Advanced Engineering Mathematics
Lecture 21

Integrating factor. Sometimes to make $Mdx + Ndy = 0$ as exact ODE, we multiply it by a function of x and y. This function is called integrating factor.

Rule 1. If $Mx+Ny \neq 0$ and the equation is homogeneous, then $\frac{1}{Mx+Ny}$ is an integrating factor of $Mdx + Ndy = 0$

Example 0.1. Solve $(x^2y - 2xy^2)\, dx + (3x^2y - x^3)\, dy = 0$

Sol. Comparing with $Mdx + Ndy = 0$, we get $M(x, y) = (x^2y - 2xy^2)$ and $N(x, y) = (3x^2y - x^3)$. $\frac{\partial M}{\partial y} = x^2 - 4xy$ and $\frac{\partial N}{\partial x} = 6xy - 3x^2 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the given ODE is not exact.

Now, $Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y = x^2y^2 \neq 0$. The integrating factor is $\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$. Multiplying the given ODE by the IF, then

$$
\begin{aligned}
&\frac{dx}{y} - \frac{2}{x}\, dx + \frac{3}{y}\, dy - \frac{x}{y^2} dy = 0\\
&\Rightarrow \frac{ydx - xdy}{y^2} - \frac{2}{x}\, dx + \frac{3}{y}\, dy = 0\\
&\Rightarrow d\Big(\frac{x}{y}\Big) - 2d(\log x) + 3d(\log y) = 0\\
&\Rightarrow d\Big(\frac{x}{y} - 2\log x + 3\log y\Big) = 0\\
&\Rightarrow \frac{x}{y} - 2\log x + 3\log y = c
\end{aligned}
$$

Rule 2. If $Mx - Ny \neq 0$ and the equation can be written as $f(x, y)y\, dx + F(x, y)x\, dy = 0$, then $\frac{1}{Mx-Ny}$ is an IF of the equation $M\, dx + N\, dy = 0$.

Example 0.2. Solve $\frac{(xy\sin xy+\cos xy)}{x}\, dx + \frac{(xy\sin xy-\cos xy)}{y}\, dy = 0$

Sol. Given

$$
\begin{aligned}
&\frac{(xy\sin xy + \cos xy)}{x}\, dx + \frac{(xy\sin xy - \cos xy)}{y}\, dy = 0\\
&\Rightarrow y\,(xy\sin xy + \cos xy)\, dx + x\,(xy\sin xy - \cos xy)\, dy = 0 \qquad (0.1)
\end{aligned}
$$

Comparing with $Mdx + Ndy = 0$, we get $M(x, y) = y\,(xy\sin xy + \cos xy)$ and $N(x, y) = x\,(xy\sin xy - \cos xy)$. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the ODE is not exact.

Now, $Mx-Ny = y\,(xy\sin xy+\cos xy)\, x-x\,(xy\sin xy-\cos xy)\, y = 2xy\cos xy$. Multiplying the equation (0.1) by $\frac{1}{Mx-Ny} = \frac{1}{2xy\cos xy}$, we obtain

$$
\begin{aligned}
&\tan xy\,(y\, dx + x\, dy) + \frac{1}{x}\, dx - \frac{dy}{y} = 0\\
&\Rightarrow \tan xy\, d(xy) + \frac{dx}{x} - \frac{dy}{y} = 0\\
&\Rightarrow \log(\sec(xy)) + \log x - \log y = \log c\\
&\Rightarrow \frac{\sec(xy)x}{y} = c\\
&\Rightarrow cy = x\sec(xy)
\end{aligned}
$$

Rule 3. If $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone (say $f(x)$), then $e^{\int f(x)\,dx}$ is an integrating factor.
Rule 4. If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is a function of y alone (say $\phi(y)$), then $e^{\int \phi(y)\,dy}$ is an integrating factor.

Example 0.3. Solve $(x^2+y^2+2x)\,dx+2y\,dy=0$.

Sol.

$$M(x,y)=x^2+y^2+2x,\ N(x,y)=2y \Rightarrow \frac{\partial M}{\partial y}=2y,\ \frac{\partial N}{\partial x}=0 \Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}.$$

Now, $f(x)=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2y}(2y-0)=1$. Then, the required IF$=e^{f(x)}\,dx=e^x$.

Multiplying the given differential equation by e^x, we get

$$\begin{aligned}
&(x^2+y^2+2x)e^x\,dx+2ye^x\,dy=0\\
&\Rightarrow (x^2+y^2)e^x\,dx+y^2e^x\,dx+2ye^x\,dy=0\\
&\Rightarrow d(x^2e^x)+d(y^2e^x)=0\\
&\Rightarrow (x^2+y^2)e^x=c
\end{aligned}$$

Example 0.4. Solve $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0$

Sol. Comparing with $Mdx+Ndy=0$, we get $M(x,y)=(3x^2y^4+2xy)$ and $N(x,y)=(2x^3y^3-x^2)$.
Here $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. Now, $\phi(y)=\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{xy(3xy^2+2)}(6x^2y^3-2x-12x^2y^3-2x)=-\frac{2}{y}$.
The required IF is $e^{\int\phi(y)\,dy}=e^{-\int\frac{2}{y}\,dy}=e^{-2\log y}=\frac{1}{y^2}$.

Multiplying the given differential equation by $\frac{1}{y^2}$, we obtain

$$\begin{aligned}
&\frac{1}{y^2}(3x^2y^4+2xy)\,dx+\frac{1}{y^2}(2x^3y^3-x^2)\,dy=0\\
&\Rightarrow 3x^2y^2\,dx+2\frac{x}{y}\,dx+2x^3y\,dy-\frac{x^2}{y^2}\,dy=0\\
&\Rightarrow d(x^3y^2)+\frac{2xy\,dx-x^2\,dy}{y^2}=0\\
&\Rightarrow d(x^3y^2)+d\left(\frac{x^2}{y}\right)=0\\
&\Rightarrow x^3y^2+\frac{x^2}{y}=c
\end{aligned}$$