the first variable z at z not comma w not is nonzero okay. Then the implicit function tells you that in a neighbourhood of w not you can solve for z as a function of w okay. This is what so, let me write that then the Implicit Function then the Implicit function theorem says that we can find g of w for w in a neighbourhood of w not.

Such that f of g f w, w is 0 okay which means what are you doing is which means you are saying that you can solve for f of z, w with you can solve for the z. Such that f of z, w as a function of w in other words let is so let me write this in other words we can solve for z is equal to g of w. For w in a for w in a neighbourhood of w not and z in a neighbourhood of z now.

Basically so, the whole idea is the function of two variables has a 0 at a point with respect to the first variable, if the derivative is non-zero at that point then you can solve for the first variable as the function of the second variable. In a neighbourhood in a suitable neighbourhood of the point correspond to the first variable and the suitable neighbourhood of a point corresponding to the second variable. This is what it is right**.**

so So, in fact of course I have stated with this for the first variable inverting a stated for a second variable also okay. So, the general idea of the Implicit function theorem is your function of several variables and assume that the partial derivative with respect to one of the variables does not vanish. Then that variable can be solved for as a function of the remaining variables in a suitable neighbourhood, that is what it is.

So, in this case the first variable z and the partial derivative with respect to the first variable does not vanish. So, you can solve for the first variable as for the second variable, which is the second variable is the remaining variable. And in fact the truth is at this whole spots several variables not for just for a function of two variables. It actually holds for several variables and of course there is also real .

You know there is a version of this for real differentiable functions also which is the visual Implicit function theorem for real differentiable functions okay. So, here I am considering complex function of two variables, but you can also consider complex function of n variables. Similarly I can also consider real differentiable function of n variables and the statement of the Implicit function theorem is always the same.

It is just that you can solve for that variable with respect to it is the partial derivative does not vanish, that is all okay. Now let us see how this is achieved. So, you know the first thing I want to start with this see now.

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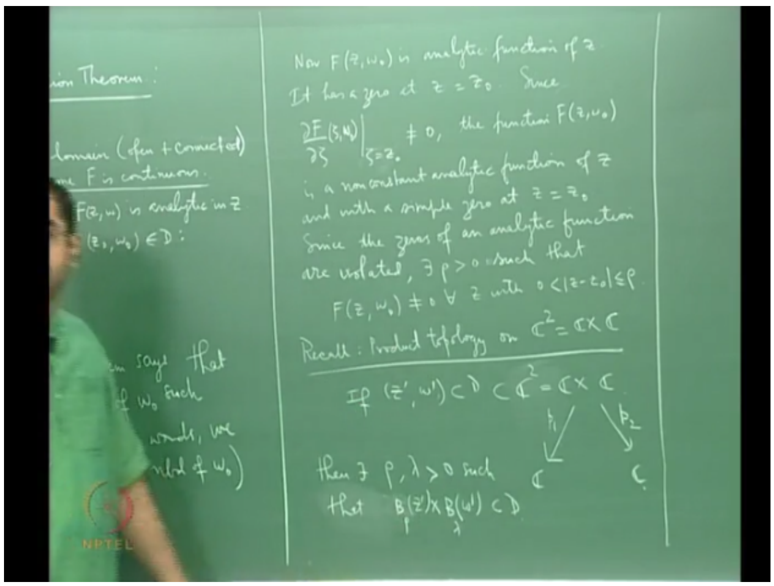

You see F of z, w not is an analytic function of z okay. It has a 0 of course analytic is the same as holomorphic okay complex differentiable in a neighbourhood it has a 0 at z equal to z now. Because F of z not, w not is 0 that is our assumption. So, z not is a 0 of the analytic function and mind you this is not a constant analytic function because it is first derivative with respect to the first variable is non-zero at that point okay.

So, since the first variable is 1 0 what it tells you is that this 0 it is a simple 0 and it also tells you that the analytic function is non-constant. Consider as a function of the first variable okay since so, this is by assumption 1 and by assumption 2. Since dou f/douz okay douf/ douzeta zeta, eta at z not you put w not here at z not , w not or in just like infinities z equal to z not at zeta equal to z not is not 0 okay.

So, you take this function as a you freeze the second variable you free you put neta equal to w not then it becomes only a function of first variable. You take the derivative with respect to that variable, then substitute the point z not, w not. I mean is essentially means you have to substitute z not because you have plugged in w not. Then at the first version derivative is non-zero implies .

What did you implies is that tells you that the function F of z,w not is a non-constant analytic function of subz**.** Because I did been a constant analytic function then it is derivative would have been 0 which is not the case. So, it is a non-constant analytic function of z and with a simple 0 at z=zokay. . And the simple the simplest of the 0 is preciously the fact that the derivative does not vanish.

If the derivative vanishes then you know the order of the 0 is more than 1 okay. So, that the derivative does not vanish with respect to the first variable tells you that in the first variable the 0 is simple 0 okay. Now you know that the zeroes of a non-constant analytic function are isolated so, I can find a discs centred at z not and sufficiently a small radius. So that for all points in the disc, including the boundary circle.

The function does not vanish okay except at the centre of the disc where it vanishes. That is i can isolate that 0 by closed disc okay. So, since the 0 the zeroes of analytic function are isolated there exist row positive and such that F of z, w not is not 0 for all z with 0 less mod z-z not strictly lesser than or equal to row. So, I can find this row okay yeah.

But they something that I have instead of told you earlier the function that I start with okay so, this assumption that of told you. But any way it is not late say that so, the function that I start with this to begin with I at least assume that it is a continuous function in both variables okay. I assume F is continuous that is something that yourself.

It is a of course you do not want to work with functions of several variables is not a continuous. So, F is continuous function of both these variables okay I mind that is in general that is very strongwhen compared to assuming that F is continuous separately in each variable okay. So, I am saying F is continuous on D okay .Now you see to give you a picture of the domains of z and w.

I want you to just recall the so called product topology, you see recall product topology on c2 this is c cross c. You see what you so, you know the complex plain which is r2, it has a same topology as r2 okay. And the topology is given by taking for a base the open discs centred at point end having positive radius right. And you know that this is a base because you take any open set in the complex plain.

It can be return as a union of such discs okay, because any open set if you give me a point of any open set there is sufficiently small discs surrounding that point which lies in that open set. And now if you take all such discs corresponding to various points in take the union you will get the open set okay. So, the topology on the complex plane is given by discs as a base for the topology open discs okay.

Now what happens in the product topology in the product topology what you do is that you take when you take a product of two topological spaces. And make it into topological space, what you do is you do the following thing to prescribe open sets there. So, what you do is you take open set in from each topological space okay, you pull it back by the projection maps okay.

And you take products alright and then you and union if you take a unions of all such products that will give you all the open sets okay. So, all I want to say is that any if you give me any point in the plane I mean c2 then you know you can and if it is inside an open set, then you can find that point sitting inside a product of discs, which is in that open set okay.

This is a fact from topology. So if so, you know if z not, w not is point of is contained in D which is contained in c cross c which contain in c2 which is so, you know I will even put z, w c cross c and there are these projections p1 the first projection, projection and the first coordinate p2 is projection on to the second coordinate okay.

Then you know there exist well I should say row there exist row and lamda positive such that you know balls entered at the open balls entered at z prime radius row cross the open ball centred at w prime radius lamda is contained in D okay. This is a fact from the definition of product topology.

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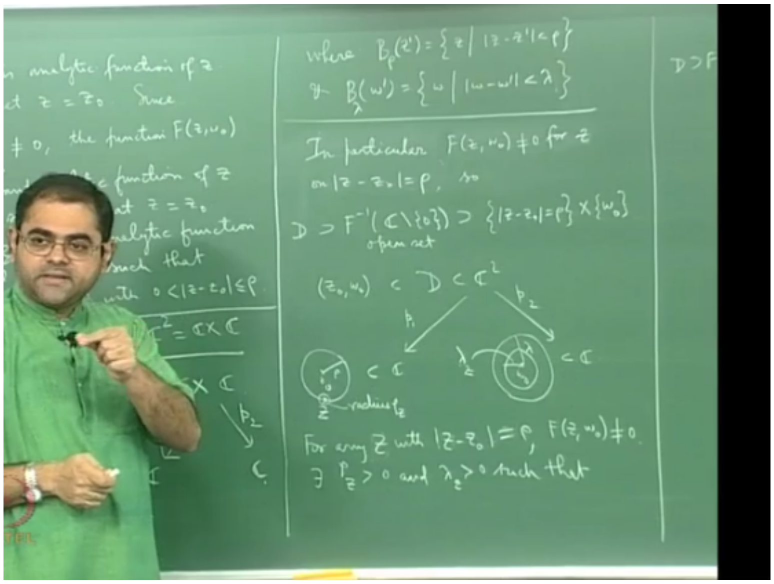

Where of course you know they disc centred at row this centred at z prime radius row is set of all z such that mod z-z prime is less than row. And the disc centred at w prime radius lamda is z of all w such that mod w-w prime is less than lamda. So, it will contain a product of discs okay. So, this is so, this kind of so, what you must understand is that whenever you are working with a point here you can always think of a disc.

You can think of this point as being as sitting in to in a product of discs, a discs centred at the first coordinate cross a disc centred in the second coordinate okay, for sufficiently small values of the radii right. This is because of product topology and in fact product topology tells you that this these projection maps are actually. You know the projection maps are continuous and the projection maps are even open maps.

They take open sets to open sets. So, **so,** the fact is that you know if I take the image of the open set d under p1 the result is an open set in c which will contain z prime. Therefore it has to contain a disc alright and if i take the image of under D of p2 I will get an open set again in c which will contain w prime and so it will contain a discs disc containing a w prime alright.

And then you can choose this radii as small enough so, that this product is inside d okay. Now so, all I am trying to say is that you know when I say that you can find a row such that for all z in a discs centred at z not radius row F of z, w not is defined it makes sense, because z not, w not is a point of D. And therefore you know I can find such a disc okay.

The image of D under the first projection will contain z not, so it will contain a disc centred at z not right. So, let us go ahead with this. So the point is now I have this disc in which this disc centred at z not radius row in which F as a function of z has only one 0 at the centre, which simple 0 okay. So, in particular a mod so, F of z, w not.

You see this is non-zero for z on mod z-z not equal to, see on the boundary of the disc the function does not have any zeroes okay, see this F of z , w not has only a 0 at z equal to z not on the boundary circle it does not have okay. So, this is the boundary circle centred at z not radius row on this boundary circle it does not have any zeroes. So, it is a and mind you therefore if you take the mod of F.

You will see that of course I forgot to write not equal to 0 so, F is not 0 so, mod F is not 0. And mind you mod of is a continuous function okay. And this continuous function is being taken on this compact set. This is a circle which is closed and bounded. So, it is compact okay in Euclidian's spaces closed and boundary subsets of précised give the compact sets.

So, you have a continuous function namely mod of F of z, w not on this compact sets. So, it is uniformly continuous and it has it attains it is bounds. So, it has a certain minimum value okay so, **so** it has a set minimum value but let me not worry about that, what I want to say is so, I want to say that F inverse of c star which is c-0 contains the product mod z- z not is equal to row cross w not okay see for each this is a same.

I am saying that the value of F at any z, w not is non-zero, so, long as z lies on the boundary circle is what it is. Now I want you to you know really look at this diagram. So here is my D which is sitting inside c2. And you know I have z not, w not here and basically I have in the so, I have these two projections p1 and p2 and under this projection into c.

I have this disc centred at z not and radius row I have this disc. And in p2 of course I have a disc centred at w not and radius lamda okay, which is inside this copy of c okay. I have this picture and the function is defined for every z here and every z that right. Now you see but you see this F inverse see c-0 is an open subset of c okay. And therefore F inverse of this is an open subset of D.

After is a set of point where F does not vanish F inverse of this is a F inverse this is a compliment of F inverse of 0. This is just the compliment of F inverse of 0 so, it is set of points where F does not vanish and therefore this is an open set**.** So, you have the open set which contains this product okay. Now the point about this product is this is a product this product is the actually on one side.

This is the circle the unit circle on L side this is the point it is a point cross the circle cross the point. It is a circle cross the point and that is holomorphic to the circle if you want. But the fact is that in any case it is I will not to point out this is a compact set because if you want product of compacts it is compact. If you want to use taken of theorem but that is too much.

And it is very clear that this cross settle cross of point is just the same as the circle topology okay. And because of circle cross of point 2 if you project it on to the circle that will be topological isomorphism. If you take this first projection circle cross a point to the circle that will be a topological isomorphism homeomorphism and under homeomorphism the image of compact set is a compact set.

Therefore any case this is a compact set okay that is the fact I want it to a remember now what you do is well you take any point zeta on the circle okay. Then at that point if I take F of zeta , w not you will see that F of zeta , w not is non-zero okay for any zeta with mod zeta. So , I am **I** seem to be okay let me not change notations let me not change notation.

Let me keeps still keeps it as z for any point z let me continuous using z with mod z-z not less than I mean sorry equal to row okay F of z, w not is non-zero this is something that we are already this is what we noted first okay. And you see so what happens is that you see I have this point z, w not in D I have this z , w not which is inside D okay.

And that is point it which the function does not vanish so, it is in the open set so, it is an open subset so, it is an open set it is a point not only in deep. But it is in this open subset of D set of points where F does not vanish and therefore there is a product neighbourhood of this there is a product neighbourhood consisting of a product of disc rounding this point and disc rounding w not, which is contained in this by whatever I told you about product of (()) (30:27) okay.

So, there exist delta z so, I let me keep you using let me use row z greater than 0 and lamda z greater than 0 such that well so, let me write it here.

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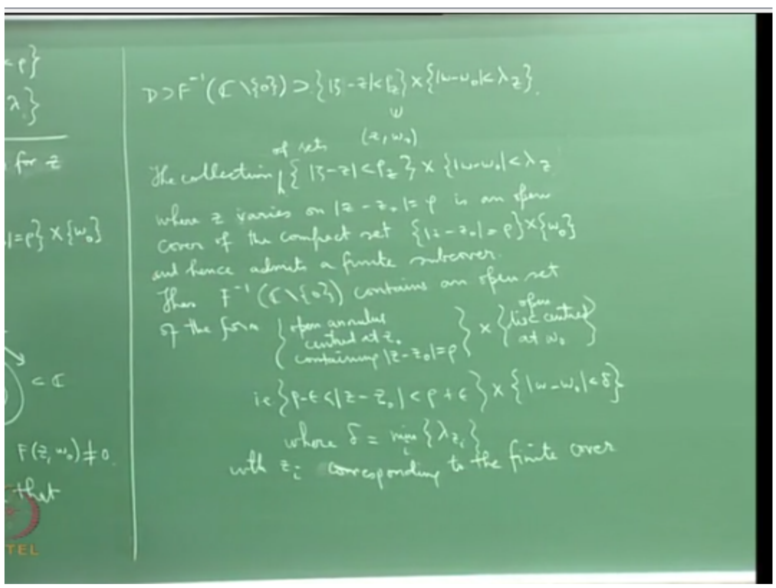

D contains F inverse c-0 which contains well this disc centred at z radius row z and this disc centred at w not radius delta z. So, this is delta z I am sorry this is lamda z. And here it is row z. So, this disc centred at z, this is radius row z okay. There is a disc like this I mean the point is that I can make this I assume that the function does not vanish on the boundary of the disc okay.

And that means that there is an open set which contains this disc okay where the function . SO , you know I can always choose the row z sufficiently small given z. But the function does not vanish so, what will happens D contains this open set D on which F is defined contains is open subset where F does not vanish that is the open set. Because F is continuous and the inverse image of F and open set is open under a continuous function.

And this contains the this product neighbourhood which is mod z. So, now I will have to use another some other notation. So, it is mod zeta- z lesser than row z cross on the other side it will be mod w-w not lesser than deltaz lamda z. So, it will contains this which I mean which of course contains point which of course contains point well z, w not there F does not vanish right.

So, all I am doing is that the set of points where F does not vanish is an open set. Because F is continuous and you take a point where F does not vanish. This point lying the first coordinate lying on the on this boundary circle there second coordinate fixed w not. Then I am just saying that you can find this point inside a product neighbourhood okay inside this okay.

And what you must understand is that the fact is that the main fact I am using is that this is the open set and a point inside an open set is always contained in a product neighbourhood in that open set. That is the fact about the product topology that I am using okay. And now what you do is now you do this for all you do it is for all z varying on this circle you do this for all z varying on this circle.

Then what will you get is you will get an open covering of this compact sets, because of you covered every point on this circle cross w not. So, what you will get is if I now vary is z then all these product neighbourhoods will give me an open covering of this products said this is compact. And therefore it is possible by the definition of compactness to extract a finites to extract a finites of cover okay.

So, I should remind you that the standard definition of compactness is that given any open cover. You can always extract a finites cover okay that is z is called compact if it is contained in union of open sets then it is enough to take only finitely many of those open sets. And take the union it will be contained in that sub union okay which is called sub cover. Cover simply means that the z is contained in the union of your collection okay.

And the fact that every cover has an open cover has a finite sub cover is compactness and we prove in a first quotient topology on included spaces that this compactness of a set is equivalent to it is being both closed as well as bounded. The boundedness is of course that it is contained in a finite large enough ball okay that means all of it is points if you measure the distance from the origin the distances are bounded okay.

And the other the closeness is the condition that it contains it is boundary and there is no if you take a sequence of points in the set, then the limit will also means set. There is no boundary point, limit point which is not in that set that is the closeness. And we prove in a first quotient topology that subset of Euclidian's spaces compact different only if it is close and boundary okay.

So, that is you tell you that for any open cover for this admits a finites sub cover. So, you know that means that I can just take finitely many z is okay. And if I take finitely many z is that means I will get finitely many row z is okay. The union of which will cover this disc on the side and here I will get finitely many lamda z is. And among them I can take the least.

And then you will see that what you will get is that an annular open annular region containing this circle cross a disc is where the function will not being not vanish okay. You will get that okay. So, let me write that down the collection mod zeta-z less than row z cross mod w-w not less than lamda z where the collection let me write the collection of this the collection of sets .

Where z lies z varies on mod z-z not equal to is equal to row is an open cover covering of the compacts set given by this product mod z- z not is equal to row cross w not and hence admits a finite cover a finite sub cover okay. Thus F inverse of c-0 contains an open an contains a an open set of the form open annulars centred open annulars centred at z not containing this circle mod z-z not equal to row cross a disc centred open disc at open disc centred at w not.

So, which is that is I can write you know again write mod z- z not greater than some I can write row+fslong I can put row-fslong cross mod w-w not less than delta where of course is this delta is actually the delta is a minimum of the lamda the minimum of lamda is zetai over i with zetai coming from the finite cover corresponding to the finite cover okay.

So, basically all this is to say to that if the function does not vanish from the circle cross this point, then it cannot vanish on open annulus containing this circle cross small disc surrounding that point okay. That is what I am saying right. So, in particular what it means is that for all w z mod w-w not is ct less than delta okay for all w in this disc alright. The function F of z, w does not vanish with z here that is what it means okay.

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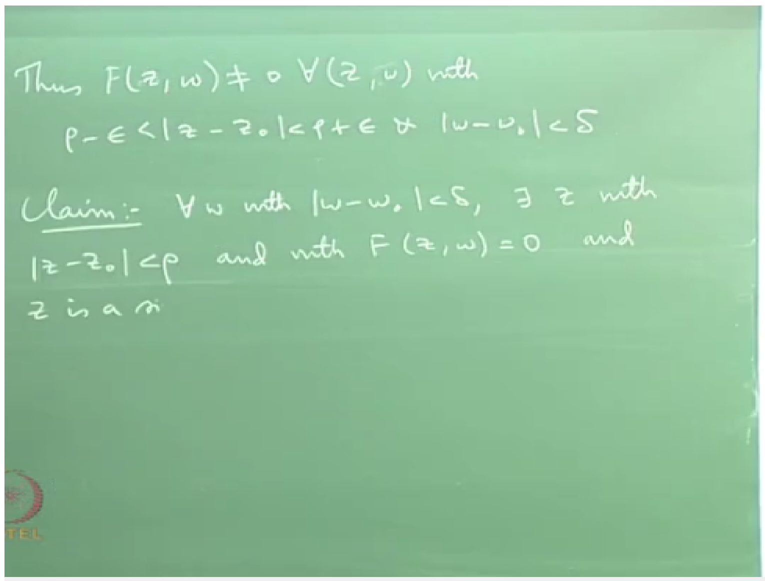

The F of z , w is not 0 for all z in for all z, w with this condition -row-epsilon less than mod z-z znot less than row+epsilon. And mod w-w not (()) (41:58) this is what happens right. Now so, this is the annulus containing this circle cross disc centred at w not okay. The function does not vanish z. We are already know that you see F of z not, w not is 0 you know that where is that term where it is, there it is we have F of z not, w not is 0 okay.

That is F of z not F of z, w not has is 0 at z not and that 0 is a simple 0 we know this okay. What I am now going to say is that I am going to say for every w with w line in that disc mod w-w not less than delta going to say that there is exactly one z lying inside mod z-z not +less than row where F of z, w is 0. And I am going to say that that 0 is simple 0 namely 0 of order 1 okay.

So, the claim is the following claim, for every w with mod w-w not less than delta there exist z with mod z-z not less than row. And with F of z, w equal to 0 and z is a simple 0 so, I should say and in fact I should say the following thing and in fact**.**

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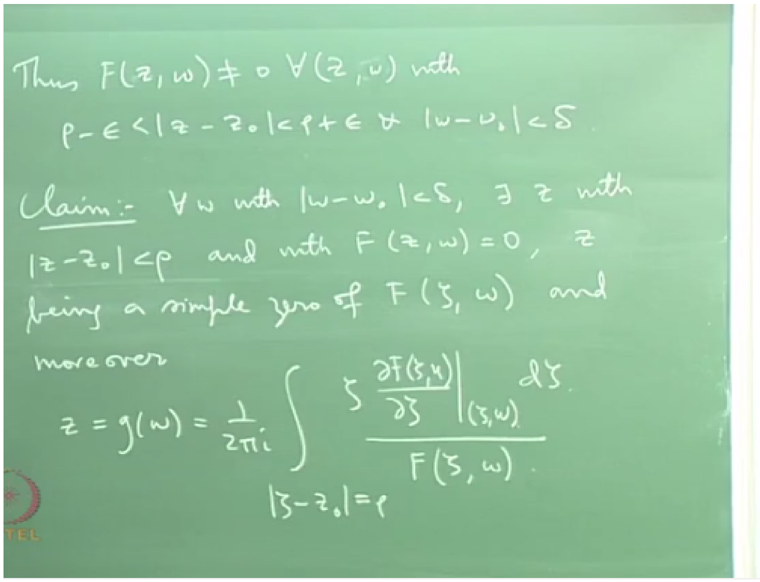

Z being a simple 0 of F of zeta, w and more over z is given by g of w which is 1/2pie i integral over mod zeta-z not equal to row zeta dou F zeta, eta over dou zeta evaluated at zeta, w not w D zeta/F of zeta, w okay. So, here is the claim. So all I am saying is that you take this w in this disc okay, take w in this disc, then I can find a z inside this disc such that z equal to g of w okay.

And this how this z depends on w is that z will be a function of w and what is that function of w it is given by this formula okay. So, saying that F of z, w is 0 and saying that z is g of w the z is given by g of w which is this function is the same as saying that F of g of w, w 0. So, what I have done is a I solved z for w z equal to 0w and this is the. So, what I have done is the explicitly F of z, w equal to 0 has being solved as z equal to g of w.

For w in a neighbourhood of w not okay and this is being done provided you see you are solving for the first variable. And remember that the partial derivative with respect to the first variable is not 0 okay. So, this is the point. So, this is what the Implicit function is. So, now we will have to prove this statement okay. And mind you in all these.

Before I end this lecture I let me tell you that is not necessary is that this g is analytic okay. You can at ensure that this g is analytic but then if you what will see later is that if this function capital F is also analytic in the second variable for every fixed value as the first variable. Then you can show that this g also is analytic has a function of w and there is a nice formula for the derivative of g okay. This will see in next lecture.