Advanced Engineering Mathematics
Lecture 38

Example 2. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ are similar matrices.

Proposition 1. *Let $A = (a_{ij})_{n\times n}$ with eigenvalues $d_1, d_2, \ldots, d_n \in \mathbb{F}$, which are not necessarily distinct. Let $D = diag(d_1, d_2, \ldots, d_n)$. Then, for any matrix P, $AP = PD$ holds if and only if the ith column vector of P be an eigenvalue of A corresponding to d_i.*

Example 1. (continued...) Diagonalise the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$.

Solution. *Step 3.* For $\lambda = 4$, let $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ be the eigenvector.

$$
\begin{aligned}
& AX = \lambda X \\
\implies & \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
\implies & \begin{bmatrix} x_1 - 3x_2 + 3x_3 \\ 3x_1 - 5x_2 + 3x_3 \\ 6x_1 - 6x_2 + 4x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
\implies & \begin{bmatrix} -3x_1 - 3x_2 + 3x_3 \\ 3x_1 - 9x_2 + 3x_3 \\ 6x_1 - 6x_2 + 0x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}
$$

From here we get $x_1 = x_2$, $x_1 + x_2 = x_3$ and $x_1 + x_3 = 3x_2$. If $x_1 = c = x_2$, where $c \in \mathbb{F}$, then $x_3 = 2c$. Therefore, the required eigenvector corresponding to the eigenvalue 4 is $c \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $c \in \mathbb{F}$.

Step 4. The required matrix $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Now we need to check whether $\det(P)$ is 0 or not, i.e., whether the matrix P is non-singular or not. It is easy to find out that the determinant comes out to be a non-zero value.

Next, we check $D = P^{-1}AP$ holds or not, where $P = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. We use the Proposition 1 and check $DP = AP$

Example 3. Diagonalise $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

Solution. *Step 1.* Let $\lambda \in \mathbb{F}$ be the eigenvalue. Then

$$\begin{aligned}
&\det(A - \lambda I) = 0 \\
\implies &\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0 \\
\implies &\lambda = -1, 1, 2.
\end{aligned}$$

Step 2. The eigenvectors corresponding to $\lambda = 1, 2, -1$ are $c_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $c_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, respectively, where $c_1, c_2, c_3 \in \mathbb{F}$. Hence, the required matrix $P = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Finally, we consider

$$\begin{aligned}
AP &= \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 0 \\ 1 & 2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
&= PD, \quad \text{where } D = \operatorname{diag}(1, 2, -1).
\end{aligned}$$