Adavnced Engineering Mathematics
Lecture 60

Poles, Residues

Regular point and Singularities: A point $z = a$ is called a regular point for a complex-valued function f if f is analytic at a. A point $z = b$ is called a singular point or a singularity if f is not analytic at b but every neighborhood of b contains at least one point at which f is analytic.

A singular point b is said to be an isolated singular point if f is analytic in some deleted neighborhood of b. Otherwise, b is non-isolated singular point.

(i) Removable singularity: An isolated singularity at $z = a$ of f is said to be a removable singularity if f if $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$.

Example. $f(z) = \sin z$, $z \neq 1$. Then $z = 1$ is removable singularity and we define the function as

$$f(z) = \begin{cases} \sin z, & z \neq 1 \\ \sin 1, & z = 1. \end{cases}$$

(ii) Pole: An isolated singularity $z = a$ of f is called a pole if $\lim_{z\to a} f(z) = \infty$.

(iii) Essential singularity: An isolated singularity $z = a$ of f is called an essential singularity for f if $\lim_{z\to a} f(z)$ does not exist in $\mathbb{C}$.

Theorem 1. *If f is analytic for $0 < |z - z_0| < R$ and has a pole of order m at z_0, then*

$$\begin{aligned} Res\,(f|z_0) = Res\,(f : z_0) =& \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z - z_0)^m f(z)\Big|_{z=z_0} \\ =& \frac{1}{(m-1)!}\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}(z - z_0)^m f(z)\right]. \end{aligned}$$

Theorem 2 (Cauchy Residue Theorem)**.** *If f is analytic on and inside a simple closed curve C except at finitely many singular points $z_1, z_2, \ldots, z_n$ then*

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} Res\,(f : z_k).$$

Example. Suppose $f(z) = \frac{\sin z}{(z^2-1)^2}$. Determine the order of the pole.

Solution: Here $f(z) = \frac{\sin z}{(z-1)^2(z+1)^2}$. Clearly, $z = 1$ and $z = -1$ are the poles of the function f.

Let us consider $z = 1$:

$$f(z) = \frac{g(z)}{(z-1)^2},$$

where $g(z) = \frac{\sin z}{(z+1)^2}$. Since g is analytic at 1 amd $g(1) = \frac{\sin 1}{4} \neq 0$. We can conclude that $z = 1$ is a pole of order 2. Similarly, we can show that $z = -1$ is a pole of order 2.

Example. Determine the residue at $z_0 = 1$ of $f(z) = \frac{\sin z}{(z^2-1)^2}$ and $\int_C f(z)\,dz$, where $C = \{|z - 1| = \frac{1}{2}\}$ is the circle radius $\frac{1}{2}$ and center at $(1, 0)$ oriented counter clockwise.

Solution: $z_0 = 1$ is a pole of order 2. We can also write $(z-1)\, f(z) = g(z) = \frac{\sin z}{(z+1)^2}$. Clearly, g is analytic at $z = 1$ with $g(1) \neq 0$. Then the residue of f at $z = 1$,

$$\begin{aligned} Res\,(f:1) =& \frac{1}{(2-1)!} \lim_{z\to 1} \left[\frac{d}{dz}(z-1)^2 f(z)\right] \\ =& \lim_{z\to 1} \frac{d}{dz}\left[\frac{\sin z}{(z+1)^2}\right] \\ =& \lim_{z\to 1} \frac{(z+1)^2\cos z - 2(z+1)\sin z}{(z+1)^4} = \frac{\cos 1 - \sin 1}{4}. \end{aligned}$$

Then $\displaystyle\int_C f(z)\, dz = 2\pi i Res\,(f:1) = \frac{\pi i}{2}(\cos 1 - \sin 1).$