Advanced Engineering Mathematics
Lecture 19

1 Integral Calculus

Example 1.1. Evaluate the volume cut off from the sphere $x^2+y^2+z^2=a^2$ by the cylinder $x^2+y^2=ax$

Sol. The given region is bounded by the surfaces $z=\sqrt{a^2-x^2-y^2}$ and $z=-\sqrt{a^2-x^2-y^2}$ and it's projection in the $xy-$ plane is the circular domain $D: x^2+y^2=ax$.

$$\begin{aligned}
V &= \iint_D dx\,dy \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz \\
&= \iint_D \sqrt{a^2-x^2-y^2}\,dx\,dy \\
&=2\int_0^{\pi}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\,r\,dr\,d\theta \\
&=-\frac{2}{3}\int_0^{\pi}(a^2-r^2)^{\frac{3}{2}}\int_0^{a\cos\theta} d\theta \\
&=\frac{2}{3}a^3\int_0^{\pi}(1-\sin^2\theta)\,d\theta \\
&=\frac{2}{3}a^3(\pi-2.\frac{2}{3})=\frac{2}{3}a^3(\pi-\frac{4}{3})
\end{aligned}$$

Example 1.2. Find the volume of the region above xy plane bounded by the paraboloid $z=x^2+y^2$ and the cylinder $x^2+y^2=a^2$

Sol. The given solid is bounded above by $z=x^2+y^2$ and below by $x^2+y^2=a^2$. Moreover the solids are symmetrical, its volume V four times of the volume lying in first octant.

$$\begin{aligned}
V &= 4\iint_D z\,dx\,dy \\
&= 4\iint_D (x^2+y^2)\,dx\,dy \\
&= 4\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{a} r^2\,r\,dr\,d\theta = 4\times\frac{\pi}{2}\times\frac{a^4}{4}=\frac{\pi a^4}{2}
\end{aligned}$$

Example 1.3. Find the volume of the region bounded by the paraboloid $z=(x-1)^2+y^2$ and the plane $2x+z=2$.

Sol. The required value is,

$$\begin{aligned}
V &= 4\iint_D (z_2-z_1)\,dx\,dy \\
&= 4\iint_D \Big[(2-2x)-\Big\{(x-1)^2+y^2\Big\}\Big]\,dx\,dy
\end{aligned}$$

Where D is a region given by eliminating z between two given surfaces, $(x-1)^2+y^2=2-2x$. So,

$$\begin{aligned}
V &= 4\iint_D \Big[1-(x^2+y^2)\Big]\,dx\,dy \\
&= 4\int_{\theta=0}^{2\pi}\int_0^1 (1-r^2)\,dr\,d\theta = 2\pi\Big(\frac{1}{2}-\frac{1}{4}\Big)=\frac{2\pi}{4}=\frac{\pi}{2}
\end{aligned}$$

Surface area. The area of the smooth surface $x = f(u,v), y = g(u,v), z = h(u,v), (u,v) \in D$ defined as double integral,

$$\iint_D \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, du\, dv$$

Given the surface $z = f(x,y)$ or $x = u, y = v$ and $z = f(u,v)$. If we take the surface area over the surface $z = f(x,y)$ having projection D on xy plane as S, then

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

Example 1.4. Find the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ cut of by $x^2 + y^2 = ax$

Sol. We have $z = \sqrt{a^2 - x^2 - y^2}$ and the required surface area is

$$\begin{aligned}
S &= 4 \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy \qquad \text{(where } D \text{ is the projection of sphere in } xy \text{ plane)} \\
&= 4 \iint_D \frac{a\, dx\, dy}{\sqrt{a^2 - x^2 - y^2}} \\
&= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=o}^{a\cos\theta} \frac{a\, r\, dr\, d\theta}{\sqrt{a^2 - r^2}} \qquad (\because x^2 + y^2 = ax \Rightarrow r^2 = ar\cos\theta \Rightarrow r = a\cos\theta) \\
&= 4a \int_0^{\frac{\pi}{2}} \left[-\sqrt{a^2 - r^2}\right]_a^{a\cos\theta} d\theta \\
&= 4a^2 \int_0^{\frac{\pi}{2}} (1 - \sin\theta)\, d\theta = 4a^2\left[\frac{\pi}{2} - 1\right] = 2a^2(\pi - 2)
\end{aligned}$$

Example 1.5. Find the area of the part of the surface of the cylinder $x^2 + y^2 = a^2$ which is cut of by the cylinder $x^2 + z^2 = a^2$.

Sol. The given cylinder is $y = \sqrt{a^2 - x^2}$. Then, $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \frac{az}{a^2 - x^2}$. The required surface area is (doubt in integration)

$$\begin{aligned}
S &= 4 \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy \\
&= 2a \iint_D \frac{dx\, dz}{\sqrt{a^2 - x^2}} \qquad (D : x^2 + y^2 = a^2) \\
&= 2a \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2}} dz \\
&= 8a \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} \times \sqrt{a^2 - x^2} = 8a^2
\end{aligned}$$