Advanced Engineering Mathematics
Lecture 20

1 Differential Equation.

$$f(x,y,y')=0, y'=\frac{dy}{dx}$$
$$f(x,y,y',y'',\cdots y^n)=0$$
$$y''+xy'+2y=0 \text{ order 2, degree 1}$$

Example 1.1. From the relation $y(x)=Ae^x+Be^{-x}+x^2$, find differential equation where A,B are arbitrary constants.

Sol.

$$y(x)=Ae^x+Be^{-x}+x^2$$
$$\Rightarrow y'(x)=Ae^x-Be^{-x}+2x$$
$$\Rightarrow y''(x)=Ae^x+Be^{-x}+2 \Rightarrow\ y''=y-x^2+2 \Rightarrow\ y''-y=2-x^2$$

It is the required differential equation.

Example 1.2. (Solution by Separation of variables) $x^2\frac{dx}{dy}+y=1$

Sol. Given,

$$x^2\frac{dx}{dy}+y=1$$
$$\Rightarrow x^2\frac{dx}{dy}=1-y$$
$$\Rightarrow \frac{dy}{1-y}=\frac{dx}{x^2}$$
$$\Rightarrow -\int\frac{-dy}{1-y}=\int\frac{dx}{x^2}+\log c$$
$$\Rightarrow -\log_e(1-y)=-\frac{1}{x}+\log c$$
$$\Rightarrow -\log_e\frac{1-y}{c}=\frac{1}{x}$$
$$\Rightarrow \frac{1-y}{c}=e^{\frac{1}{x}}$$
$$\Rightarrow (1-y)=ce^{\frac{1}{x}}$$
$$\Rightarrow y=1-ce^{\frac{1}{x}}$$

It is the required solution of the given differential equation.

Example 1.3. Solve the differential equation $\sqrt{1-x^2}\,dy+\sqrt{1-y^2}\,dx=0$. Also find a particular solution if $y(x=0)=1$.

Sol.

$$\frac{dy}{\sqrt{1-y^2}}+\frac{dx}{\sqrt{1-x^2}}=0$$
$$\Rightarrow \int\frac{dy}{\sqrt{1-y^2}}+\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}c$$
$$\Rightarrow \sin^{-1}x+\sin^{-1}y=\sin^{-1}c$$
$$\Rightarrow \sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})=\sin^{-1}c$$
$$\Rightarrow x\sqrt{1-y^2}+y\sqrt{1-x^2}=c$$

Putting $x=0$ in the above equation, we obtain

$$0 \cdot \sqrt{1-1} + 1 \cdot \sqrt{1-0} = c \Rightarrow c = 1$$

Thus,

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$$

Theorem 1.1. *The neccessary sufficient condition for the ODE to be exact is that* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example 1.4. Verify whether the ODE $a^2 - 2xy - y^2 dx - (x+y)^2 dy = 0$ is exact or not.

Sol. Comparing with $Mdx + Ndy = 0$, we get $M(x,y) = a^2 - 2xy - y^2$ and $N(x,y) = -(x+y)^2$. For exactness we need to show $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\frac{\partial M}{\partial y} = -2x - 2y$ and $\frac{\partial N}{\partial x} = -2(x+y) \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ the given ODE is exact.

Example 1.5. Verify whether the ODE $(y^2 e^x + 2xy)dx - x^2 dy = 0$ is exact or not.

Sol. Comparing with $Mdx + Ndy = 0$, we get $M(x,y) = y^2 e^x + 2xy$ and $N(x,y) = -x^2$. For exactness we need to show $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\frac{\partial M}{\partial y} = 2ye^x + 2x$ and $\frac{\partial N}{\partial x} = -2x \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the given ODE is not exact.

Example 1.6. Verify whether the ODE $(x^2 + y^2 + 2x)dx + 2xdy = 0$ is exact or not.

Sol. Comparing with $Mdx + Ndy = 0$, we get $M(x,y) = (x^2 + y^2 + 2x)$ and $N(x,y) = 2y$. For exactness we need to show $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the given ODE is not exact.