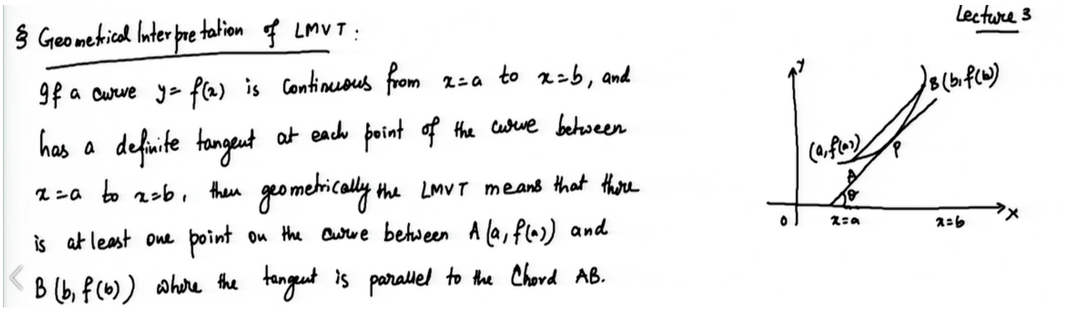
Advanced Engineering Mathematics
Lecture 3

1 Mean Value Theorem

Geometrical Interpretation of LMVT If a curve $y = f(x)$ is continuous from $x = a$ to $x = b$ and has a definite tangent at each point of the curve between $x = a$ to $x = b$, then geometrically the LMVT means that there is at least one point on the curve between $A(a, f(a))$ and $B(b, f(b))$ where the tangent is parallel to the chord AB.



Theorem 1.1. *(Cauchy Mean Value Theorem) If two real valued functions f and g defined on a $[a, b]$ are such that*
i) f and g are continuous on $[a, b]$
ii) f and g are differentiable on (a, b), i.e., f' and g' exists on (a, b)
iii) for no point in (a, b) such that $g'(x) = 0$, i.e, $g'(x) \neq 0\ \forall x \in (a, b)$.
Then, there exists at least one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Note: LMVT can be derive from Cauchy's MVT by taking $g(x) = x$ on $[a, b]$.

Example 1.1. If $f'(x)$ exists on $(0, 1)$, then show that by Cauchy's MVT that $f(1) - f(0) = \frac{f'(x)}{2x}$ has at least one root in $(0, 1)$.

Sol. We consider $g(x) = x^2$ on $[0, 1]$. Then both $f(x)$ and $g(x)$ are continuous on $[0, 1]$ and they are diffrentiable on $(0, 1)$. Also, $g'(x) = 2x \neq 0\ \forall x \in (0, 1)$. Then by Cauchy MVT, $\exists$ at least one point, say $c \in (0, 1)$, such that

$$\frac{f(1) - f(0)}{1 - 0} = \frac{f'(c)}{g'(c)} = \frac{f'(c)}{2c}$$

This implies that there exists at least one $c \in (0, 1)$ which satisfies $f(1) - f(0) = \frac{f'(x)}{2x}$.

Example 1.2. In Cauchy's MVT for the functions $\phi(x) = e^x$ and $f(x) = e^{-x}$, show that c is the arithmetic mean between a and b.

Sol. We are given that $\phi(x) = e^x$ and $f(x) = e^{-x}$. Both ϕ and f are continuous on $[a, b]$ and they are differentiable on (a, b). Also, $f'(x) = -e^{-x} \neq 0\ \forall x \in (a, b)$. The condition for

CMVT are satisfied. Therefore there exists at least one $c \in (a, b)$ such that

$$
\begin{aligned}
& \frac{\phi(a)-\phi(b)}{f(a)-f(b)} = \frac{\phi'(c)}{f'(c)} \\
\Rightarrow \ & \frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}} = -e^{2c} \\
\Rightarrow \ & \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = -e^{2c} \\
\Rightarrow \ & -e^{a+b} = -e^{2c} \\
\Rightarrow \ & a + b = 2c \\
\Rightarrow \ & c = \frac{a+b}{2}
\end{aligned}
$$