Advanced Engineering Mathematics
Lecture 34

Matrix Representation of a Linear Map

Let V and W be two vector spaces, and $T : V \to W$ is a linear map. Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be the ordered basis in V and $(\beta_1, \beta_2, \ldots, \beta_n)$ be the ordered basis in W. Then, each $T(\alpha_i)$ can be written as the linear combination of β_i's, $\forall i$, i.e.,

$$\begin{aligned} y_1 = T(\alpha_1) &= a_{11}\beta_1 + a_{21}\beta_2 + \ldots + a_{m1}\beta_m \\ y_2 = T(\alpha_2) &= a_{12}\beta_1 + a_{22}\beta_2 + \ldots + a_{m2}\beta_m \\ &\vdots \\ y_n = T(\alpha_n) &= a_{1n}\beta_1 + a_{2n}\beta_2 + \ldots + a_{mn}\beta_m \end{aligned}$$

where a_{ij}, $\forall i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ can be uniquely determined. then, the corresponding matrix A is given by

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

We see that $y = A^t B$, where $B = (\beta_1, \beta_2, \ldots, \beta_m)$.

Example 1. A linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3)$. Find the matrix representation of T with respect to the ordered/standard basis in $\mathbb{R}^3$.

Solution. Lets us take the ordered basis in $\mathbb{R}^3$ as $\{(1,0,0), (0,1,0), (0,0,1)\}$. We now write:

$$\begin{aligned} T(\alpha_1) &= T(1,0,0) = (3,1) \\ T(\alpha_2) &= T(0,1,0) = (-2,-3) \\ T(\alpha_3) &= T(0,0,1) = (1,-2) \end{aligned}$$

The corresponding matrix representation of T will be: $A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{bmatrix}_{2\times 3}$.

Example 2. For the same linear transformation as the previous example, find the matrix representation of T with respect to the ordered basis $\{(1,1,0), (1,0,1), (0,1,1)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

Solution. We have: $T(1,1,0) = (1,-2)$, $T(1,0,1) = (4,-1), T(0,1,1) = (-1,-5)$. So, we can write:

$$\begin{aligned} T(1,1,0) &= (1,-2) = 1(1,0) - 2(0,1) \\ T(1,0,1) &= (4,-1) = 4(1,0) - 1(0,1) \\ T(0,1,1) &= (-1,-5) = -1(1,0) - 5(0,1) \end{aligned}$$

and hence, the matrix representation will be $A = \begin{bmatrix} 1 & 4 & -1 \\ -2 & -1 & -5 \end{bmatrix}_{2\times 3}$.

Example 3. Let the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, x_2 + 4x_3, x_1 - x_2 + 3x_3)$. Find the matrix representation of T with respect to the ordered basis in $\mathbb{R}^3$.

Solution. Choose the basis: $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and proceed as previous examples.

Example 4. The matrix representation of a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the ordered basis $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$ is given by $A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{bmatrix}$. Find the linear transformation of A.

Solution. We observe that:

$$\begin{aligned} T(0,1,1) &= 0(0,1,1) + 2(1,0,1) + 2(1,1,0) = (4,2,2) \\ T(1,0,1) &= 3(0,1,1) + 3(1,0,1) - (1,1,0) = (2,2,6) \\ T(1,1,0) &= 0(0,1,1) - 2(1,0,1) + 2(1,1,0) = (0,2,-2) \end{aligned}$$

Let (x, y, z) be in $\mathbb{R}^3$ such that $(x, y, z) = c_1(0, 1, 1) + c_2(1, 0, 1) + c_3(1, 1, 0)$. Which gives: $c_1 = \frac{y+z-x}{2}$, $c_2 = \frac{x+z-y}{2}$, $c_3 = \frac{x+y-z}{2}$.
Therefore,

$$\begin{aligned} (x,y,z) &= \frac{y+z-x}{2}(0,1,1) + \frac{x+z-y}{2}(1,0,1) + \frac{x+y-z}{2}(1,1,0) \\ T(x,y,z) &= \frac{y+z-x}{2}T(0,1,1) + \frac{x+z-y}{2}T(1,0,1) + \frac{x+y-z}{2}T(1,1,0) \\ &= \frac{y+z-x}{2}(4,2,2) + \frac{x+z-y}{2}(2,2,6) + \frac{x+y-z}{2}(0,2,-2) \\ &= (-x+y+3z, x+y+z, x-3y+5z) \end{aligned}$$

which certainly is the required linear map.