Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology-Madras

Lecture-10
Introduction to the Implicit Function Theorem

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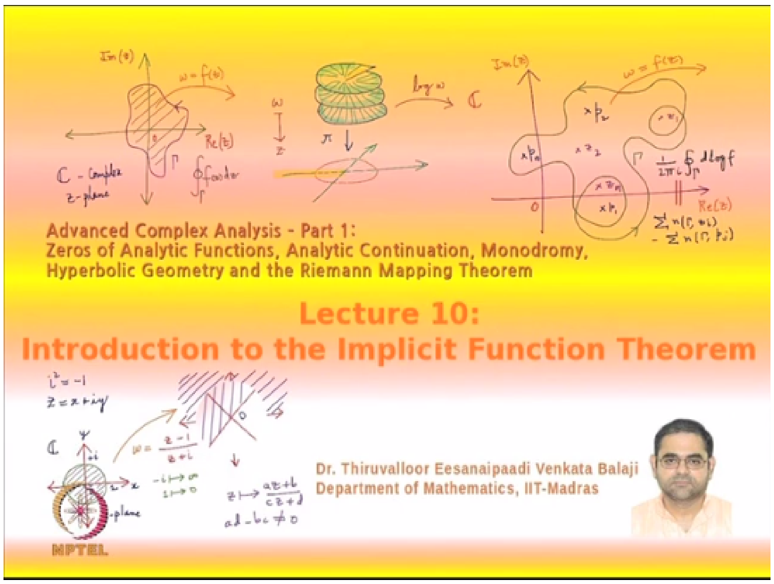

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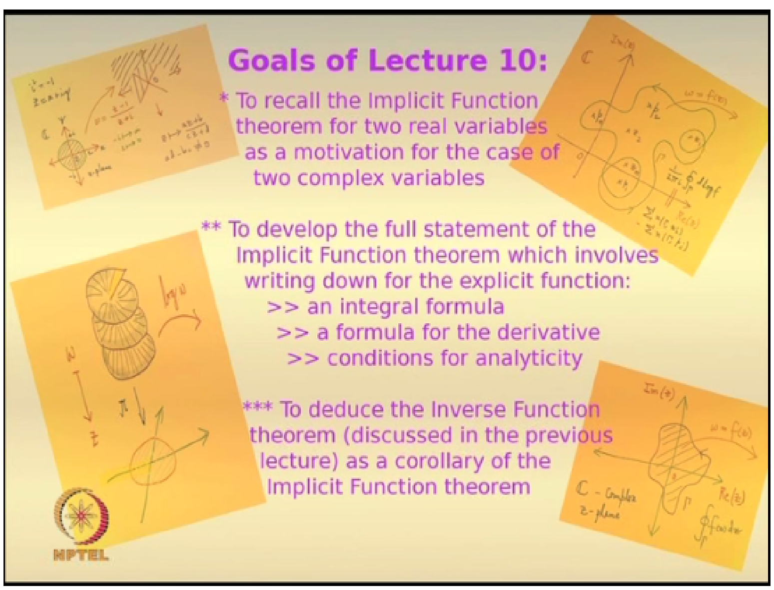

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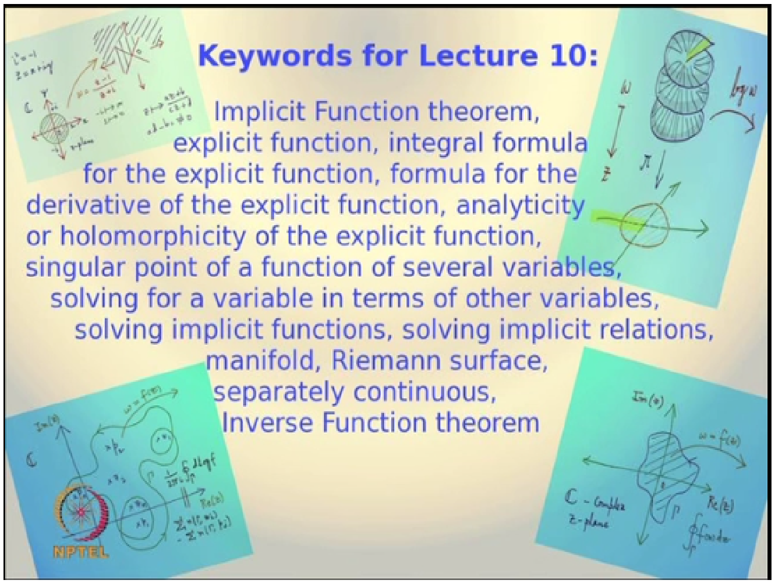

Okay, so what we will try to look at in this and the lecture is the so, called Implicit function theorem okay now let me recall what the Implicit function theorem is in the congestive real analysis okay as for as real variables are constant. This is something that you should of seen in a courses in real analysis.

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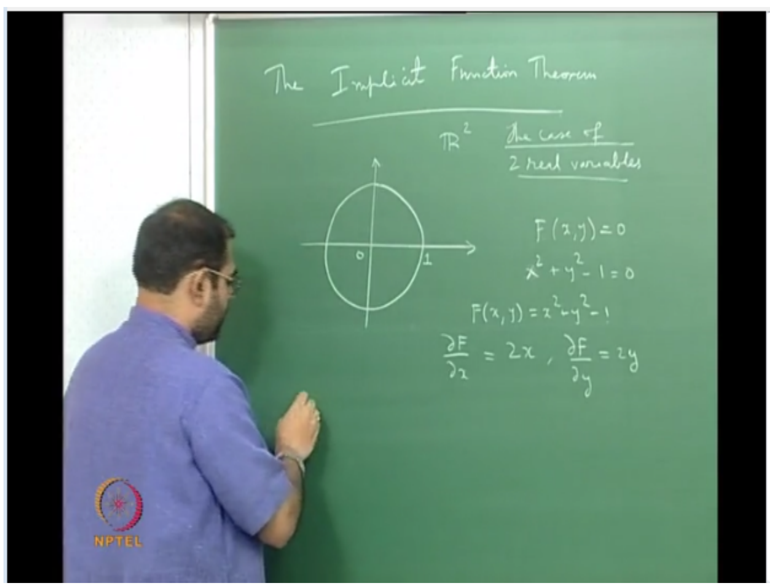

So, the so let me put the title as the implicit function theorem so, you know the so, let us look at the look at a very simple example. So, let me look at the case of two real variables this is something that you should have come across the first or second course in analysis real analysis. So, we take the plane R2 and let us look at some equation of the form capital F of x,y equal to 0.

So, this is to be thought of as an Implicit relationship between x and y because you are not able to x as a function of y or y as a function of x this is called an Implicit relation. If this can be equally equivalently written as x is equal to some function of y okay that means you have solved it explicitly for x. And if you can write it out the form y is equal to function of x.

Then you have solve it explicitly for y okay but if if in general you are given a relationship like this where one variable is not written as a function of the other variable explicitly this is called an Implicit function. So, the standard example is x square+ xsquare+ysquare-1=0 which you know is this circle is unit circle. And so, it is the unit circle centred at the origin alright.

And what you should notice is that you know if you calculate so, if you calculate so, if I take the example f of x,y is equal to xsquare+ysquare-1. And if I calculate dou f /dou x of then I will get if I partially differentiate this with respect to xi I will get 2x okay. And if I partially differentiate it with respect to y I get 2y okay and the set of points where the partial derivative with respect to x vanishes or the set of points where x equal to 0 okay.

And x equal to 0 corresponds to the y axis these are the points on the y axis and of course I am looking at points on the curve also okay. So, you know so, if you look at f equal to 0 and dou f/dou x equal to 0 if look at this set of equations okay. So, f equal to 0 means it is a point on the curve defined by f of x,y and it is also point where the first partial derivative with respect to x vanishes okay.

So, this is this happens only so, this happens only at the point x,y is equal to so, you know if I want the first partial derivative to vanish then 2x must be 0. So, x must be 0 and If x,y is lying on the curve then it should satisfy the equation and if x is 0 it means that y is + or -1. So, you get 0+ or- so, you will get so, namely you will get these points. So, it is x equal to 0 y is equal to -1 and I will get this point which is x equal to 0 y equal to +1 okay.

And so, if you take all other points on the curve these are points for which the first partial derivative with respect to x does not vanish okay. So, if you take curve circle-these two points 0+or-1 on these points of the curve what happens is that the first partial derivative with respect to x does not vanish okay. And at so, what happens that these so, at every other point of the curve what happens is that.

You can solve for x as a function of y so, if you give me any other point on the curve okay. Then you know I can take the if you give me a point x0,y0 on the curve okay. Then I can so, I get the point y0 here okay and I can find a small neighbourhood of y0 okay where x can be written as a function of y0 and for that function this is the graph of that function in that neighbourhood.

So, you know if I take this piece of the curve in this piece of the curve at say at x0,y0 locally you can write x as a function of y okay locally you can solve so, the point is so, in this case you know I can solve for this is locally at y0 okay x is a function of y in an neighbourhood of y0.So, you see x0,y0 is a point on the curve that is the point which is different from these two points okay .

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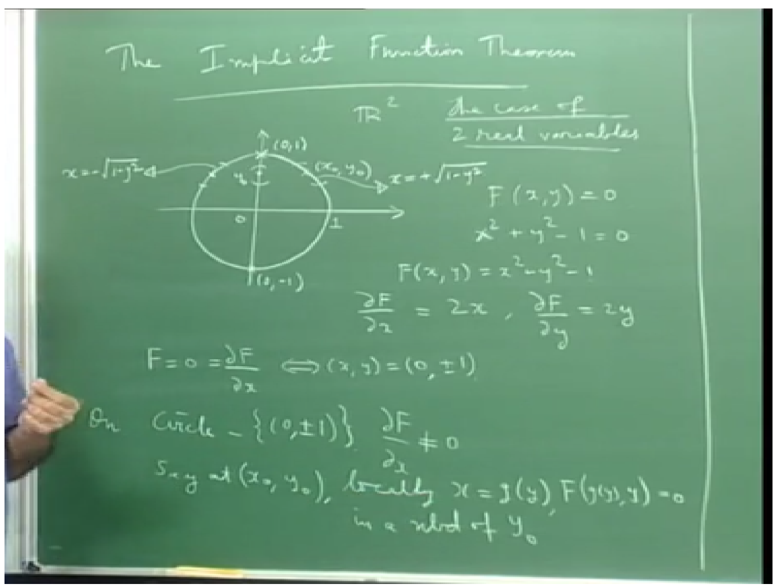

So, the first partial derivative with respect to x can does not vanish and then locally at that point I can write the curve as a function of y okay. And you know what this is in this you can write the function you can write it you know what you know the function that I am talking about you can write x as positive square root of x can be written as spot positive square root of 1-ysquare.

That will give me this branch of the curve and of course if my point is here if my x0,y0 is here then I **I** will be able to write x as negative square root of 1-ysquare okay. So, here on this piece this is the graph of x=positive square root of 1-y square and this is the graph of x=negative square root oof 1-y square, so what you see is that if you chose a point where the first partial derivative with respect to x does not vanish you can write x explicitly as a function of y.

And mind you when I write x=g of y what it means is that f of g of y, y is 0 because it is a x=g of y is actually the piece of the curve, so x=g of y will satisfy the equation of the curve okay. And this x=g of y in this case if this is if it is a point in the first quadrant then x=g of y is so it is positive square root of 1-y square, so this is the g of y okay and in this case it is negative square root of 1-y square okay.

And that is what you will get whether you take a point here or here you will always get positive square root of 1-y square you take a point here or here will get always negative square root of 1-y square and these are the 2 branches of that curve actually okay depending on the point you have chosen. But the idea is that your what you see is that you get uniquely a function which solves your equation you get a unique function.

Of course you get this branch if you start with the point here and you get this piece of the curve it start with the point here, so the piece of the curve that you get depends on the point at which you are trying to solve the equation and that point has to have the property the first that the first partial derivative with respect to x does not vanish. So, the moral of the story is the is that if the first partial derivative with respect to x I mean the partial derivative with respect to x does not vanish. Then you can solve for x okay and similarly you I can repeat the same thing with y also.

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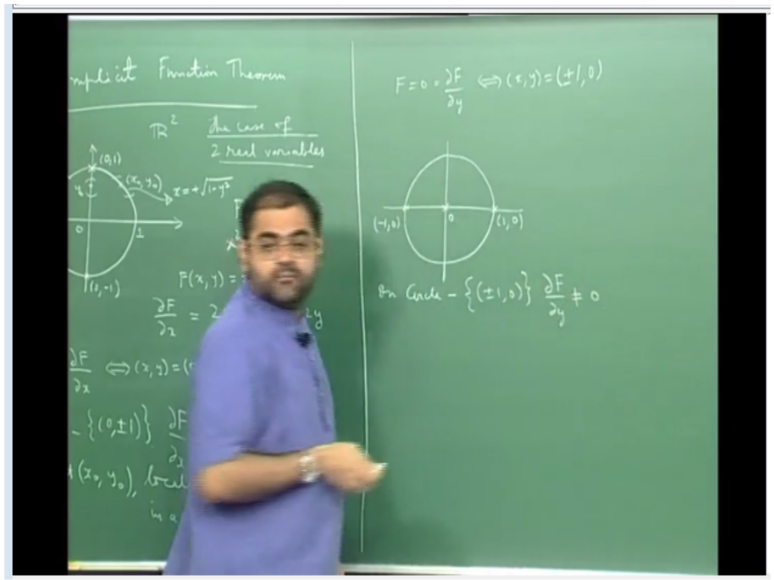

If I look at the equation f=0=tou f by tou y if I look at this then this is true if and only if the point x, y is well you know I **I** want tou f by tou y to be 0 tou f by tou y is 2y, so 2y is which means y is 0 so, the y coordinate is 0 and if y coordinate is 0 and I want the curve then I will get xquare-1=0 that will give me x equal to + or -1. So, I will get + or -1,0 so, this situation is that I will get let me draw another diagram instead of cramming that diagram with two many more things.

So, here is my so, I will get the points + or- so, I will get these two points 1,0 and -1,0. So, tell and these are the points where the first partial derivative with respect to the variable y does not vanish. And now you choose a point different from this on circle – these two points + or -1,0 the first partial derivative with respect to y does not vanish okay.

Because these are the two points where the first partial derivative with respect to y vanish and we remove them at say at now if I take a point x0,y0 on which is different from these two points see for example I could have taken a point x0,y0 here. You can solve for y as a function of x okay y can be written as a function of x and that will be a solution of f of x,ff of x,y equal to 0.

So, we will get f of x,fx is 0 in a neighbourhood of x0 okay. So, what will happen is that you see here you project down to x0 and that the small neighbourhood of x0 you can find such that the graph of the curve at in a neighbourhood of x x0 will be given by an equation you know what it is going to be y is equal to positive square root of 1-xsquare so, this is your f of x okay.

And if I had chosen the point here if I chosen the point x0,y0 of course if I chosen any point on the upper semicircle I will get this function. If I chosen a point on the lower semi circle if I chosen a point x0,y0 here the lower semi circle then you know for the corresponding x coordinate. I will get a neighbourhood such that the equation the Implicit function can be solve for y.

And it will be –root of 1-xsquare so, it will be so, here it is going to be y equal to negative square root of 1-xsquare okay. So, the moral story of the story is that to sum up you have an Implicit function in two real variables. You can solve that Implicit function for the variable x in a neighbourhood of a point where the first partial derivative with respect to a x does not vanish okay.

And that is here that is what I written here if the first partial derivative with respect to x does not vanish. I can solve for x as a function of y if the first partial derivative with respect to y does not vanish I can solve for y as a function of x. And both it will be explicitly solution of the Implicit function the Implicit equation I started it okay. So, this is what the Implicit function theorem in real analysis.

And in fact this Implicit function theorem can be generalised to several variables instead of just taking a function of two real variables okay. You can have a function of several variables and then the Implicit function theorem for several variables will tell you that you can solve for a variable as a function of the other variables provided the first partial derivative with respect to that variable does not vanish at a point on the locus where this function is defined okay.

This is what this how it generalises there is a and it is not very difficult to prove that theorem. So, in the case of two variables goes to extension case of several variables and this Implicit function theorem is a lot of importance in study of manifolds and study of Riemann surface and things like that it is a very important theorem. Now what I want to say is that there is an Implicit function theorem which is also there in for two complex variables okay.

And it is again it also extends to several complex variables and what I want to tell you is that this Implicit function theorem for two complex variables that I am going to talk about is also going to tell you the same thing it is the same statement except that instead of two real variables they are going to deal with two complex variables. And is going to say that whenever the first partial derivative with the partial derivative with respect to the certain variable is not 0.

Then you can solve for that variable in terms of the other variable that is what it is going to tell you. But the point with complex analysis is that because of Cauchy's theory you can explicit formula given by an integral which you do not get in the case of real functions. So, the advantage with complex analysis is that because of the existence of Cauchy theory you cna write down the local solution okay.

So, that is what we are going to see so, and there are I will try to see if I can give you applications of this one of the applications of the Implicit theorem for example is the fact that If you give me a an equation in two variables such that there is no singular point a singular point is a point where is a point on the locus where all the partial derivatives vanish okay.

The see to be able to solve it as a function of one variable in terms of the other the partial derivative with respect to that variable should not vanish okay. But is the partial derivative with every variable vanishes at a point. Then I am in bad shape I cannot conclude that I can solve the function for one variable in terms of the variables okay. Because all that the Implicit function theorem says is that if the partial derivative with respect to a certain variable is not 0.

Then you can solve in a neighbourhood of that point for that variable you can solve for that variable in terms of the other variables okay. So, obviously if all the partial derivatives are going to vanish then you cannot apply the Implicit function theorem and the points on the locus where all the partial derivatives vanish at the same time they are called singular points okay. And the problem is that the at singular points.

You cannot conclude whether you can solve of course both kinds of cases are there are where you have singular points and yet you can solve locally for one variable even though the first partial derivative vanishes as you can look at the curve for example ysquare equal to x cube which is a crush okay. And there are also curves for which at singular points you cannot solve locally okay.

So, the main application of manifold theory is that whenever you have an equation which has no singular points. For example the curve is this curve here is a circle in the circle has no singularities okay. And one way of seeing it, is that it has a unique tangent at every point okay. And whenever you have curve which has no singularities then the beautiful thing is that you can turn it into a manifold okay.

So, in this case the circle is a manifold one dimensional manifold if you had if I had two complex variables then I can turn the locus into what is called a Riemann surface okay. So, one of the most important applications of the Implicit function theorem is that you can for a curve for an equation which is non-singular which has no singular points. You can turn the locus of that equation the set of points satisfied by that equation into a manifold.

And you can do calculus on it okay so, that is one of the most important applications of the Implicit function theorem that is why it is very very important for higher analysis. So, let me go ahead with the complex version.

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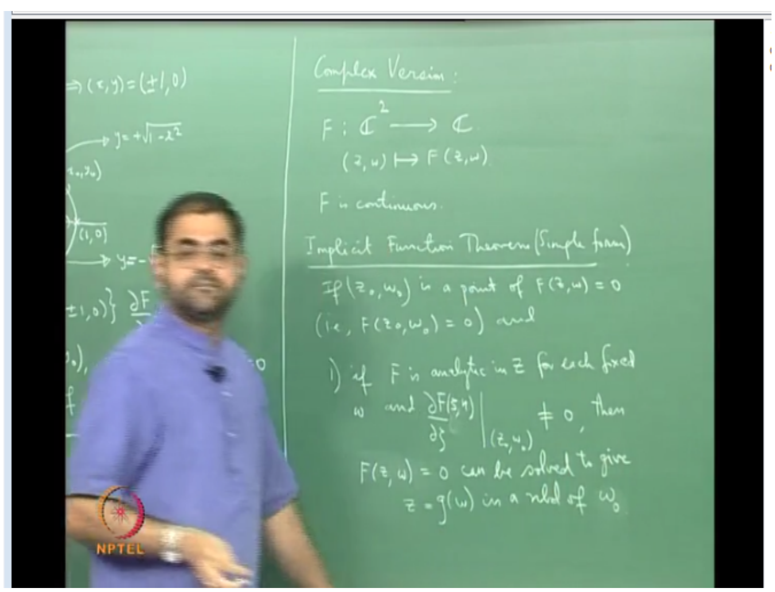

So, what I am going to do complex version so, what I am going to do is the I am going to take a function f from c2 to c written as z,w going to capital F of z,w okay. Here is my function and of course you know when I when we talk about the Implicit function theorem the function that we had dealing with has to have some obviously good properties.

For example I **I** want at least that the function is continuous in both variables and you know it is partial derivative with respect to each variable exist. And the partial derivatives are also continuous these are obvious assumptions that one makes maybe some of these assumptions can be relaxed little but this no harm in assuming that the function that you are dealing with is at least continuous in both variables.

And the first partial derivatives are continuous so, what I am going to do is at least in the complex case let me be very strict and tell you what the exact conditions are I want f to f is continuous. So, I should take at this point give you a word of caution when I say f is continuous I mean f is continuous is a map okay. And that means continuity with respect to both variables okay.

And I want you to understand that continuity with respect to both variables implies continuity with respect to each variables separately. But continuity with respect to each variables separately is not as strong enough as continuity with respect to both variables. So, this is so, this is stronger than saying that for every z fixed f of z,w is a continuous function of w and for every w fixed f of z,w is a continuous function of z okay.

That mean when I say that for example f of z,w is a continuous function of w for every fixed z that means I am saying in the variable w separately f is a continuous function. It is called separately continuous with respect to second variable similarly when I say f of z,w is continuous with respect to z for a fixed w. I am saying it the function f is separately continuous with respect to the first variable okay.

These are very very weak conditions when compared to this condition which is continuity in both variables okay. So, I am just this is what I am assume when I say f is continuous mind you it implies separate continuity it is implus continuity in each variable. But it is for for stronger than that, that is one thing the we can easily find functions which are you know continuous in separately in each variable.

But put together in two variables they will not be continuous okay. So, I am just trying to f as the point that this is not this is stronger than continuity in each variable alright. Now so, I assume this okay and then I also assume the following thing so, here is what the so, here is the Implicit function theorem so, let me just say simple form let me give you the form.

And then give you the more involve statement so, what is the simple form the simple form is well if z0,w0 is a point of f of z,w=0 okay that means is a solution of this that I sz0 f of z0,w0 is 0 okay. And if and number1 if F is analytic in z for each fixed w okay. So, if you fix the w then this function becomes an a function of z and with as a function of z and with as a function of z for a fixed w it assumes it is analytic okay.

That means I can partially differentiate this with respect to the variable z because it is analytic with respect to z okay. And the partial derivative of f with respect to the first variable okay so, you see I am let me write it very carefully f I write f as a function of zeta eta I differentiate partially with respect to the first variable okay. Then I substitute z0,w0 suppose the so, I am just saying that the partial derivative with respect to the first variable is non-zero at the point z0,w0 okay.

Then you see there exist a delta positive so, let me not say that then I will say then the first variable f of z,w equal to 0 can be solved to give z equal to g of w in a neighbourhood of z0 in a neighbourhood of in a neighbourhood of w0 okay. So, you see this is the so, of course this implies that you know f of gw,w,w0 okay. So, it is just same statement literally what is the general statement.

The statement the general idea of the Implicit function theorem is whenever you take a point on the locus in a satisfied by the equation. If whenever you the first partial derivative with respect to certain variable is not 0 then you can solve explicitly for that variable in a neighbourhood of the point correspond to the other variables okay. So, here z0,w0 is a point on this locus .

It is a point that satisfies f of z,w is 0 okay and the first partial derivative of f with respect to the first variable does not vanish at that point. Then I can solve for the first variable as a function of the second variable in a neighbourhood of th second variable point corresponding to the given point. So, the second variable point is w0 so, in a neighbourhood of w0 I can solve okay.

I can solve for z as a function of w in a neighbourhood of w0 provide the first partial derivative with respect to is not vanish. And to make sure that I can really differentiate this with respect to the first variable z with respect to the first variable I need and I want to do it in a and I want to do it not just at that point I want it do it in a neighbourhood of that point. So, I will have to assume that the function is analytic in the first variable at least in a neighbourhood of the point z0.

That is what I have assumed here okay I assuming that if you freeze the second variable then as a function of first variable it is analytic. That allows me to differentiate it with respect to the first variable oaky. So, the this is a first part of the statement then what is the second part of the statement the second part of the statement is a following second part of the statement is well .

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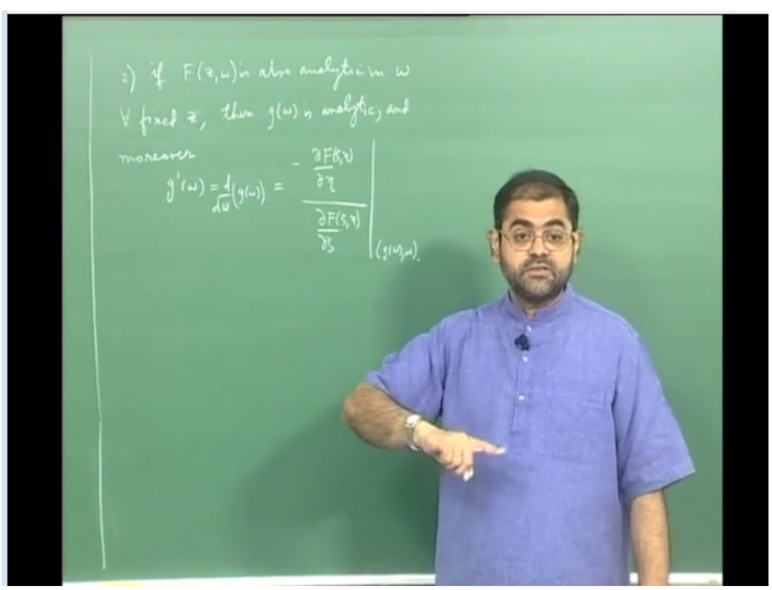

If it tells you more it tells you that so, you can ask see this function z equal to g of w is a function of w in a neighbourhood of w0 okay. You can ask when that is analytic after all let us also function of a complex variable you can ask the question when that is analytic okay and of course then you can ask a question as to what is the derivative of that so, this gives the remaining parts of the theorem gives you answers to these questions okay.

So, you know if so, if f of z,w is an also analytic in w for every fixed z okay which is a condition similar to this one okay in here we have assume taht the function f is analytic in the first variable for every fixed value of the second variable okay and now I am assuming the other way round I am assuming the function is also analytic with respect to the second variable for every fixed value of the first variable.

Then g of w is analytic g of w that I wrote that is explicit solution z equal to g of w which is explicit solution of f of z,w in a neighbourhood of w0. That g of w becomes an analytic and what is the so, if it is analytic you can ask me what is the derivative is there is as simple formula for the derivative and the answer is yes. And the derivative is given by the following formula.

And more over g dash of w okay with which by this I mean d/dz g of w sorry g/d d w g of w is given by the following formula it is – of you differentiate f partially with respect to the second variable and divide by that derivative of f with respect to the first variable and then evaluate this whole crazy thing at the point g of w, w okay. This is the derivative of the analytic function g with respect to w right mind you.

The numerator I have the derivative negative of the derivative with respect to the second variable. And the denominator I have the derivative with respect to the first variable and I am taking the quotient and then I am evaluating it at gw,w that gives me a function of w. And what is that function of w that function of w is the derivative of g of w with respect to w that is what the formula sets okay.

And mind you have you already assumed that the derivative with respect to the first variable is not 0 in a neighbourhood okay. So, what will happen is in that neighbourhood this will never you will get a suitable neighbourhood where this never vanishes and that is what justifies the dividing by this mind you dou f/dou zeta at g w0 which is z0,w0 is non-zero okay.

So, it will work in a neighbourhood of w0 okay so, the denominator will not vanish so, this division is it is division makes sense okay. And so, this is the formula for the derivative alright and fine so, what it so, if I put these two together it tells me that if you give me a complex function of two variables. Then if you want to look at this 0 locus of that function of two variables.

Then at a point on the 0 locus of the function of two variables if the first partial derivative does not vanish for which I need function with respect to the first variable to be analytic for every fixed value of the second variable. Then I can solve for the first variable in terms of the second variable as a function in a neighbourhood of the point corresponding to the second variable.

And further this solution which is the explicit solution this function is itself analytic and when does that happen that happens provided the function is also analytic as a function of the second variable for every fixed value of the first variable. And in that case you have formula for the derivative of this explicit function explicit solution of the Implicit equation. And that is given by this ratio of derivatives negative ratio of derivatives that is what it is okay.

And okay so, this is the kind of simple formula okay now what is more involve form more involve statement the more involve statement is the following right. It will tell you what this neighbourhoods are it will give you more information choose row greater than 0 such that for 0 less than mod z-z0 less than or equal to row.

F of z,w0 is not 0 okay such see such a row exist see F of z,w is a analytic function for in z for every fixed w. So, you know you put w equal to w0 so, F of z,w0 is an analytic function of z and it has a 0 at z0 because f of z0,w0 is 0. So, z0 is a 0 of this analytic function of z and you know the zeroes of an analytic function or isolated and mind you this analytic function is not identically 0 why it is not identically 0 is because that.

Because it is first partial derivative is non-zero in a neighbourhood of z0 okay. I mean it is not 0 at z0 and therefore it is not okay it cannot be 0 in a neighbourhood of z0 alright. So, it is certainly not the uh it is not the it is not an constant function which a non-constant analytic function of z. And you know non-constant analytic function has **iso** zeroes which are isolated.

So, the zeroes z0 of this function is isolated so, there is a small neighbourhood there is a deleted closed disc you know centred at z0 of positive radius on which there are no other zeroes so, that is how you get this row okay. We have already seen this several times in earlier lectures okay and then there exist a delta greater than 0 such that in fact g of w is given by 1/2piei integral over mod zeta-z0 equal to row.

The integral being taken with respect to the anti-clockwise sense of zeta d douf of zeta,eta zeta,w /dou zeta d zeta/F of zeta,w okay. So, this is the formula for g whenever w is at distance of distance less than delta from w0 okay. So, this is the more involve statement gives you a formula for the explicit function g which is the function of w okay.

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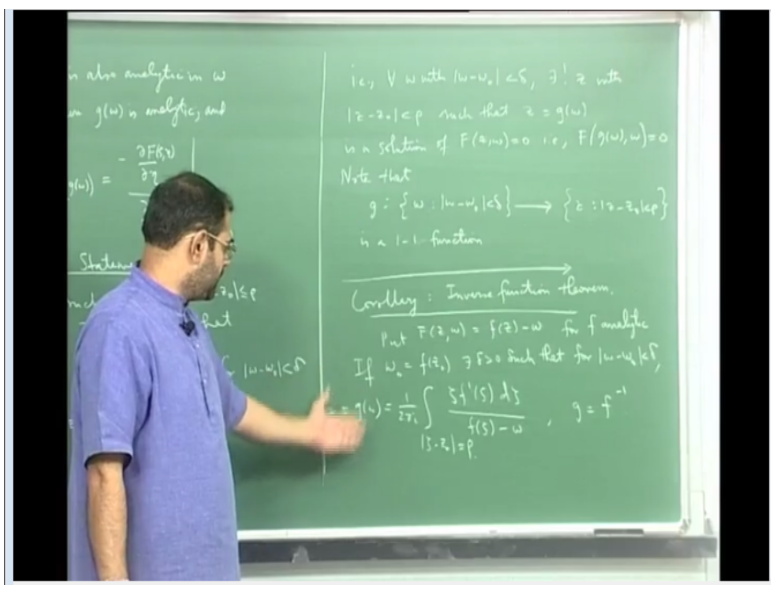

So, **so** what this tells you is that, that is for every w with mod w-w0 strictly less than delta there exist a unique z with mod z-z0 strictly less than row such that z equal to g of w is the solution of F of w equal to 0 that is F of g of w,w is 0. So, this also tells you note than note that g which is defined from set of all w such that mod w-w0 is less than delta to the set of all z such that mod z-z0 less than row is a one one function.

It is an injective function it takes z it takes w to gw which is z there exist there is a unique set okay so, you get a unique solution z equal to gw for the Implicit for the Implicit function F of z,w equal to 0. And I think I am more or less done that is right. So, **so** this is the extra information that you get this is the extra information that you get if you want the more involve statement okay.

This is a extra information that you get, now it is very easy to tell to give you a corollary every simple corollary a simple corollary of this is the inverse theorem okay. So, the corollary is inverse function theorem okay so, what you do is put f of z,w is equal to F of z-w for f analytic okay. You put F of z,w to be F of z-w where f is an analytic function of z okay.

Then F of z-w for every fixed w F of z-w is an analytic function of z and for every z fixed z f of z will become a constant and a constant-w is certainly an analytic function of w with derivative-1. So, what you get is you will get you will get the inverse function theorem so, in particular you will get that if w0 equal to F of z0 there exist delta greater than 0 such that for mod w-w0 less than delta.

You will have g of w z equal to g of w will be given by 1/2piei integral over mod zeta-z0 equal to row. And you know if I plug in here I will get zeta f dash of zeta d zeta/ f zeta-w which if you recall was the formula that we got for g is actually f inverse okay. So, this is the formula that we derived for the inverse function in the inverse theorem and we get this formula and we get this .

You get a formula for the inverse function theorem also a corollary of the implicit function theorem so, the Implicit function theorem is always a more stronger statement than the inverse function theorem okay and therefore the philosophy is that the fact that a injective analytic function is a biholomorphic map okay. if a function is injective and analytic there it is a holomorphic isomorphism is actually corollary of the you know the Implicit function theorem in some sense okay.

So, the Implicit function theorem is stronger than the inverse function theorem right so, what I will do is in the next lecture I will try to explain how to get the proof of the Implicit function theorem okay I will stop here.