Advanced Engineering Mathematics
Lecture 23

Bisection method. Let us consider the given equation $f(x) = 0$. If $f(x) > 0$ and $f(x) < 0$ or $f(x) < 0$ and $f(x) > 0$. Either way $f(a) \cdot f(b) < 0$ To find mid point $x = \frac{a+b}{2}$. Check $f(x_1) > 0$, $[a, x_1]$ there exist a root in $[x_1, b]$. $x_2 = \frac{x_1+b}{2}$

Example 0.1. Find the positive root of the equation $x^3 - 9x + 1 = 0$, correct up to two decimal places.

Sol. Let $f(x) = x^3 - 9x + 1$, $f(0) = 1 > 0$ and $f(1) = -7 < 0$. There is a sign change in $[0, 1]$, so there exists a root between $x = 0$ to $x = 1$, i.e, in $[0, 1]$. Let $x_1 = \frac{0+1}{2} = \frac{1}{2}$. Then, $f(\frac{1}{2}) = -3.37 < 0$

n	a_n	b_n	x_n	$f(x_n)$
0	0	1	0.5	−3.37
1	0	0.5	0.25	−1.23
2	0	0.25	0.125	−0.123
3	0	0.125	0.06255	0.437
4	0.0625	0.125	0.09375	0.155
5	0.09375	0.125	0.1193755	0.016933
6	0.109375	0.125	0.11718	−0.053

The required root is 0.11.

Iterative method/Fixed Point argument. Let us take $f(x) = 0 \Rightarrow x = \phi(x)$. Let us say our initial guess is x_0. Then, we formulate a numerical scheme,

$$x_1 = \phi(x_0)$$
$$\mathrm{x}_2 = \phi(x_1)$$
$$\mathrm{x}_3 = \phi(x_2)$$
$$\cdots$$
$$\mathrm{x}_{n+1} = \phi(x_n)$$

Theorem 0.1. *If $\phi(x)$ is continious function on $[a, b]$, which contains the root of $f(x) = 0$ and if $|\phi'(x)| < 1$ $\forall x \in (a, b)$, then for any choice of $x_0 \in [a, b]$, the sequance $(x_n)_n$, determined as $x_{n+1} = \phi(x_n)$ $\forall n$ converges to the root of $x = \phi(x)$.*

Example 0.2. In order to find the root of $x^3 - x - 1 = 0$ near $x = 1$ which one of the following scheme will be useful. $i) x = x^3 - 1$ $\quad ii) x = \frac{x+1}{x^2}$ $\quad iii) x = \sqrt{\frac{x+1}{x}}$

Sol. $i)$ Given equation is $x^3 - x - 1 = 0 \Rightarrow x = x^3 - 1 = \phi(x) \Rightarrow \phi'(x) = 3x^2$ Clearly $|\phi'(x)| = 3x^2 > 1$ in nbd of 1. The scheme does not converge.
$ii)$ Given equation is $x = \frac{x+1}{x^2} = \phi(x) \Rightarrow \phi'(x) = -\frac{1}{x^2} - \frac{2}{x^3} \Rightarrow |\phi'(x)| \le \frac{1}{x^2} + \frac{2}{x^3} \le \frac{1}{x^3} + \frac{2}{x^3} = \frac{3}{x^3} \ge 3 > 1$. The scheme does not converge.
$iii)$ Given equation is $x = \sqrt{\frac{x+1}{x}} = \phi(x) \Rightarrow \phi'(x) = -\frac{1}{2x^2}\sqrt{\frac{x}{x+1}} \Rightarrow |\phi'(x)| = \left|-\frac{1}{x^2}\sqrt{\frac{x}{x+1}}\right| \le \frac{1}{2} \cdot \frac{1}{x^3}\frac{1}{\sqrt{x+1}} < 1$. The scheme will converge.

Example 0.3. Let $f(x) = x^3 + x - 5 = 0$

Sol. since $f(1) = -3$ and $f(2) = 3$. So there is a root $[1, 2]$. $x = (5 - x)^{\frac{1}{3}} = \phi(x) \Rightarrow |\phi'(x)| = \left|-\frac{1}{3}\frac{1}{(5-x)^{\frac{2}{3}}}\right| \le \frac{1}{3}\frac{1}{(5-x)^{\frac{2}{3}}} \le 1$