Advanced Engineering Mathematics
Lecture 6

1 Functions of two or more than one variable

In a 2-dimensional space a point is represented by an ordered pair (x, y) and the set of all such points constitutes a 2-dimensional Euclidean space $\mathbb{R}^2$. For $\mathbb{R}^n$, a point will be of form $(x_1, x_2, ..., x_n)$. If in 2D space the variables x and y are independent such that a variable $z = f(x, y)$ is a function of x and y. Here z is dependent variable. The subset D of $\mathbb{R}^2$ is the set of points for which the function $f(x, y)$ is defined, then D is called domain of f.

Example 1.1. $z = \frac{xy}{x^2-y^2}$. Here z is not defined for $x = y$.
$D = \{(x, y) : x \neq y\} = \{(x, y) \in \mathbb{R}^2 : x \neq y\} = \{x \in \mathbb{R}, y \in \mathbb{R} : x \neq y\}$

Example 1.2. $z = \sqrt{1 - x - y}$. Here $D = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1\}$

Limit of $f(x, y)$. Let D be a subset of $\mathbb{R}^2$ which is domain of definition of a function f. Let (a, b) be a point in D. Let $(x, y) \in D$ and $(x, y) \to (a, b)$ in any manner. f is said to tend to a limit A if $\forall \varepsilon > 0\ \exists\ \delta > 0$ such that $|f(x, y) - A| < \varepsilon$, whenever $(x, y) \in$ a nbd of the point (a, b). We may choose the nbd as

$$N_{(a,b)} = \{(x, y) \in \mathbb{R}^2 : 0 < |x - a| < \delta, 0 < |y - b| < \delta\} \quad \text{or,}$$
$$N_{(a,b)} = \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < \delta^2\}$$

Then the limit, $\lim\limits_{(x,y)\to(a,b)} f(x, y) = A$ is called double limit or simultaneous limit.

Iterated or Repeated Limit. Let N be an nbd of (a, b) and $f(f(x, y))$ is defined on N. For a fixed value of y, $\lim\limits_{x\to a} f(x, y)$, if exists, will involve y. Therefore, $\lim\limits_{x\to a} f(x, y)$, will be different for different values of y. That is $\lim\limits_{x\to a} f(x, y) = \phi(y)$. Again if $\lim\limits_{y\to b} \phi(y)$, exists and equal to A, then we write $\lim\limits_{y\to b} \lim\limits_{x\to a} f(x, y) = A$.

Example 1.3. Find the limit $\lim\limits_{x\to a} f(x, y)$, where $f(x, y) = xy\frac{x^2-y^2}{x^2+y^2}$.

Sol. Let us take $x = r\cos\theta, y = r\sin\theta$. Then $x^2 + y^2 = r^2$ and as $(x, y) \to (0, 0)$, $r \to 0$ $f(x, y) = r^2 \sin\theta \cos\theta \cos 2\theta$,

$$|xy\frac{x^2 - y^2}{x^2 + y^2}| = |\frac{r^2}{4}\sin 4\theta| \leq \frac{r^2}{4} < \varepsilon, \quad \text{if} \quad r^2 < 4\varepsilon \quad \text{or} \quad 0 < x^2 + y^2 < \delta^2.$$

So, $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$ (using definition).

Example 1.4. Let, $f(x, y) = \begin{cases} x\sin\frac{1}{y} + \frac{x^2-y^2}{x^2+y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$
Show that $\lim\limits_{y\to 0} \lim\limits_{x\to 0} f(x, y)$ exists but neither of $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ nor $\lim\limits_{x\to 0} \lim\limits_{y\to 0} f(x, y)$ exists.