Advanced Engineering Mathematics
Lecture 54

Green's Theorem: Let R be a closed bounded region in xy-plane whose boundary C consists of finitely many smooth curves. Let M and N be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in R. Then,

$$\iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\, dx\, dy = \oint_C \big(M\, dx + N\, dy\big),$$

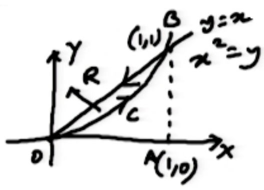
where the line integral being taken along the entire boundary C of R such that R is on the left side as one advances in the direction of integration.

Example. Verify Green's theorem in the plane for $\oint_C \big((xy+y^2)\, dx + x^2\, dy\big)$, where C is the closed curve of the region bounded by $y = x^2$ and $y = x$.

Solution: Given that $M(x,y) = xy + y^2$ and $N(x,y) = x^2$. To verify Green's theorem, we need to show

$$\iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\, dx\, dy = \oint_C \big(M\, dx + N\, dy\big).$$

$\frac{\partial M}{\partial y} = x + 2y$, $\frac{\partial N}{\partial x} = 2x$. The curve intersect at $(0,0)$ and $(1,1)$. Therefore, the left hand side



$$\begin{aligned}\iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\, dx\, dy &= \int_{x=0}^{1}\int_{y=x}^{x^2} (x - 2y)\, dx\, dy \\ &= \int_{x=0}^{1} (x^4 - x^3)\, dx = -\frac{1}{20}.\end{aligned}$$

Right hand side implies

$$\begin{aligned}\oint_C \big(M\, dx + N\, dy\big) &= \oint_C \big((xy+y^2)\, dx + x^2\, dy\big) \\ &= \int_{C_1} \big((xy+y^2)\, dx + x^2\, dy\big) + \int_{C_2} \big((xy+y^2)\, dx + x^2\, dy\big).\end{aligned}$$

Along $C_1 :\ x^2 = y$ implies $2x\, dx = dy$. Along $C_2 :\ x = y$ implies $dx = dy$.

$$\begin{aligned}\oint_C \big(M\, dx + N\, dy\big) &= \int_0^1 \big((x^3 + x^4)\, dx + x^2 \times 2x\, dx\big) + \int_0^1 \big((x^2 + x^2)\, dx + x^2\, dx\big) \\ &= \int_0^1 (x^4 + 3x^3 + 3x^2)\, dx = -\frac{1}{20}.\end{aligned}$$

Stoke's Theorem: Let S be a piecewise smooth open surface bounded by a piecewise simple closed curve C. Let $\vec{F}(x,y,z)$ be a continuous vector function which has continuous first order partial derivatives in the region of space which contains S in its interior. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\, ds = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\, ds,$$

where $\hat{n}$ is the outward unit normal on S.

Example. Find the value of $\oint_C \vec{r} \cdot d\vec{r}$, where C is any closed curve bounding a surface S.

Solution: $\oint_C \vec{r} \cdot d\vec{r} = \iint_S \left(\vec{\nabla} \times \vec{r}\right) \cdot \hat{n}\, ds = \iint_S \vec{0} \cdot \hat{n}\, ds = 0.$

Example. Verify Stoke's theorem for $\vec{F}(x, y, z) = (2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution: By Stoke's theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, ds.$

The boundary C of S is the circle in xy-plane of radius unity and center at origin. Let $x = \cos t$, $y = \sin t$, $z = 0$, $t \in [0, 2\pi]$.

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \oint_C \left[(2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2 z\,\hat{k}\right] \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right) \\
&= \int_C (2x - y)\, dx \\
&= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\, dt = \pi.
\end{aligned}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \hat{k}$$

$$\begin{aligned}
\iint_S \left(\vec{\nabla} \times \vec{F}\right) \cdot \hat{n}\, ds &= \iint_S \hat{k} \cdot \hat{k}\, ds \\
&= \iint_S ds = \pi.
\end{aligned}$$