Course Name: Essentials of Topology
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Welcome to Lecture 6 on Essentials of Topology. In the previous lectures, we have recalled the concepts associated with sets and functions. In this lecture, we will recall the notions associated with metric spaces. The concepts which will be covered in this lecture are metric spaces. We will discuss some of the examples. We will also discuss the concept of open ball and open set.

Begin with the motivation behind the concept of a metric space. The idea is inspired by the concept of distance. Accordingly, let us take a non-empty set X and use this symbol $d(x, y)$, which denotes the distance between two elements x and y of X. Let us review the concept of distance.

As we know, distances are always positive. The notation we have taken, it can be denoted by $d(x, y)$ is greater than zero. The second feature about distance is we know that two points are zero distance apart if and only if they are the same, i.e., we can write $d(x, x) = 0$. The third feature of the distance is the distance from x to y is the same as the distance from y to x, and in terms of our notation, we can write $d(x, y) = d(y, x)$. One more feature of distance is the distance from x to z via another element y is greater than or equal to the distance from x to z directly, that is, the distance from x to z is always less than or equal to the sum of the distance between x and y and the distance between y and z.

These are the familiar notions of distance that we all know. Having these ideas, let us define the concept of metric space formally. A metric space is a pair (X, d), where X is a non-empty set and $d : X \times X \to \mathbb{R}_{\geq 0}$ is a function, which is called metric such that the properties mentioned herein, that is, $d(x, x) = 0$ and $d(x, y)$ is greater than zero, when $x \neq y$, $d(x, y)$ is equal to $d(y, x)$ and $d(x, z)$ is less than or equal to $d(x, y) + d(y, z)$, for all $x, y, z \in X$. So, we can conclude that the features, which we have seen for distance can be given in terms of a function from an arbitrary set to a set of non-negative real numbers, and provide the notion of metric space.

This is one of the examples that if X is a non-empty set and $d(x, y)$ is given as 0 if $x = y$ and 1 if $x \neq y$, then it can be easily checked that (X, d) is a metric space. Even this one is clear from the definition of d. This is also simple from there, and the third one can be easily obtained; therefore, (X, d) is a metric space. If we want to see how this metric is defined, let us take a set X, and if we are taking two elements of this set, what is the distance from a point? These are some of the elements of this set. Then, whether we are finding out $d(x, y)$, or we are computing $d(x, z)$, or we are computing $d(x, w)$, all the distances will be 1 as per the definition of this metric.

Moving ahead, let us take another example. Let $X = \mathbb{R}$, the set of real numbers and $d(x, y) = |x - y|$. Then $(\mathbb{R}, d)$ is also a metric space. It can be seen that this $d(x, y)$ is nothing but the distance of two points on the real line.

Let us take some more examples. This example is on $\mathbb{R}^2$. Let x as ordered pair (x_1, x_2), y as ordered pair (y_1, y_2) in $\mathbb{R}^2$. Then, $d(x, y)$ is defined as the $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, is a metric on $\mathbb{R}^2$, known as Euclidean metric. If we want to visualize this d on $\mathbb{R}^2$, then this d is nothing but given here. This is $d(x, y)$, if x is this point on $\mathbb{R}^2$, and y is this point. Let us take another example on $\mathbb{R}^2$ again, $d_M(x, y)$, this is defined as:

$$max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

It can be easily seen that this is also a metric on $\mathbb{R}^2$, and this metric is known as max metric, that's why we are using this notation, d_M. If we want to visualize this metric on $\mathbb{R}^2$, let us take the first point, that is, (x_1, x_2), and this is another point (y_1, y_2) in $\mathbb{R}^2$. Then it can be seen that this distance is greater than this one, and therefore this is actually $d_M(x, y)$. Let us take one more example on $\mathbb{R}^2$ again. This is $d_T(x, y)$, which is equal to $|x_1 - y_1| + |x_2 - y_2|$. It can also be seen that this d_T is a metric on $\mathbb{R}^2$, and this is known as taxicab metric. This nomenclature is interesting; just go through the literature and think on it, that's why this is a taxicab metric.

If we want to visualize this metric on $\mathbb{R}^2$, let us take this as our point x, and this is y. Then this $d_T(x, y)$, as this is defined as the sum of these two, therefore this is the first one, that is, $|x_1 - y_1|$, and this will be $|x_2 - y_2|$, and their sum, that will be given by $d_T(x, y)$. Moving to the next concept, the concept of open ball, which plays a key role in the study of metric spaces. For a metric space (X, d), let us take x as an element of X alongwith $r > 0$, as a real number. The open ball of radius r centered at x is denoted as $B(x, r)$, which is a set $\{y \in X : d(x, y) < r\}$.

Let us take some examples of open balls. The first example, which we have seen, that was for a non-empty set X, we have taken a metric d that was $d(x, y)$ as 0, if $x = y$, and 1 if $x \neq y$. Let's take any r greater than 0 and less than or equal to 1, and we want to find an open ball centered at x with radius r, as per the definition. This is a collection of $y \in X$ such that $d(x, y)$ is less than r, or in our case, this is $y \in X$ and $d(x, y) < r$. Note that what r we are taking, r is less than or equal to 1 and greater than 0. So $d(x, y)$ will always be less than 1, or in the other sense, under this condition, this is a collection of $y \in X$ such that $d(x, y) = 0$, and that is nothing but a singleton set $\{x\}$. Similarly, if we are taking $r > 1$, and we are computing this open ball centered at x with radius r. This is $y \in X$ such that $d(x, y) < r$, as r is always greater than 1; we can say that this is nothing, but $y \in X$ such that $d(x, y)$ equals to either 1 or 0, or $d(x, y)$ equals to 0, and therefore this will be the set X.

Finally, we can conclude that in the case of this metric, an open ball centered at x and radius r is given as a singleton set $\{x\}$ if r lies between 0 to 1, and X otherwise. This metric is known as a discrete metric, which we have to use. Let us take another example, which we have seen that was on the set of real numbers with $d(x, y) = |x - y|$. If we want to find out the open ball here centered at x and radius r, this is a collection of $y \in \mathbb{R}$ as $X = \mathbb{R}$ such that $d(x, y) < r$, and this is a collection of $y \in \mathbb{R}$ such that $|x - y| < r$. This is a collection of $y \in \mathbb{R}$ such that $y \in (x - r, x + r)$, or, this is, precisely the open interval $(x - r, x + r)$.

So, this is the open ball in the case of this metric. Moving ahead, we have seen a metric defined as $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where (x_1, x_2), and (y_1, y_2) both were in $\mathbb{R}^2$. If we want to see the structure of the open ball in this case, let us fix the center at the origin and take the radius as 1. Let us compute an open ball centered at $(0, 0)$ and radius 1. This is given by ordered pair $(x, y) \in \mathbb{R}^2$ such that the distance between this (x, y), and origin, that is $(0, 0)$ is less than 1, or this is a collection of or $(x, y) \in \mathbb{R}^2$ such that $\sqrt{x^2 + y^2} < 1$, and this set is nothing but the points or the elements of this circle centered at origin and radius 1.

Let us take another example, which we have already seen that $d_M(x, y)$, defined like $max\{|x_1 - y_1|, |x_2 - y_2|\}$, where this $x = (x_1, x_2)$ and $y = (y_1, y_2)$, both are from $\mathbb{R}^2$. Again, if we are taking the center at the origin and let us take the radius as 1, and we want to visualize the nature of an open ball here, that can be given as this is ball centered at origin and radius 1, that will be collection of $(x, y) \in \mathbb{R}^2$ such that $d((x, y), (0, 0))$, this is less than 1, or if we simplify it, this is nothing but this is a collection of all ordered pair $(x, y) \in \mathbb{R}^2$ such that the maximum of $|x|$ and $|y|$ is less than 1. If we simplify this condition, this is precisely $|x| < 1$ or $|y| < 1$, or, in other sense, x lies in $(-1, 1)$ and y lies in $(-1, 1)$. Therefore, we can conclude that the interior of this square, that is, the points inside this open square form this set, that is, an open ball is given like this one.

Moving ahead, let us take one more example; we have also seen that $d_T(x, y)$ defined in this fashion is a metric on $\mathbb{R}^2$, where (x_1, x_2), and (y_1, y_2) are members of $\mathbb{R}^2$. If we want to find out the open ball, let us fix the center at the origin and again take the radius as 1. Then the open ball centered at origin and radius 1 is given by the collection of ordered pair $(x, y) \in \mathbb{R}^2$ such that $d_T(x, y)$ is less than 1, or this is a collection of $(x, y) \in \mathbb{R}^2$ such that this $|x| + |y| < 1$. If we simplify this condition, this is precisely $x + y < 1$, $x - y < 1$, $-x + y < 1$, and $-x - y < 1$. That is, this is representing the four inequalities: the first one is $x + y < 1$, the second is $x - y < 1$, $-x + y < 1$, and $-x - y < 1$, and if we are taking $x + y = 1$, that is, this line, similarly $x - y = 1$, $-x + y = 1$, and $-x - y = 1$. Finally, the points that satisfy these four conditions lie inside this diamond type of shape. So, the open ball with respect to the taxicab metric is given here.

By using the concept of an open ball, let us move to a new concept, which is known as an open set. So, for a given metric space (X, d) and a subset G of X, we say that this G is open if for all $x \in G$, there exists $r > 0$ such that the open ball centered at x and radius r is a subset of G, meaning is that, if this is the set X and let us take a subset G of X, so when we say that G is open if these are elements of G. If we can find some open ball centered at these points and which lies inside G, then we say that G is an open set.

Let us take one example, which is inspired from the notion of the open ball itself. This is precisely that, in a metric space (X, d), the open ball centered at x and radius r itself is an open set. If we want to justify that this is an open set, let us take an element y in this open ball. Our motive is to justify that there exists some $r' > 0$ such that $B(y, r')$ is a subset of an open ball centered at x and radius r. In order to achieve this one, first, we have to find out r', and second, we have to show that this is a subset of this one. So how do you find out this r'? Let us take $r' = r - d(x, y)$. Note that y belongs to an open ball centered at x and radius r. Therefore, this $d(x, y)$ is always less than r, and because of this, we can conclude that r' is always greater than 0.

See this picture, this is an open ball centered at x with a radius r. Now, if we are taking an element y of this ball, then the distance between x and y, that is, denoted by $d(x, y)$ here, we are taking this r', and this r' is nothing, but this is $r - d(x, y)$, so this is the r', which is given here. Now, in order to justify that the open ball centered at y is a subset of the open ball centered at x, let z belong to the open ball centered at y and radius r, then $d(y, z)$ is always less than r', try to find out $d(x, z)$, $d(x, z)$ will always be less than or equal to $d(x, y)$ plus $d(y, z)$, or this is less than $d(x, y) + d(y, z)$ is less than r', so let us write r' here, or this equals to $d(x, y) + r'$ is given here. So, this is nothing but $r - d(x, y)$, or which is precisely r, meaning is that we have got $d(x, z)$ is less than r, or z belongs to open ball centered at x and radius r. So, what finally, we have shown that every element of this open ball centered at y and radius r' is also an element of an open ball centered at x and radius r, and therefore, this $B(y, r')$ is a subset of $B(x, r)$, and hence this $B(x, r)$ is an open set.

Moving ahead, let us discuss some of the properties of an open set, so the first such property is an empty set, and X are open sets; this is an open set, and this is trivially true because an empty set doesn't contain any element, this X is open set, this is simple one because for all $x \in X$, this definition of open ball, whatever r we are taking, the definition tells that this is collection of $y \in X$ such that $d(x, y)$ is less than r, which itself is a subset of X. Therefore, X is an open set.

The next one, in a metric space (X, d), an arbitrary union of open sets is open. In order to justify this result, let us take a family of open sets. These are subsets of X, and this family is indexed by a set I. Our motive is to justify that the $\cup\{G_i : i \in I\}$ is an open set. Note that each of the G_i is open. So, in order to justify this, let us take an element x of this $\cup\{G_i : i \in I\}$, then $x \in G_i$, for some $i \in I$.

Note that G_i is an open set, so one can find out some real number greater than 0 such that this open ball centered at x and radius r is a subset of G_i, and also if G_i is a subset of $\cup\{G_i : i \in I\}$. Therefore we found an $r > 0$ such that $B(x, r)$ is a subset of $\cup\{G_i : i \in I\}$, and hence this $\cup\{G_i : i \in I\}$ is an open set. Finally, let us see one more result that in a metric space (X, d), the finite intersection of open sets is also an open set. Let us take sets G_1, G_2, up to G_n, these are subsets of X, and these all are open sets. Our motive is to justify that their intersection, that is, $G_1 \cap G_2 ... \cap G_n$, this is also an open set.

It is to be noted here that if this intersection is an empty set, there is no need to prove that the empty set is always an open set. So, we are assuming that this intersection is non-empty, and if this intersection is non-empty, let us take an element x in this intersection; that is, $x \in G_1 \cap G_2 ... \cap G_n$. Then $x \in G_i$, for all $i \geq 1$ and less than or equal to n. Now, note that every G_i is an open set; we can find that there exists a greater than 0 such that the open ball centered at x and radius r_i is a subset of G_i. This is true for all i greater than or equal to 1 and less than or equal to n.

Let us construct r, which is the minimum of r_i, where i is greater than or equal to 1 and less than or equal to n. Then obviously, this r will greater than 0, and $B(X, r) \subseteq B(x, r_i)$, and which is a subset of G_i, for all $1 \leq i \leq n$, or in other sense, or we can conclude that $B(x, r)$ is subset of $G_1 \cap G_2 ... \cap G_n$, because $B(x, r)$, that is, the open ball centered at x and radius r is a subset of every G_i, where $1 \leq i \leq n$. Therefore, $G_1 \cap G_2 ... \cap G_n$, is an open set. So, what we have seen, we have seen three interesting properties in a metric space that empty set and X are open sets, arbitrary union of open sets is an open set, and finite intersection of open sets is also open set. By using these properties, in the next lecture, we will begin with the concept of topology.

These are the references.

That's all from today's lecture. Thank you.