Advanced Engineering Mathematics
Lecture 45

Example 1. In what direction from the point $(1,1,-1)$ is the directional derivative of $f(x,y,z)=x^2-2y^2+4z^2$ a maximum? Also find the value of maximum directional derivative.

Solution: We are given that $f(x,y,z)=x^2-2y^2+4z^2$. $\vec{\nabla}f(x,y,z)=2x\,\hat{\imath}-4y\,\hat{\jmath}+8z\,\hat{k}$.

The directional derivative of f is maximum in the direction of $\vec{\nabla}f$. Since the point P is given by $(1,1,-1)$, therefore $\vec{\nabla}f(1,1,-1)=2\hat{\imath}-4\hat{\jmath}-8\hat{k}=\vec{a}$.

The maximum directional derivative is given by

$$\frac{df}{ds}=\vec{\nabla}f(1,1,-1)\cdot\hat{a}=\frac{|\vec{\nabla}f|^2}{\vec{\nabla}f}=|\vec{\nabla}f|=2\sqrt{21}.$$

Example 2. For the function $f(x,y)=\frac{y}{x^2+y^2}$. Find the value of the directional derivative making an angle 30° with the positive x-axis at the point $(0,1)$.

Solution: The directional derivative is given by

$$\begin{aligned}\vec{\nabla}(x,y)=&\frac{\partial f}{\partial x}\,\hat{\imath}+\frac{\partial f}{\partial y}\,\hat{\jmath}\\ =&-\frac{2xy}{(x^2+y^2)^2}\,\hat{\imath}+\frac{x^2-y^2}{x^2+y^2}\,\hat{\jmath}\\ \vec{\nabla}(0,1)=&-\hat{\jmath}.\end{aligned}$$

The unit vector which makes an angle 30° with the positive x-axis is given by $\cos 30^\circ\hat{\imath}+\sin 30^\circ\hat{\jmath}$. Hence the corresponding directional derivative is

$$\vec{\nabla}f(0,1)\cdot(\cos 30^\circ\hat{\imath}+\sin 30^\circ\hat{\jmath})=-\sin 30^\circ=-\frac{1}{2}.$$

Example 3. What is the greatest rate of increase of $u=xyz^2$ at $(1,0,3)$?

Solution: $\vec{\nabla}u(x,y,z)=yz^2\,\hat{\imath}+xz^2\,\hat{\jmath}+2xyz\,\hat{k}$. Hence $\vec{\nabla}u(1,0,3)=9\hat{\jmath}$.

The greatest rate of increase of u at $(1,0,3)$ = the maximum value of $\left.\frac{df}{ds}\right|_{(1,0,3)}=|\vec{\nabla}u(1,0,3)|=9$.

Example 4. Find the equation of the tangent plane and normal to the surface $2xz^2-3xy-4x=7$ at the point $(1,-1,2)$.

Solution: The given surface $f(x,y,z)=2xz^2-3xy-4x-7=0$. Then

$$\begin{aligned}\vec{\nabla}f(x,y,z)=&(2z^2-3y-4)\,\hat{\imath}-3x\,\hat{\jmath}+4xz\,\hat{k}\\ \vec{\nabla}f(1,-1,2)=&7\hat{\imath}-3\hat{\jmath}+8\hat{k}\end{aligned}$$

Here $7\hat{\imath}-3\hat{\jmath}+8\hat{k}$ is the vector along the normal to the surface at $(1,-1,2)$.

If $R=(X,Y,Z)$ is the position vector of any point in tangent plane at $(1,-1,2)$. Then the vector $\vec{R}-(\hat{\imath}-\hat{\jmath}+2\hat{k})$ is perpendicular to the vector $\vec{\nabla}f(1,-1,2)$. Therefore, the required equation of tangent is

$$\begin{gathered}\big((X-1)\hat{\imath}+(Y+1)\hat{\jmath}+(Z-2)\hat{k}\big)\cdot\vec{\nabla}f(1,-1,2)=0\\ 7(X-1)-3(Y+1)+8(Z-2)=0\\ 7X-3Y+8Z=20.\end{gathered}$$

The equation of the normal to the surface at the point $(1, -1, 2)$ is

$$\frac{X-1}{\frac{\partial f}{\partial x}} = \frac{Y+1}{\frac{\partial f}{\partial y}} = \frac{Z-2}{\frac{\partial f}{\partial z}}$$
$$\frac{X-1}{7} = \frac{Y+1}{-3} = \frac{Z-2}{8}.$$