Advanced Engineering Mathematics
Lecture 55

Example. Verify Stokes theorem for $\vec{F}(x,y) = (x^2+y^2)\,\hat{i} - 2xy\,\hat{j}$ taken around the rectangle bounded by $x = \pm a$, $y = 0$, $y = b$.

Solution:

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,ds\\ &= \int_{x=-a}^{a}\int_{y=0}^{b}\big(-4y\hat{k}\big)\cdot\hat{k}\,ds = -4ab^2.\end{aligned}$$

Example. By converting the surface integral into a volume integral, evaluate

$$\iint_S (x^3\,dy\,dz + y^3\,dz\,dx + z^3\,dx\,dy),$$

where S is the surface of $x^2+y^2+z^2 = 1$.

Solution: By divergence theorem,

$$\iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iiint_V \Big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\Big)\,dx\,dy\,dz.$$

Clearly, $\vec{F}(x,y,z) = x^3\,\hat{i} + y^3\,\hat{j} + z^3\,\hat{k}$, then $\frac{\partial F_1}{\partial x} = 3x^2$, $\frac{\partial F_2}{\partial y} = 3y^2$, and $\frac{\partial F_3}{\partial z} = 3z^2$.

$$I = 3\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz.$$

V is the closed unit sphere $x^2+y^2+z^2 \le 1$. Consider the parametric form of the region by

$$x = r\cos\theta\cos\phi, \qquad y = r\cos\theta\sin\phi, \qquad z = r\sin\theta,$$

where $0 \le r \le 1$, $0 \le \theta \le \pi$, and $0 \le \phi \le 2\pi$. By the change of variable, we have

$$\begin{aligned}I =& 3\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2\cdot r^2\sin\theta\,dr\,d\theta\,d\phi\\ =& 3\int_{r=0}^{1} r^4\,dr\int_{\theta=0}^{\pi}\sin\theta\,d\theta\int_{\phi=0}^{2\pi}d\phi = \frac{12\pi}{5}.\end{aligned}$$

Example. Evaluate

$$\iint_S (x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy),$$

where S is the closed surface bounded by the plane $z = 0$, $z = b$, and the circle $x^2+y^2 = a^2$.

Solution: Using Gauss divergence theorem, we have

$$\begin{aligned}I_S =& \iiint_V (3x^2+x^2+x^2)\,dx\,dy\,dz\\ =& 5\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=0}^{b} x^2\,dx\,dy\,dz = \frac{5}{4}\pi a^4 b.\end{aligned}$$

Example. Evaluate

$$\iint_S (x^2\,dy\,dz + y^2\,dz\,dx + 2z(xy - x - y)\,dx\,dy),$$

where S is the surface of the cube $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$.

Solution: Using Gauss divergence theorem, we have

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} xy\,dx\,dy\,dz = \frac{1}{2}.$$

Example. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ over the entire surface of the region above xy-plane bounded by the curve $z^2 = x^2 + y^2$ and the plane $z = 4$ if $\vec{F}(x,y,z) = 4xz\,\hat{\imath} + xyz^2\,\hat{\jmath} + 3z\,\hat{k}$.

Solution: Using Gauss divergence theorem, we have

$$\begin{aligned}
\iint_S \vec{F}\cdot\hat{n}\,ds &= \iiint_V (4z + xz^2 + 3)\,dx\,dy\,dz \\
&= \int_{z=0}^{4}\int_{y=-z}^{z}\int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (4z + xz^2 + 3)\,dx\,dy\,dz \\
&= 2\int_{z=0}^{4}\int_{y=-z}^{z} (4z+3)\sqrt{z^2-y^2}\,dy\,dz \\
&= \pi\int_{z=0}^{4} (4z+3)z^2\,dz = 320.
\end{aligned}$$

Example. Evaluate by Green's theorem $\int_C \big[(x^2 - \cosh y)\,dx + (y + \sin x)\,dy\big]$, where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,1)$, and $(0,1)$.

Solution: With the comparison with Green's theorem, we have

$$\begin{aligned}
M &= x^2 - \cosh y, & \tfrac{\partial M}{\partial y} &= -\sinh y, \\
N &= y + \sin x, & \tfrac{\partial N}{\partial x} &= \cos x.
\end{aligned}$$

Therefore, by Green's theorem

$$\begin{aligned}
\oint_C \big(M\,dx + N\,dy\big) &= \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\,dx\,dy \\
&= \int_{x=0}^{\pi}\int_{y=0}^{1} (\cos x + \sinh y)\,dx\,dy \\
&= \pi(\cosh 1 - 1).
\end{aligned}$$