Advanced Engineering Mathematics
Lecture 40

Equivalent Matrices and Elementary Matrices

An $n \times n$ matrix B is said to be equivalent to an $n \times n$ matrix A over the same field $\mathbb{F}$ if B can be obtained from A by a finite number of elementary rows and columns operations.

Definition. *For a given $m \times n$ matrix A, a row-reduced echelon matrix B can be obtained by applying on A a finite number of elementary row operation on A. On B, we can apply column operations to reduce it into a column reduced echelon matrix, say C. This C has the following properties:*

(a) No zero row is followed by a non-zero row.

(b) No zero column is followed by a non-zero column.

(c) The leading 1 in each non-zero row is the only non-zero element.

(d) The leading 1 in each non-zero column is the only non-zero element.

(e) The leading 1 in the kth row is the leading 1 in the kth column.

Thus, C has the form $C = \begin{bmatrix} I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix}$, where I is the identity matrix of order r, O_{oq} is the zero matrix of order $p \times q$. Then, C is called the Fully-Reduced-Normal-Form.

Example 1. Obtain the fully reduced normal form of the matrix A give by $A = \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{bmatrix}$.

Solution. Let us first apply the elementary row operations.

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_4-3R_1} \begin{bmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1-R_2,\ R4-R_2]{R_3-2R_2} \begin{bmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1+2R_3]{R_2-2R_3} \begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= B \rightarrow \text{row equivalent to } A$$

$$\xrightarrow[c_2-3c_1]{c_5-2c_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[c_{34}]{c_{23}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= C = \begin{bmatrix} I_3 & O_{3,2} \\ O_{1,3} & O_{1,2} \end{bmatrix} \rightarrow \text{fully reduced normal form.}$$

Example 2. Show that the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & -3 \end{bmatrix}$ is non-singular and express it as the product of elementary matrices.

Solution. We start the process and apply operation as follows:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 0 \\ 6 & 2 & -3 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 3 & 3 & 0 \\ 6 & 2 & -3 \end{bmatrix} \xrightarrow[R_3-6R_1]{R_2-3R_1} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 3 & -\frac{3}{2} \\ 0 & 2 & 0 \end{bmatrix}$$
$$\xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1-\frac{1}{2}R_3]{R_2+\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

Clearly, A is row-equivalent to I_3. Since, $|I_3| \neq 0$, this implies A is non-singular. In short,

$$\begin{aligned} I_3 &= \left[\left(R_2 + \frac{1}{2}R_3\right)\left(R_1 - \frac{1}{2}R_3\right)(R_3 - 2R_2)\left(\frac{1}{3}R_2\right)(R_2 - 3R_1)(R_3 - 6R_1)\left(\frac{1}{2}R_1\right)\right] A \\ &= \left[E_{23}\left(\frac{1}{2}\right) E_{13}\left(-\frac{1}{2}\right) E_{32}(-2)\, E_2\left(\frac{1}{3}\right) E_{21}(-3)\, E_{31}(-6)\, E_1\left(\frac{1}{2}\right)\right] A \\ &= \left[\left(R_2 + \frac{1}{2}R_3\right)\left(R_1 - \frac{1}{2}R_3\right)(R_3 - 2R_2)\left(\frac{1}{3}R_2\right)(R_2 - 3R_1)(R_3 - 6R_1)\left(\frac{1}{2}R_1\right)\right] A \end{aligned}$$

$$\begin{aligned} A &= \left[E_{23}\left(\frac{1}{2}\right) E_{13}\left(-\frac{1}{2}\right) E_{32}(-2)\, E_2\left(\frac{1}{3}\right) E_{21}(-3)\, E_{31}(-6)\, E_1\left(\frac{1}{2}\right)\right]^{-1} I_3 \\ &= \left\{E_1\left(\frac{1}{2}\right)\right\}^{-1} \{E_{31}(-6)\}^{-1} \{E_{21}(-3)\}^{-1} \left\{E_2\left(\frac{1}{3}\right)\right\}^{-1} \{E_{32}(-2)\}^{-1} \left\{E_{13}\left(-\frac{1}{2}\right)\right\}^{-1} \left\{E_{23}\left(\frac{1}{2}\right)\right\}^{-1} I \\ &= \left[E_1(2)\, E_{31}(6)\, E_{21}(3)\, E_2(3)\, E_{32}(2)\, E_{13}\left(\frac{1}{2}\right) E_{23}\left(-\frac{1}{2}\right)\right] I. \end{aligned}$$

Definition. *An $n \times n$ matrix obtained by applying a single elementary row operations on I_n is said to be an Elementary Matrix of order n. There are three types of elementary matrices.*

1. *The elementary matrix obtained by applying R_{ij} on I_n is denoted by E_{ij}.*
2. *The elementary matrix obtained by applying cR_{ij} on I_n is denoted by $E_i(c)$.*
3. *The elementary matrix obtained by applying $R_i + cR_j$ on I_n is denoted by $E_{ij}(c)$.*

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ E_2(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_{13}(c) = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$