Advanced Complex Analysis-Part1: Zeros of Analytic Functions, Analytic Continuation, Monodromy, Hyperbolix Geometry and the Riemann Mapping Theorem

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Lecture-01
Fundamental Theorems Connected with Zeros of Analytic Functions

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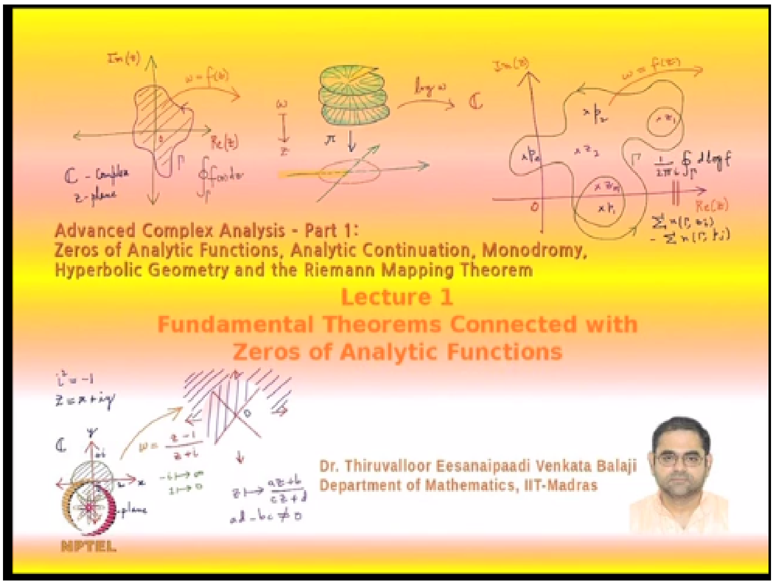

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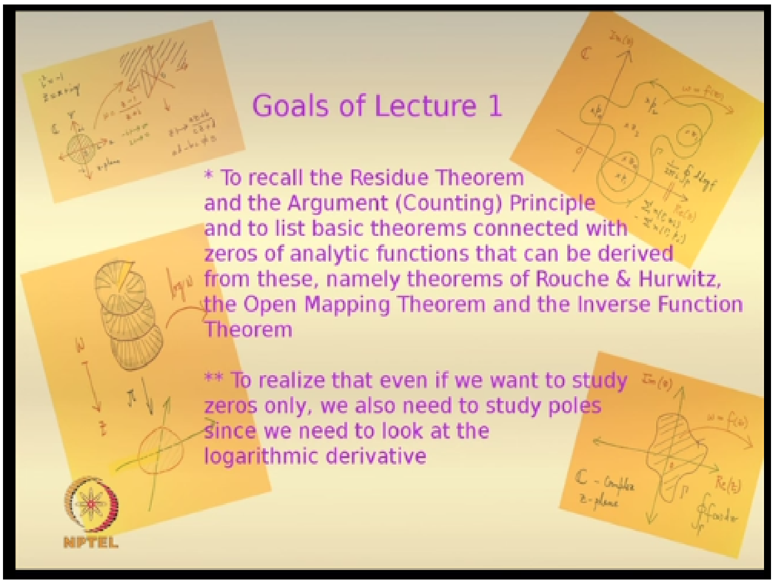

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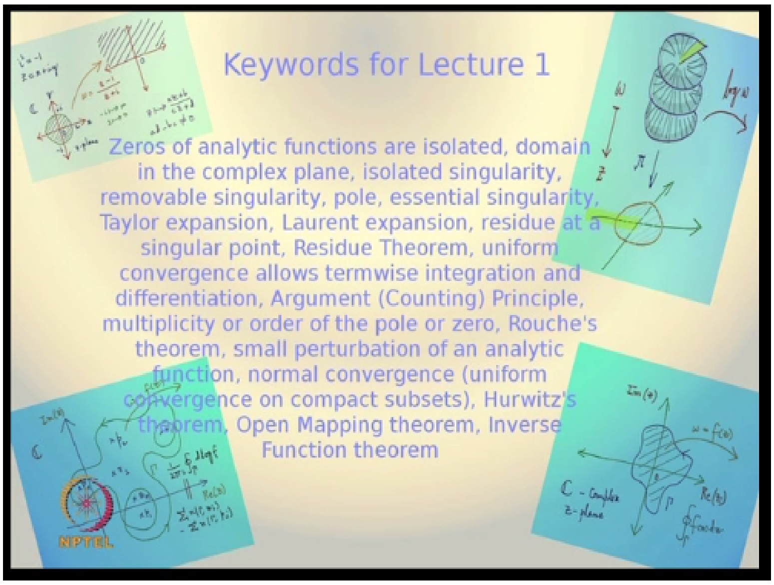

So welcome to this course on advance complex analysis. So what we intend to do is to give you a given the course selection of topics from advanced complex analysis. So of course we assume that you have already done a first course in complex analysis basically you know covering the motion of an analytic function and then Cauchy theorem and then the idea of Taylor series Laurent series, ideal similarity in this new theorem ok.

So of course we have chosen for the topics to be presented certain important theorems, certain landmark theorems which are you free not stressed upon in a first portion complex analysis and whose proofs are also not all that easy ok but they are very interesting theorems and we are of very geometric in nature and that is what we will try to cover. So so the of course so let me start with what will be doing in the first few lectures and that is about trying to look at zeros of analytic functions ok.

So basically you know so we are interested in zeros of analytic functions. So this is the broad topic for the first few lectures right and of course there are what I am going to do is state some important theorems connected with this theme ok, so of course let me first of all remind you I am when I see analytic function I think of a function which is defined on a domain in the complex plain, ok .

So the domain is open connect that set, ok so the fact that it is open means that the given every given a point in the set that there is a small disk surrounding that point which is contained in that set. So the fact that the set is open is being is the same as saying that the set is a union of disks ok and of course you know we always work with open sets because if you want to study the properties of function at a point.

Especially if you want to take a limit at a point then you should be able to approach that point from all directions and so you must have a nice disk surrounding that point where the function is defined so you can actually take the limit ok, so we always study only functions at points where in the neighbourhood of its functions defined ok, that is the reason why we always study a functions defined open sets.

And of course we are also study functions define and connectors sets because and if it is not connected then it falls into two pieces and essentially a function on such a set is a different function on each piece ok. So you can reduce the study of functions to just studying functions on single piece, ok that is why we always study a functions define an open connected sets which are do mix, alright.

Now so we take function defined on a domain in the complex plane and we of course assume the function takes complex service so again the co domain of a function is complex numbers and if you remember from the first course in complex analysis there are several ways of trying to define when the function is analytic at a point in the in the domain.

So of course the simplest definition is that of and it is the most common definition it is in the function should be differentiable at not only at that point but in a small disk surrounding the point ok and we also use the word holomorphic function instead of the word analytic function that is common in the literature and we have say function is holomorphic or analytic on the whole domain.

If it is analytic at every cardinality cannot hold me enough credit every point of a function being analytic at a point can also be described in several other ways 1 day is the way that I told you that it is differentiable enable the point that is the first derivative exists in about the point the other way of defining a function to be analytic at a point is saying that the function can be expressed as a convergence power series centre at that point in a small disk surrounding the point.

And this should and if this happen happens for every point then we could call that function an analytic function. So basically this one definition of analytic function which says that at the function is differentiable once in the neighbourhood in a small neighbourhood of the point. There is another definition that says that it is represented by convergent power series centre at that point.

And relationship between these two definitions is that they required and that is the great thing about complex valued functions because power series if you would have learnt in a first course in complex analysis is infinitely differentiable within its region of convergence the region of convergence of power series always is a disk centre at the point and then probably some points of boundary may or may not be included.

But between the disk the power series always represents an analytic function and not only this one's differentiable it is differentiable infinitely many times. So this is one of the striking features that differentiate the real valued differentiable functions in complex valued and differentiable functions ok. So if you are seeing a real valued function on a subset of real line see an interval open interval is differentiable throughout the interval.

There is no reason that other higher derivatives exist in that there is no reason even though the derivatives continuous where as if you assume a complex variable function of the complex variable on at a point is differentiable in neighbourhood to the point, the amazing thing is that it becomes infinity difference which means all the derivatives of all orders exist in the rock cutting.

This is the greatness and this is amount of power that one time differentiability gives you infinite differentiability in the neighbourhood ok and that is the characteristic main featuring certain analytic function and of course the power series if you take the portion of the power series they going to be related to the they are going to be related to the Taylor coefficient ok.

And this Taylor coefficient can be gotten by using the Cauchy integral formula ok. So you have this notion of an analytic function either you define that something that is locally given by a convention power series or something that is a function that is differentiable everywhere ok, differentiable one and of course the usual way of checking in a function is analytic is the is my checking the so-called Cauchy Riemann equations.

So what you do is that you check the you take the real and imaginary parts and functions this is how you write a check function is analytic you should you take the real and imaginary parts of function and then you check you take the real and imaginary parts of the function and this is how you try to check function analytics usually take the real and imaginary parts of the function and then you write the Cauchy Riemann equations.

And then you check the Cauchy Riemann questions are satisfied and then you also probably check that the first partial differential continuous and then you conclude the function analytic. Now so there is a way of checking a function is analytic using Cauchy Riemann equation as well but now there is the point is that we are interested in zeros of analytic functions.

And the first important all of you should I study in a first course in complex analysis has this is it the zeros of analytic function isolated ok so that is the first important factor that means given a 0 analytic function there is always a small disk surrounding that zero where there are no other zeros ok, so this is called so if you have zeros they can be separated from each other by small open disks centre at those disk and those zeros.

And this is and this is what we say this is what we say is what we need and you say that zeros are isolated, so zeros of analytic function are isolated ok . Now you see then of course the comes the questions when you what is the problem with looking at the zero of analytic function well you take but you take a function which is having zero at a certain point which is analytic at the point .

If you take a small neighbourhood you know there is a small neighbourhood when there is no other zero because zero is isolated, now if you invert the function in that disk when you know that the reciprocal of a function is defined except for the zero ok and then this gives rise to a whole at the point and this si one example of what is called singular ok. So analytic functions there are bad there are points on the boundary of the reason.

Why the function is analytic where the analytic functions of superstars similar, so this super similar and again you would have tried about singular points in the first close encounter on this analysis. So basically one is always one always worries about so-called isolators similarities because one does not want to be the case of non isolated similarities is far more complicated to analyse .

So for example if you look at the function lock Z then you know it has several branches you have define various branches of logarithm but to define a branch of the logarithm you will have to make a script in the on the plane for example how to split the plane along negative relaxes and then you can define a branch of the logarithm and then the whole negative realise this becomes points of similarities for this function.

So this tells you that this similarities are not isolated because they are continuously lie because they are continuously lie on the negative real axis but of course these are not the kind of similarities one always study is isolated similarities and these are isolated similarities basically of three types, if you recall the first one is called the removable similarities and remove the similarities essentially the is a losing is a non similarity ok.

For example of function like sign Z of Z there is 1/z if you look at Z=0 if you try to directly substance function will get 0/0 which is not defined value but of course no limited Z at 0 signs and Z is 1. So if you define the function to be at to take the value 1 Z=0 this gives rise to analytic function and the therefore the point Z=0 is what is called a removable z, and we could sign said by 1000 into sign Z.

And how this is reflected of course it is reflected by looking at the power series expansion if you power series expansion for sin Z divided by whole by Z you see that you essentially do not get any negative power itself, and that tells me the essentially these are Taylor not a (()) (13:06) and therefore this is short really a similarity and therefore and you will also see that if you take that power series for sig Z at the origin.

And divide by zero input Z=0 you get 1 and that will tell you that one should be the value that you should be fine for the function to become analytic at the point at the origin. So this is what is called an isolated remove this analytic, then of course dumb so called ports of a function and the poles are well they are supposed to be thought of a zero to the denominator ok, so I mean the simplest examples if you take if you given a point z not.

When you look at the function 1/Z-Z power of 10 where n is a positive integer and then you know if z not is zero of set of this function of the denominator this function which is z-z 0 power n, so 1/z-z to the power n has a poor again add z ok. So the pole is basically 0 in the denominator, so that is why you think of it and well so the pole is really similarity it is something that you cannot tinker with to make the function analytic at the point.

Ok and in a way the verse kind of similarity is called an essential similarity and that the similarity of in the for example you take e power you take exponential of 1/z at Z=0 that is an essential similarity and both poles and essential similarities are really the bonofied similarity or immovable similarities are actually non similarities because you can always get rid of them.

You can get rid of immovable similarities by read if n function at that point, but you cannot get rid of a pole or an essential similarity at a given point ok. And of course you would have also learnt how do you distinguish between a pole and an essential similarity and they there is a so-called Laurent theorem which is an analogue or you even called an extension of the Taylor theorem.

So the Taylor theorem is a theorem that if a function is analytic at a point then you can express it as a convergent power series around that point. That is you can find a convergence power series centre in that point, which point wise convergence to a given function in a good neighbourhood of the point. This is Taylor's theorem and this is the theorem that actually tells you that once differentiability implies infinite differentiability.

This is what you see the equivalent of the seemingly weaker definition of an electricity being once differentiable and the stronger definition and the stronger you know implication that once difference between price infinitely, infinitely many times, so that is the Taylor theorem. But the Laurent theorem is kind of expression the Taylor theorem, it tells you that if you also include negative powers.

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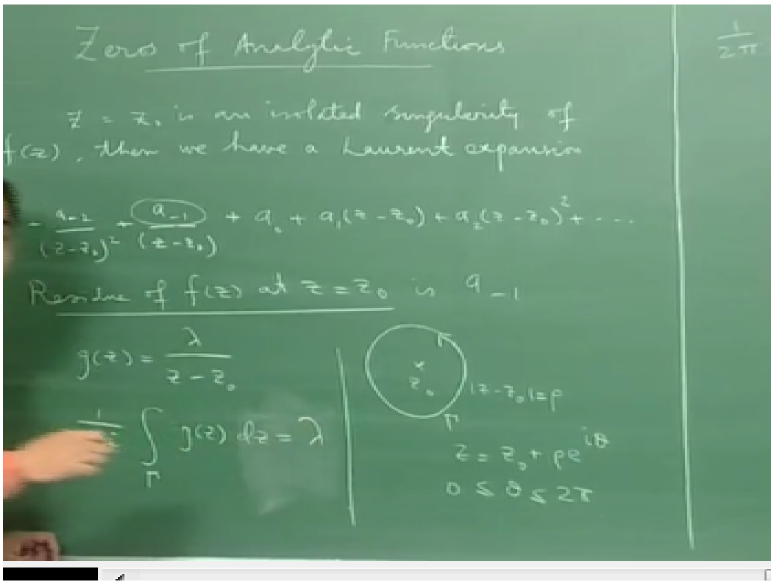

Then you can get a series involving also negative powers in a deleted neighbourhood of the point and that is called the Laurent series and for all you know the Laurent series may have negative powers of arbitrary order ok. So you know so if you look at if you if Z=z0 is and well isolated singularity of f of z of course I am assuming at this analytic function and z0 is a similar point is an isolated similarity .

Then we have not we have Laurent expansion, so well f(z)=let me write like this a0+a1 z-z0+a2z-z0 squared and so on. This is the this what is called the the analytic part for the Laurent expansion and then you get the negative powers you get a-1/z-z0 when you get a-2/, so this is a substitute -1 it is not a-1 ok. And this a subscript-2z-z0 squared and so on. So this is called a Laurent series centred exact not in the function converges to this.

This equality means that this Laurent series converges if you plug in a value of Z in a small disc surrounding z0 then this series converges to a value which is equal to the function value at the at that point and of course the point should not be z0 because you cannot substitute z0 here because you will be divided by 0 always negative terms. And the fact and the fact that if z=z0 is there the removable similarity.

Then all these negative questions will be 0, so Laurent expansion actually be a Taylor expansion that is exactly what happens when you look at 1/zxsinz for example if Z=0 ok and ok so this is so called Laurent expansion and the point is that among the important thing lattice namely the poles and the essential similarities you cannot distinguish the type of similarity by looking at the Laurent expansion.

If you get infinitely many of these negative powers ok, then the similarities is an isolated essential similarity. for example exponential of 1/z and z=0 and if you get only finitely many of these negative terms, then it is a pro ok and the order of the pole will be equal to the minus of the largest negative subscript you get here ok. a. So that is the another that is one way of you know trying to distinguish between a pole.

And an essential similarities is one more way of distinguishing between a pole and essential similarity and that is by taking limits ok if you take the limit of the function as a point tense to the similarity and if the limit exist at equal to infinity from all directions. So of course this means you have to make sense of what limit=infinity means I mean the limit of complex quantity you will say it is equal to infinity if the modulus of the quantity because (()) (20:19) ok.

No matter how you approach the limiting point, so you know if the similarity z0 is such that as zx to z0 in no matter in whatever direction the mode of F(z) goes infinity then you say the limit of F(z) and z tense is z0 infinity and this is a situation exactly when z0 of pole, ok and if there could be and what happens in the case a essential similarity the limit will not exist. In the sense that you might get limits differently limits as you approach different directions.

For example you can take exponential of 1/z and try to calculate the limit from by approaching the point z=0 from the positive axis from the negative axis the real axis positive real axis, we will see we will get different values the fact that you get different values from different directions tells you limit does not exist. And there is precisely is that condition that tells you that it is an essential similarity ok.

So the most important thing about similarities is what is called the residue of the infinity the residue of the function at at an isolated is at an isolated is at an isolated similarity and that is supposed to be the value of this this of course a-1 which is a coefficient of 1/z-z ok. So you know residue of f(z) at z=z0 is a-1 ok and this is the very very important value that the function .

Because it is connected with the residue theorem ok, which tells you that this is what you will this is what you get if you should try to integrate if you integrate the function over a curve surrounding over a closed curve that surrounds this point is a simple for example a circle a circle or arc circle that surrounds this point if you integrate the function what you will get is 2pieix 2pieix this point.

So for example you know can take very very example take g(z) to be 1/z-z0 and this is interesting you can think of ok and you here is Z0 on the complex plain and then if you want draw a circle look at the circle mod z-z0=rho is the circle and obviously this is an analytic function except for the 0.z=z0 everywhere else the denominator never vanishes z0 is z of order 1.

So it is a simple pole ok the pole of order 1 is called the simple pole and if it is order is greater than one is called multiple pole and when if you try to integrate if you calculate 1/2piei integral over if I call this circle as gamma f(gz) I will end up with I will end up with 1 ok. So in fact you know if I if you want I can even put lamda here where lambda is any complex number ok.

And if I integrate it what I will end up with is well if you so if you would have done this several times so if you want integrative over a contour the method is that you first parameterize that on to and then you make a change of variable mind you whenever you integrate something when you integrate the function over a contour you must understand that the variable lies on the counter ok.

So that means that you should write an equation for the contour and that is called parameterization of the contour. So the parameter of this contour z=z0+rho e to the i theta where theta varies from 0-2pie so this integral becomes well if you write it down fact already we have done it and let me just recall quickly you will just get lambda. So you will get 1/2pie i integral theta from 0-2pie.

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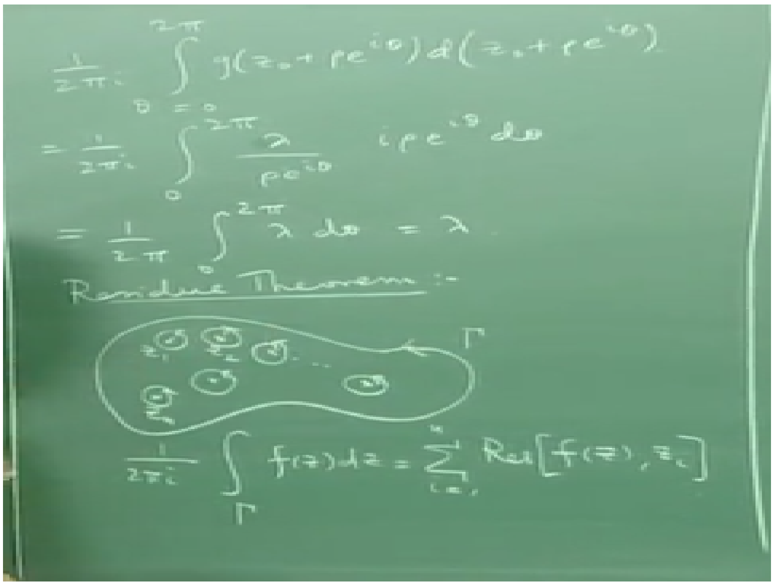

I am going to get g(z) z is z0 the rho e to the i theta of course I forgotten to write Dz there which is the variable of integration ok. So I will get D t z0+rhoe to the i theta and this, this will turn out to D1/2pie i integral 2pie what you will get here is 1/z-z0 to substitute this I will get rho e to the i theta and if I differentiate this I will get rho e to the i theta D theta ok.

And what will I end up with I will end up with my rho to the ei theta cancels my i cancels so I get 1/2pie i integral 0-2 pie I think I forgot lamda there, so that lamda on top so I get lamda t theta and that is just lamda ok. So the moral of the story is that you see if I look at this function and integrate it over the small over the small circle surrounding this point, this a simple pole.

I pick up this discussion and you know actually about this point if you try to write the Laurent expansion this is a Laurent expansion. The Laurent expansion for lambda/z-z0 is lamda/z-z0 ok. it is already the Laurent expansion and a-1 is the coefficient of 1/z-z0 and that is lambda and that is what shows us going to calculate 1/2piei and that is the residue of the function. So this is, this is a simplest illustration of the philosophy behind the residue theorem.

There is new theorem there is new theorem says that if you integrate over you know the point which is isolated similarity and assume that there are no other similarities then what you get is the residue and you will get again ½ pie so let me write that so to be more precise you have the residue theorem which is the residue theorem is a starting point fighting for our discussion.

So you know so basically you have let us assume that you have a nice contour like this and you have function F defined on this on this on this on a domain which contains contour and the interior of the contour and assume that you know there are there are well several isolated similarities z1, z2 and so on zn. If you integrate the function if you write 1/2 piei integral over this contour of Fz Dz what you will get is summation 1=1-n.

Residue of f(z) at zr this is the residue theorem. So what have done here is have taken I have simply taken the function to have only one similarity isolated similarity and in the I do this integration and what I end up this is the residue at that point, but if you have several and of course you should assume that there are no similarities on the contour over which you are integrating ok.

So you know the assumption for this is that the function is analytic in the interior and also on the boundary which means that to say the functions is analytic on the boundary actually means that is analytic in small disc surrounding every point on the boundary which means it is actually analytic in an bigger open space bigger domain which actually contains this boundary and the interior ok.

So then this is a so-called residue theorem and the simplest case it reduces this and you can also see that you know if you if you take this function and you instead of taking lamda by z-z0 suppose I took this power series I mean I have no power series if I take this Laurent series and I integrated ok around contour like this then of course the first thing is that the integral of this whole series is a same as integrating term by term.

That is you can integrate term by term and then take the resulting series that this is correct because you can interchange integration and summation provided the series of functions converges uniform ok and it is a theorem that if you take a Laurent series then we think in the reason why the Laurent series define if you take a close risk in that region then the Laurent series will convey uniformly ok.

And of course whenever I say Laurent series you should be deleted the neighbourhood you should not include the point of course because you cannot substitute the point because you will be dividing by zero for the negative terms, but but of course this is a similar theorem for power series you says that whenever you have power series we just converging in a in a disc.

Then you take any close disc inside that disc the power will converge there in fact absolutely and uniform ok. So because in his uniform convergences the integral if I calculate this integral for this function I can actually integrate term by term and you know if I integrate term by term from here onwards each terms will give you 0 because it is Cauchy's theorem.

Cauchy's theorem says that if you integrate analytic function over simple closed curve there is the I means the integral gone vanish cannot going to get anything ok and so the integral of all these terms will go away ok the integral of this term will give you a-1 of course ok and the integrals of all these terms was also go away because 1/z-z0 to the power of for example 1/z-z0 the whole squared has an anti derivative which is just -1/z-z0 to the power of 1 ok.

So all these negative terms from power 2 onwards they all have anti derivatives and it is a version of fundamental theorem of calculus that whenever a function has an anti derivative then the integrals just anti derivative evaluated at the final point-the anti derivative evaluated at the initial point, but then this the close curve the final point is same as initial point, therefore you get zero.

So if you integrate this term by term the only thing you will pick up way -1 and that is the proof of the residue theorem if you had a single similarity ok and if you have several similarities this follows because of Cauchy's theorem because what Cauchy's theorem will tells you is that Cauchy's theorem tells you basically that if you take analytic function and integrate it over simple closed curve the integral is 0 ok.

But this is so called simply connected version of Cauchy's theorem which is over simple closed curve where a which means the region inside the curve has no ports ok, but then there is a different version of Cauchy's theorem which says that you know if your domain is has an outer curve like this and you have inner curves you have inner curves and of course you know in all these issues the orientation of the curve is very important.

We always take the curves to be oriented in the anticlockwise and in that called the positive while intention ok and if you change orientation the sign of the integral will change that is how it goes, and find if you apply Cauchy's theorem to the region which is this the interior of this curve and the exterior of all these little curves that the functions of course analytic and therefore you will get that the integral is 0.

But that will amount said that integral over the the outer region the outer curve is the sum of the integral of the inner curves ok and but then some of each in but each integral will give you the residue at that point as I have explained here and therefore this si the residue theorem the that there is new theorem from I have to give some Cauchy's theorem and literally this kind of argument ok fine.

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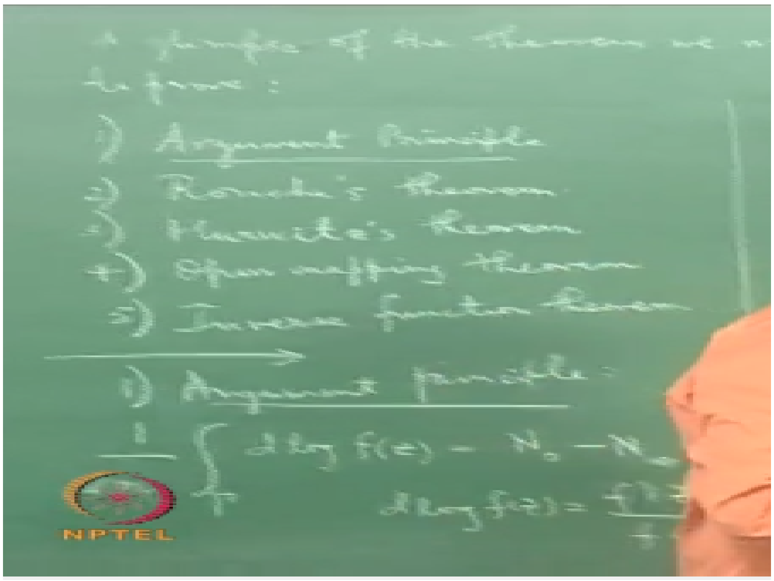

So so you have the residue theorem now the you see the so let me having told you so far let me also tell you what kind of theorems we going to prove ok I think I do not know how many lecture may take but probably a few lectures. So you see the kind of theorems we want to prove are actually theorem of both 0 of analytic functions and the so yeah so glimpse of the theorem you would you like to prove.

So probably some of you who have done a little bit of further reading beyond first course might have seen proofs of these theorems, but this is where I would like to start the course. So the first theorem is so argument principle ok and the this si so-called argument principle and the and what is the argument principle, so let me state the theorem that that I have want to prove.

One is the argument principle probably it is just a some kind of corollary to the residue theorem if you look at the logarithmic integral if you remember right from first course but anyway I will recall it. The second thing is using the argument principle or even otherwise you can prove the so called Rouche's theorem ok then we would like to study we would like to Horvitz's theorem .

And of course and thereafter one would like to study you know the open mapping theorem through open mapping theorem and of course also one would like to prove inverse function theorem ok, so the these are the I mean first set of theorem we would like to prove ok and probably you would have seen the first and second maybe, but anyways they are the starting point.

So I will make it up a point to recall them ok. So let me explain let me explain what these what these theorems are so the first one is is argument principle . So what did so what does so what does it say so I briefly describe what the statements of this theorems are and you will see that they are actually connected with I mean they are the right theorems that will come under the topic trying to study zeros of analytic principle ok.

So the argument principle so the argument principle is well you know 1/2pie i integral over a simple close curve so when a simple closed curve let me explain what that means. First of all the curve is set be close to it its initial point is same as terminal point ok.

Of course by a curve generally we mean the image of an introvert, a closed interval if you want the closed interval 0, 1 of the real line, a continuous image of that on the complex plane is called a curve. For example the circle the curve because it is the image of the interval 02pie under this function theta going to if you want z0 is to the i theta ok. So it is a continuous image of interval.

And it is the close curve if the initial point is same as final point. The fact that there is an initial point and there is a final point tells you that the curve is already oriented ok that means there is direction for curve and that is that direction is given by the direction of that that is given by the direction of increase of the parameter, the variable that is used to write the equation of the curve ok.

And when I say simple curve it means that the curve is does not need cross itself does not intersect itself. So it is not something like a figure 8 or more complicated curve that cross themselves once segment of the curve gives in turn and comes back and hits itself at some point again crosses itself ok there is no such self crossings, so such a curve is called simple curve and since we are going to do since we are going to do integration ok.

The curves that we were always deal with will be piecewise smooth, that means the curves if you are write down the parameters parameterization for the curve ok then the parameterization will always come will be defined over some inflow and the fact that you can divide this interval into sub intervals in each of which the function can you write down is actually differential.

It is differentiable and continuous ok, so this is what called piecewise smooth curve. So for example here the functions theta going to Z0+going to the i theta that theta lies from zero to 2pie is of course a continuously differentiable function of theta which is the parameter ok, but in more but more generally the curve need not be given by single parameterization in could just break down into several pieces.

And each piece may have a different parameterization ok, one piece maybe say part circle and another piece may be part of parabola, the third piece may be part of the line but it does not matter the point is piecewise it has to be smooth. So whenever I say simple it was counter when I see whenever I say contour it is always a something that is piecewise smooth ok.

So the argument principle basically tells you that if you are looking at a function which is function defined on domain like this ok with the property that this the function is analytic on this domain which contains full region ok excepted finitely many points which lying the interior which are only ports ok, you assume that they are only ports ok.

And the function should not have the function should not have any zeros on the boundary of the part ok, then ah ½ pie i integral over gamma of D log F(z) it is called the logarithm ok which is will give you the number of zeros-the number of the ports inside the inside the region, so this is the this is basically logarithm and of course D log F(z) of course means D log F(z) means F-F(z)/Fz.

What you must understand is that because F is analytic wherever F is analytic and F dash is also analytic because this I told you a function that is it is analytic infinitely different. So and this is a coefficient of analytic functions we be analytic that the only problem is the denominator might vanish, so wherever you have zeros of F if F-./F the logarithmic so far logarithmic derivative of F will have a port.

Wherever F has a 0 and of course if F has a pole then F dash have a pole ok, so the only poles for this function we assume the only poles for this function are some zeros inside zeros should not lie on the boundary and some poles of that inside and they also should not lie on the boundary, so the boundary should be free from both zeros and poles and there only finitely means zeros and poles inside inside the boundary ok.

And so this si the argument principle, so computing the log, so integrate the derivative, the argument principle tells you that you get the difference between the number of zeros and number of poles. So that is that is the so called argument principle ok and then let me quickly tell you about what these other theorems have to say. So so I am I am just now giving whole view of screen.

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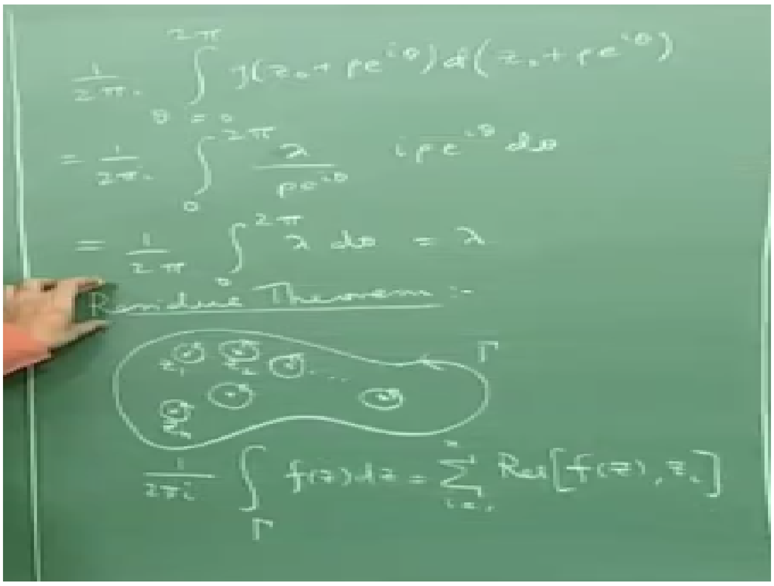

And then we will go into them more deeply so what is Rouche's theorem, so as I told you this whole exercises to somehow study zeros of analytic functions ok and basically for example you want to count the number of zeros in a insider inside a in a region which is bounded by a HCL curve a simple close curve. So the point is that the point is that you do not get in general a 0 F could be a well of course be a pole for F dash/F so you cannot avoid considering poles also ok.

So that is the reason the argument principle gives you zeros and poles, so in particular if there are not ports then you will be counting the number of zeros and of course when you count number of zeros mind every is zero has to be counted with multiplicity. For example if you take the function Lambda/z-z0 this has a zero of order 1 at z, so the number of zeros wull be one, you should take care a simple close curve enclosing z1.

If you now going to enclose z0 then because of zero ok, but if I replace it by lamda/z-z0 to the power of 10 then the number of zeros will be m, though physically there is only one zero of z0, but it is order is there. So it is also the order of vanish ok, it is the number of times the factor z-z0, z-0 appears. So whenever you say zeros of course you have to count them with multiplicity.

Zeros also have to be count them with multiplicity zero. For example if I take the function z-z0 to the power of 10, where m is possible then z0 is 0 of that function, but it is a should be counted as mos. So zeros and poles have to be counted with multiplicities and then this called of this, so this si the kind of counting ok that is one thing. Then Rouche's therorem is something more the philosophy of Rouche's theorem is that you know you take a you take analytic function in a domain.

And suppose you are interested in this zeros inside the region the domain that is you know that you get the interior of a simple closed curve ok. Then Rouche's theorem says that you know if you if you put up the analytic function little bit ok, even after perturbations the number of zeros will not change ok. That is if you add to the analytic function another analytic function which is small enough.

That means you add to the analytic function as smaller analytic function, of course you know there is nothing called smaller or bigger in complex numbers because complex numbers are not ordered, but then whenever we says smaller or bigger we always refer to the models. So you know what Rouche's theorem says is that you take a function F(z) this si analytic in a in say a in say a boundary region surrounded by simple close curve .

Then the number of zeros will be the same for F(z) and F(z)+gz where gz is smaller, smaller on the boundary ok, so that is Rouche's theorem and you think of adding g-f as a small perturbation ok. So I just write it in words the number of zeros of zeroes is not affected inside a simple close contour is invariant is invariant means it does not change, is invariant under small perturbation.

So you take the analytic function and add to it a small analytic function, function that is smaller than this function on the boundary, the boundary. Then ever after adding it the number of zeros not going to change ok. So that the addition of another analytic function which is dominated by the given analytic function on the boundaries si called perturbation if you want ok.

And it is a small perturbation because what you are adding in modular systems strictly less than the modular is given function on the boundary ok. So this is this is Rouche's theorem right, so one version of Rouche's theorem will tell you that suppose you want you have 2 analytic functions of F(g) ok, how can you conclude that they have the same number of zero ok, so that answer to that is you calculate.

You know if you if you cannot mod of F+g triangle will be inequality will always give you mod of F+g is less than or equal to mod F+modg ok. Now the question is on the boundary if you get stitched inequality if you get mod of F+g is strictly less than mod F+modg on the boundary, then both the F and G will have a same, that another that is another avatar of this Rouche's theorem.

So it tells you to come also to compare the it tells you the 2 analytic function will have the same on zeros if the sum of the modelling dou is strictly greater than dominates minutes the modulus of their sum on the boundary, that is another avatar of this ok, it helps to compare number of 0s. Then third one is Horvitz's theorem. So this Horvitz's theorem is again you know it is again a very beautiful theorem.

What it says is that you know if you have sequence of analytic functions which is convergence to a given function ok in a domain and assume that the convergent is going to be uniform on every close disk in the domain. So this is called uniform convergence on compact subsets okay, the other word that use in literature is called normal convergence ok. So if you have normal convergence ok which means uniform convergence of compact subsets.

For example the convergence is uniform on every close disc in your domain, so if this is if this is normal convergence and if F as 0 of order N at z0 then what happen is I draw a diagram so z0 is a point where the limit function F(z) has 0 ok, now some fundamental complex analysis will tell you that because of the convergence may form and since each of this function is already analytic.

F will also convert to analytic, ok that is again an exercise that you can easily try to do and in fact the derivatives of all these laws to convert derivatives ok and all this is just because of normal convergence uniform convergence on compact subsets and of course in the key in that case integrals derivatives will also come because the movement you have normal convergence integrals derivatives everything will behave with respect limits ok.

So so you suppose limit so the limit function will be analytic and if it has 0 of order n at the point z0 what happen is you can find a small enough disc surrounding z0 with small said that beyond a certain stage on the FMs they will all if you take each Fn it will have a N 0s inside this each Fn will have N0s with multiplicities which means that some of the zeros will multiple zeros.

And the beautiful thing is as you if you plot this M0s ok so I was so you know if you plot those Ms those capital N0 and if you make this small n become larger and large then these various zeros will slowly come they will go less they will all tend to the point z, ok so what it tell you is that when you take a nice limit of analytic functions then the 0 will limit comes from zeros is the limit of zeros of the same number of zeros of the functions in sequence ok.

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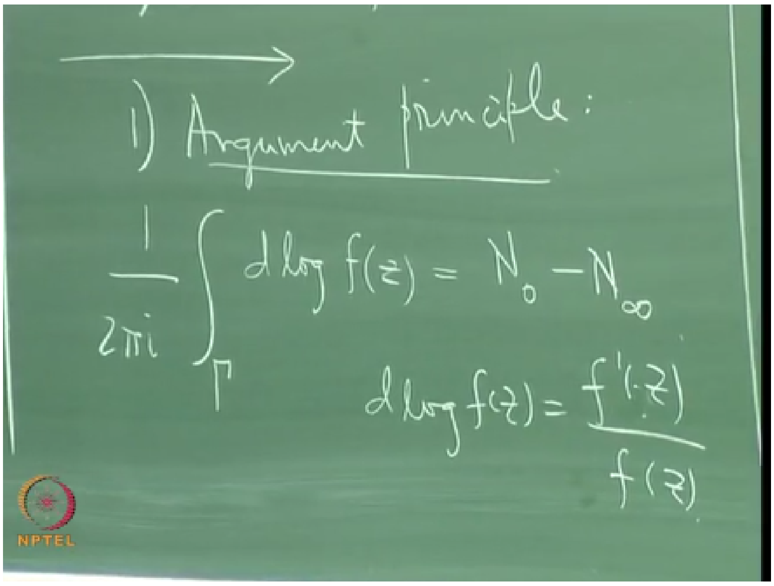

So if F has zero order of N z0 then you know then let me write it out somewhere here wow so then that exist row greater than 0 sets that for large N F n has N zeros in mod z-z0 less than rho which come which converts z0 as n to n. So what Horvitz's theorem says that the zero of the limit comes by you know you take zero of beyond a certain stage is zeros of the functions that giving the limit.

Zeros of the function that we are taking limit of is those 0s that slowly in together and give you the zero of your function ok. So this is this is Horvitz's theorem and then I quickly tell you what they open map in the inverse function theorem are the open mapping theorem is very beautiful theorem, it tells you that if you take an analytic function and you take a point where the function is non zero.

The the derivative of the function is not zero then there is a neighbourhood of the point where the function is open mapping which means the map open set so, and this is very deep result ok because it is very rarely interplays that you will get open maps ok and for example the objective continuous map will not be a (()) (54:56) open the map what he tells you that an open map along with infectivity will tell you the inverse map.

So you know you know what it tells you is that an open map is as good as an isomorphism accept that you need to know that this inject, you know is injective genesis compressibility conditions that you know. And all this is through analytic complex analytic also, so the only condition is that you know the derivative should vanish and then the neighbourhood of the point everything goes.

It is a localizing some of them ok, that is essentially the open map and let me quickly tell you about the inverse function the inverse function theorem is that again in a sense you can think of as another variant of open mapping theorem, what it says is that whenever the derivative is non 0 ok, at a point there is a small neighbourhood where can you invert ok and the inverse function can be written again using Cauchy's integral.

There is a integral form to writing more details, so you can invert there is an expressive formula ok. So so this is I am not I am not writing more details will go into them in the succeeding lecturers. So the point about these two theorems is that you must be essentially looking well there is more general version of open mapping will tell you that you do not even need the derivative to be non vanishing.

Essentially you need analytic function ok, so you need a non consonant analytic then it always map open set to open set and in particular see if the derivative is non zero then your neighbourhood of that point has a derivative of non 0 the function is actually holomorphic analytic isomorphism, which means is an isomorphism, it is injective on to it need which is open.

And if you take the inverse function that also holomorphic that is also analytic and that is given by form and end of the form, that what the inverse functions theorem ok. Now all these have somehow connect with zeros of analytic functions and they can all be derived starting from the argument principle which essentially is I should say the residue theorem applied not to F but it is residue theorem apply to the logarithmic derivative.

So the root for everything is there given, so we will do this entire fore coming exercise correct, so I will let me stop here.