Advanced Engineering Mathematics
Lecture 43

1 Partial Derivative

Suppose $\vec{r}$ is a vector function depending on more than one scalar variable. Let $\vec{r} = \vec{f}(x, y, z)$, i.e., $\vec{r}$ is a function of variable x, y and z. Thus partial derivative of $\vec{r}$ with respect to x is defined by

$$\frac{\partial \vec{r}}{\partial x} = \lim_{h \to 0} \frac{\vec{r}(x+h, y, z) - \vec{r}(x, y, z)}{h}.$$

Similarly, we can defined $\frac{\partial \vec{r}}{\partial y}, \frac{\partial \vec{r}}{\partial z}, \frac{\partial^2 \vec{r}}{\partial x \partial y}, \dots$.

Vector Differential Operator: The vector differential operator $\vec{\nabla}$ (nabla) and it is defined as

$$\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\,\hat{j} + \frac{\partial}{\partial z}\,\hat{k}.$$

Gradient of a scalar function: Let f be defined and differentiable at each point (x, y, z) in a certain region of space. Then the gradient of f is denoted by

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k}.$$

Example 1. Let the scalar function $f(x, y, x) = x^3yz^2$. Then the gradient of f is

$$\vec{\nabla} f(x, y.z) = 3x^2yz^2\,\hat{i} + x^3z^2\,\hat{j} + 2x^3yz\,\hat{k}$$
$$\vec{\nabla} f(1, 1, 2) = 12\hat{i} + 4\hat{j} + 4\hat{k}.$$

Properties: Let f and g be a multi-variable scalar function.

1. $\vec{\nabla}(f \pm g) = \vec{\nabla} f \pm \vec{\nabla} g$
2. $\vec{\nabla}(fg) = g\vec{\nabla} f + f\vec{\nabla} g$

Definition 1. Let $f(x, y, z)$ be a scalar field over a region $\mathcal{R}$. Then the points satisfying an equation of the type

$$f(x, y, z) = c$$

constitute a family of surface in 3-dimensional space is called *level surface.*

Lemma 1.1. *Let f be a scalar function. Then $\vec{\nabla} f$ is a vector normal to the surface $f(x, y, z) = c$, where c is a constant.*