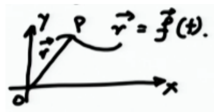
Advanced Engineering Mathematics
Lecture 50

Application of vector calculus in Mechanics

Velocity: The velocity of a particle relative to a suitable frame of reference is the time rate of change of the position vector $\vec{r}$ of the particle relative to the given frame of reference.

Let $\vec{OP} = \vec{r}$. At any time interval Δt, the increment in $\vec{r}$ be $\Delta\vec{r}$, then $\frac{\Delta\vec{r}}{\Delta t}$ is the average velocity of P relative to 0 during the interval Δt. Therefore, the velocity of the particle P at time t is given by



$$\vec{v} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = \vec{r}_t.$$

Example 1. Let $\vec{r} = e^t\,\hat{i} + t^2\,\hat{j} + \sin t\,\hat{k}$ be the position of a particle at time t. Then velocity $\vec{v} = \frac{d\vec{r}}{dt} = e^t\,\hat{i} + 2t\,\hat{j} + \cos t\,\hat{k}$.

At time $t = 1$,

the velocity of the particle: $\left.\frac{d\vec{r}}{dt}\right|_{t=1} = e\,\hat{i} + 2\,\hat{j} + \cos 1\,\hat{k}$.

magnitude of the velocity: $|\vec{v}| = \sqrt{e^2 + 4 + \cos^2 t}$.

Acceleration: It is a time rate of change of velocity. If $\vec{OP} = \vec{r}$ is the position of a particle then acceleration of the particle at time t is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\Big(\frac{d\vec{r}}{dt}\Big) = \frac{d^2\vec{r}}{dt^2}.$$

Example 2. Let $\vec{r}(t) = (t^2 - 1)\,\hat{i} + \sin^2 t\,\hat{j} + t^3\,\hat{k}$ be the position of a particle at time t.

$$\text{Velocity: } \vec{v} = \dot{\vec{r}} = 2t\,\hat{i} + 2\sin t\cos t\,\hat{j} + 3t^2\,\hat{k}$$
$$\text{Acceleration: } \vec{a} = \ddot{\vec{r}} = 2\hat{i} + 2\cos 2t\,\hat{j} + 6t\,\hat{k}$$

Equation of motion for a particle

Momentum: By momentum $\vec{p}$ of a moving particle P at any time t, we mean the vector $m\vec{v}$, where m is the mass of the particle and $\vec{v}$ is its velocity, i.e.,

$$\vec{p} = m\vec{v}.$$

Moment of momentum: If $\vec{p}$ is the linear momentum of a particle P at any instant of time t, then $\vec{OP}\times\vec{p} = \vec{H}$ is called the moment of momentum, or angular momentum of the particle with respect to O, i.e.,

$$\vec{H} = \vec{OP}\times\vec{p} = \vec{r}\times m\frac{d\vec{r}}{dt} = m\Big(\vec{r}\times\frac{d\vec{r}}{dt}\Big).$$

Newton's 2nd Law: The time rate of change of linear momentum of a particle is proportional to the applied/imposed force and takes place in the direction in which the force acts.

$$\vec{F} \propto \frac{d\vec{p}}{dt}$$
$$\vec{F} = km\frac{d\vec{v}}{dt} = km\frac{d^2\vec{r}}{dt^2}$$
$$\vec{F} = m\vec{a}.$$

Example 3. (Motion under gravity) If a moving particle of mass m be subject to the action of gravity alone then the equation of motion of the particle is

$$m\frac{d^2\vec{r}}{dt^2} = -mg\hat{k},$$

where $\hat{k}$ is the unit vector drawn vertically upwards.

$$\begin{aligned}\frac{d^2\vec{r}}{dt^2} &= -\, g\hat{k}\\ \vec{r}(t) &= -\,\frac{g}{2}t^2\hat{k} + t\vec{e} + \vec{f},\end{aligned}$$

where $t = 0$, $\vec{v} = \vec{u_0}$, and $\vec{r} = 0$, then $\vec{e} = \vec{u_0}$, and $\vec{f} = \vec{0}$. Hence, $\vec{r}(t) = -\frac{1}{2}gt^2\hat{k} + \vec{u_0}t$. The locus of $\vec{r}$ is a plane curve determined by the vectors $\vec{u_0}$ and $\hat{k}$.

Kinetic energy: If a particle of a mass m moves with a velocity $\vec{v}$ then the kinetic energy is given by

$$T = \frac{1}{2}m|\vec{v}|^2.$$

Potential energy: Let a particle of mass m be placed at height h with respect given frame of reference and let g be the gravity. Then the potential energy of the particle is given by

$$V = mgh.$$